Time Series Analysis

Solutions to problems in Chapter 5

IMM

Solution 5.1

Question 1.

$$V[X_t] = V\left[\epsilon_t + c(\epsilon_{t-1} + \epsilon_{t-2} + \cdots)\right] = \left[1 + c^2 \sum_{i=1}^{\infty} 1\right] \sigma_\epsilon^2 = \infty$$

The variance of $\{X_t\}$ is not limited and therefore $\{X_t\}$ is not stationary.

Question 2.

$$\begin{aligned} Y_t &= X_t - X_{t-1} \\ &= \epsilon_t + c(\epsilon_{t-1} + \epsilon_{t-2} + \cdots) - \epsilon_{t-1} - c(\epsilon_{t-2} + \epsilon_{t-3} + \cdots) \\ &= \epsilon_t + (c-1)\epsilon_{t-1} \end{aligned}$$

I.e. a MA(1)-process and therefore stationary (independent of c).

Question 3.

The auto-covariance is

$$\begin{aligned} \gamma_Y(0) &= \left(1 + (c-1)^2\right)\sigma_\epsilon^2 = \mathrm{Var}[Y_t] \\ \gamma_Y(1) &= \mathrm{Cov}\left[\epsilon_t + (c-1)\epsilon_{t-1}, \epsilon_{t+1} + (c-1)\epsilon_t\right] \\ &= (c-1)\sigma_\epsilon^2 = \gamma_Y(-1) \\ \gamma_Y(k) &= 0 \qquad\qquad |k| \geq 2 \end{aligned}$$

The auto-correlation function of $\{Y_t\}$ is

$$\rho(k) = \begin{cases} 1 & k = 0 \\ \frac{c-1}{1+(c-1)^2} & k = \pm 1 \\ 0 & |k| \geq 2 \end{cases}$$

Solution 5.2

Question 1.

The process is a first order Markov chain with two states and jump-intensity λ. The process can be visualised as shown in figure 1

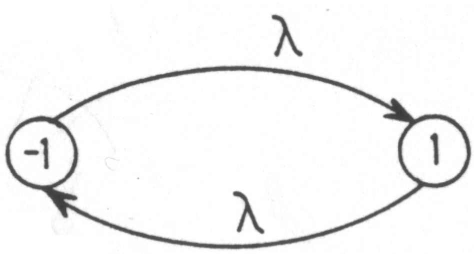

Figure 1: A two states Markov chain.

Investigation of stationarity:

Because of symmetry the following is given

$$\mathrm{P}\{X(t) = 1\} = \mathrm{P}\{X(t) = -1\} = \frac{1}{2} \tag{1}$$

From this follows

$$E[X(t)] = 1 \cdot \mathrm{P}\{X(t) = 1\} + (-1) \cdot \mathrm{P}\{X(t) = -1\} = 0$$

Notice that if the process was not initiated with $\mathrm{P}\{X(t) = -1\} = \frac{1}{2}$ equation (1) would only hold after infinitely long time. The covariance can be determined as

$$\begin{aligned}\gamma(u) &= \mathrm{Cov}[X(t), X(t+u)] = E[X(t)X(t+u)] \\ &= 1 \cdot \mathrm{P}\{X(t) \text{ and } X(t+u) \text{ has same sign}\} + \\ &\quad (-1)\mathrm{P}\{X(t) \text{ and } X(t+u) \text{ has opposite sign}\}\end{aligned}$$

Jumps between state -1 and 1 can be described by an Poisson process with intensity λ. This means that the number of jumps in the time interval $[t, t+$

$u]$ will follow a $P(\lambda u)$-distribution. The frequency function for a $P(\lambda u)$-distribution is

$$f(x) = \frac{(\lambda u)^x}{x!} e^{-\lambda u} \qquad , \qquad x = 0, 1, 2, \ldots$$

The probability of $X(t)$ and $X(t+u)$ having the same sign is equal the probability for an equal number of jumps in the interval $[t, t+u]$. I.e.

$$\begin{aligned}
\gamma(u) &= 1 \cdot \text{Prob}\{\text{even number of jumps}\} + \\
&\quad (-1) \cdot \text{Prob}\{\text{an uneven number of jumps}\} \\
&= \sum_{x \text{ even}} \frac{(\lambda u)^x}{x!} e^{-\lambda u} - \sum_{x \text{ uneven}} \frac{(\lambda u)^x}{x!} e^{-\lambda u} \qquad (u \geq 0) \\
&= e^{-\lambda u} \left(1 - \frac{\lambda u}{1} + \frac{(\lambda u)^2}{2!} - \frac{(\lambda u)^3}{3!} + \cdots \right) \\
&= e^{-\lambda u} e^{-\lambda u} \\
&= e^{-2\lambda u} \qquad (u \geq 0)
\end{aligned}$$

Since $\gamma(u) = \gamma(-u)$ the following applies

$$\gamma(u) = e^{-2\lambda |u|} \quad \Rightarrow \quad \rho(u) = \frac{\gamma(u)}{\gamma(0)} = e^{-2\lambda |u|}$$

The spectrum is determined from theorem 5.6 (5.58) to

$$\begin{aligned}
f(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma(u) e^{-i\omega u} du \qquad (-\infty < \omega < \infty) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2\lambda |u|} e^{-i\omega u} du \\
&= \frac{1}{2\pi} \left(\int_{-\infty}^{0} e^{(2\lambda - i\omega)u} du + \int_{0}^{\infty} e^{(-2\lambda - i\omega)u} du \right) \\
&= \frac{1}{2\pi} \left(\frac{1}{2\lambda - i\omega} - \frac{1}{-2\lambda - i\omega} \right) \\
&= \frac{1}{2\pi} \left(\frac{2\lambda + i\omega}{4\lambda^2 + i\omega} - \frac{-2\lambda + i\omega}{4\lambda^2 + \omega^2} \right) \\
&= \frac{1}{2\pi} \frac{4\lambda}{4\lambda^2 + \omega^2} \qquad (-\infty < \omega < \infty)
\end{aligned}$$

Solution 5.3

Question 1.

$Z_t = X_t + Y_t$, X_t and Y_t are independent and X_t and Y_t are stationary processes, which means that the sum of the two process is also stationary. The autocovariance of Z_t is

$$\begin{aligned}\gamma_Z(k) &= \text{Cov}[Z_t, Z_{t+k}] = \text{Cov}[X_t + Y_t, X_{t+k} + Y_{t+k}] \\ &= \text{Cov}[X_t, X_{t+k}] + \text{Cov}[Y_t, Y_{t+k}] = \gamma_X(k) + \gamma_Y(k)\end{aligned}$$

We get the following spectral density

$$\begin{aligned}f_Z(\omega) &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_Z(k)e^{-i\omega k} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}(\gamma_X(k) + \gamma_Y(k))e^{-i\omega k} \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_X(k)e^{-i\omega k} + \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_Y(k)e^{-i\omega k} = f_X(\omega) + f_Y(\omega)\end{aligned}$$

Question 2.

Now

$$Z_t = X_t + Y_t \qquad , \qquad X_t = \alpha X_{t-1} + W_t \ , (|\alpha| < 1)$$

X_t is an AR(1)-process, which can also be written on the form

$$(1 - \alpha B)X_t = \pi(B)X_t = W_t$$

W_t and Y_t are independent $\Rightarrow$ X_t and Y_t are independent. I.e. the previous result $f_Z(\omega) = f_X(\omega) + f_Y(\omega)$ can be used. The spectral density of X_t is found using (5.85)

$$f_X(\omega) = \frac{b^2}{2\pi}\frac{1}{(1 - \alpha e^{i\omega})(1 - \alpha e^{-i\omega})} = \frac{b^2}{2\pi}\frac{1}{1 + \alpha^2 - 2\alpha\cos\omega} \ ,$$

and the spectral density of Y_t is

$$f_Y(\omega) = \frac{b^2}{2\pi} \ ,$$

which leads to

$$f_Z(\omega) = f_X(\omega) + f_Y(\omega) = \left(\frac{b^2}{2\pi}\right)\frac{2 + \alpha^2 - 2\alpha\cos\omega}{1 + \alpha^2 - 2\alpha\cos\omega}$$

Solution 5.4

Question 1.

The process X_t defined by

$$\nabla X_t = \epsilon_t \ ,$$

where ϵ_t is white noise with the mean value μ_ϵ and the variance σ_ϵ^2, is called a random walk. We set $X_1 = \epsilon_1$ and consider only the time indices $t | t \geq 1$, i.e.

$$\begin{aligned} &X_t = X_{t-1} + \epsilon_t \Rightarrow \\ &X_1 = \epsilon_1 \\ &X_2 = X_1 + \epsilon_2 = \epsilon_1 + \epsilon_2 \end{aligned}$$

i.e.

$$X_t = \sum_{i=1}^{t} \epsilon_t$$

The mean value of X_t is

$$\mathrm{E}[X_t] = \mathrm{E}[\sum_{i=1}^{t} \epsilon_i] = \underline{\underline{t\mu_\epsilon}}$$

The variance of X_t is

$$\begin{aligned} V[X_t] = \mathrm{Cov}[X_t, X_t] &= \mathrm{Cov}[\sum_{i=1}^{t} \epsilon_i, \sum_{i=1}^{t} \epsilon_i] \\ &= (V[\epsilon_1] + V[\epsilon_2] + ... + V[\epsilon_t]) = \underline{\underline{t\sigma_\epsilon^2}} \ , \end{aligned}$$

as $\mathrm{Cov}[\epsilon_s, \epsilon_p] = 0$ for $s \neq p$.
The covariance is

$$\begin{aligned} \mathrm{Cov}[X_{t_1}, X_{t_2}] &= \mathrm{Cov}[\sum_{i=1}^{t_1} \epsilon_i, \sum_{i=1}^{t_2} \epsilon_i] \\ &= \underline{\underline{\min(t_1, t_2)\sigma_\epsilon^2}} \end{aligned}$$

Question 2.

X_t is not stationary as for instance $\mathrm{E}[X_t]$ depends on t.

Solution 5.5

Question 1.

The process X_t is given by

$$\begin{aligned} &X_t - \phi X_{t-1} = \epsilon_t \Rightarrow \\ &(1 - \phi B) X_t = \epsilon_t \ , \end{aligned}$$

which in MA-form is

$$\begin{aligned} X_t = (1 - \phi B)^{-1} \epsilon_t &= \sum_{n=0}^{\infty} (\phi B)^n \epsilon_t \\ &= (1 + \phi B + \phi^2 B^2 + .. + \phi^i B^i + ...) \epsilon_t \end{aligned}$$

Question 2.

a) The process

$$\begin{aligned} &X_t = \frac{5}{6} X_{t-1} - \frac{1}{6} X_{t-2} + \epsilon_t - \frac{1}{4} \epsilon_{t-1} \Rightarrow \\ &(1 - \frac{5}{6} B + \frac{1}{6} B^2) X_t = (1 - \frac{1}{4} B) \epsilon_t \end{aligned}$$

is an ARIMA(2,0,1) process, i.e. an ARMA(2,1) process. The transfer function is

$$H(z) = \frac{\theta(z^{-1})}{\phi(z^{-1})} = \frac{1 - \frac{1}{4} z^{-1}}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}} \ .$$

The roots of $\phi(z^{-1})$ is $\frac{1}{2}$ and $\frac{1}{3}$, which is within the unit circle, thus the ARMA(2,1) process is stationary. Since the root to $\theta(z^{-1})$ is $\frac{1}{4}$ the ARMA(2,1) process is invertible.

b) The process

$$\begin{aligned} &X_t = \frac{4}{3} X_{t-1} - \frac{1}{3} X_{t-2} + \epsilon_t - \frac{1}{4} \epsilon_{t-1} \Rightarrow \\ &(1 - \frac{4}{3} B + \frac{1}{3} B^2) X_t = (1 - \frac{1}{3} B)(1 - B) X_t = (1 - \frac{1}{4} B) \epsilon_t \end{aligned}$$

is an ARIMA(1,1,1) process. The transfer function is

$$H(z) = \frac{\theta(z^{-1})}{\phi(z^{-1})(1-z^{-1})^d} = \frac{1-\frac{1}{4}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-z^{-1})} \ .$$

The process is invertible as the root of $\theta(z^{-1})$ is $\frac{1}{4}$ as found above. The process is not stationary as the transfer function has one pole in z=1.

c) The process

$$X_t = 2X_{t-1} - X_{t-2} + \epsilon_t - \frac{1}{4}\epsilon_{t-1} \Rightarrow$$
$$(1-B)^2 X_t = (1 - \frac{1}{4}B - \frac{1}{4}B^2)\epsilon$$

is an ARIMA(0,2,2) process. The transfer function is

$$H(z) = \frac{\theta(z^{-1})}{\phi(z^{-1})(1-z^{-1})^d} = \frac{1-\frac{1}{4}z^{-1}-\frac{1}{4}z^{-2}}{(1-z^{-1})^2} \ .$$

Again the ARIMA process is not stationary as the transfer function has two poles in z=1. The process is invertible as the zeros of the system 0.64 and -0.39 are within the unit circle.

Solution 5.6

An ARMA process is given by

$$(1-\phi B)X_t = (1-\theta B)\epsilon_t \ ,$$

with E[ϵ_t]=0 and Var[ϵ_t]=σ_ϵ^2

Question 1.

The process can be rewritten into AR-form in the following way

$$\begin{aligned}
(1-\phi B)(1-\theta B)^{-1}X_t &= \epsilon_t \\
\left((1-\phi B)\sum_{n=0}^{\infty}(\theta B)^n\right)X_t &= \epsilon_t \\
\left((1+\theta B+\theta^2B^2+\ldots)-(\phi B+\phi\theta B^2+\phi\theta^2B^3+\ldots)\right) &= \epsilon_t
\end{aligned}$$

Question 2.

The process can be rewritten into MA-form in the following way

$$\begin{aligned}
X_t &= (1-\theta B)(1-\phi B)^{-1}\epsilon_t \\
&= \left((1-\theta B)\sum_{n=0}^{\infty}(\phi B)^n\right)\epsilon_t \\
&= \left((1+\phi B+\phi^2B^2+\ldots)-(\theta B+\theta\phi B^2+\theta\phi^2B^3+\ldots)\right)\epsilon_t
\end{aligned}$$

The variance of X_t can be found as

$$\begin{aligned}
V[X_t] =& \text{Cov}[X_t, X_t] \\
=& \text{Cov}[(\epsilon_t+\phi\epsilon_{t-1}+\phi^2\epsilon_{t-2}+\ldots)-(\theta\epsilon_{t-1}+\theta\phi\epsilon_{t-2}+\theta\phi^2\epsilon_{t-3}+\ldots), \\
&(\epsilon_t+\phi\epsilon_{t-1}+\phi^2\epsilon_{t-2}+\ldots)-(\theta\epsilon_{t-1}+\theta\phi\epsilon_{t-2}+\theta\phi^2\epsilon_{t-3}+\ldots)]
\end{aligned}$$

since $\mathrm{Cov}[\epsilon_s, \epsilon_p] = 0$ for $s \neq p$ we get

$$\begin{aligned}
V[X_t] =& ((1 + \phi^2 + \phi^4 + \ldots) + (\theta^2 + \theta^2\phi^2 + \theta^2\phi^4 + \ldots) - 2(\theta\phi + \theta\phi^3 + \theta\phi^5 + \ldots))\sigma_\epsilon^2 \\
=& (1 + \phi^2 + \phi^4 + \ldots)(1 + \theta^2 - 2\theta\phi)\sigma_\epsilon^2 \\
=& \underline{\underline{\frac{1 + \theta^2 - 2\theta\phi}{1 - \phi^2}\sigma_\epsilon^2}}
\end{aligned}$$

Solution 5.7

Question 1.

$$Y_t = \nabla_s X_t = X_t - X_{t-s}$$

$$\begin{aligned}
\gamma_y(k) &= \mathrm{Cov}[Y_t, Y_{t+k}] = \mathrm{Cov}[X_t - X_{t-s}, X_{t+k} - X_{t+k-s}] \\
&= \mathrm{Cov}[X_t, X_{t+k}] + \mathrm{Cov}[X_{t-s}, X_{t+k-s}] - \mathrm{Cov}[X_t, X_{t+k-s}] - \mathrm{Cov}[X_{t-s}, X_{t+k}] \\
&= 2\gamma_x(k) - \gamma_x(k-s) - \gamma_x(k+s)
\end{aligned}$$

Question 2.

$\{X_t\}$ is now an AR(1)-process. I.e. the auto-covariance funktion of $\{X_t\}$ is

$$\gamma_x(k) = \alpha^{|k|}\sigma_x^2 = \frac{\alpha^{|k|}}{1-\alpha^2}\sigma_\epsilon^2$$

Using the result from question 1 leads to the following auto-covariance function of $\{Y_t\}$

$$\begin{aligned}
\gamma_y(0) &= 2\gamma_x(0) - 2\gamma_x(s) = 2(1-\alpha^s)\sigma_x^2 & \\
\gamma_y(k) &= 2\gamma_x(k) - \gamma_x(s-k) - \gamma_x(k+s) & 0 \le k < s \\
&= [2\alpha^k - \alpha^{s-k} - \alpha^{k+s}]\sigma_x^2 & 0 \le k < s \\
\gamma_y(k) &= 2\gamma_x(k) - \gamma_x(k-s) - \gamma_x(k+s) & s \le k \\
&= [2\alpha^k - \alpha^{k-s} - \alpha^{k+s}]\sigma_x^2 & s \le k \\
&= -\alpha^{k-s}(\alpha^s - 1)^2\sigma_x^2 & s \le k \\
\gamma_y(k) &= \gamma_y(-k) & k < 0
\end{aligned}$$

Question 3.

$$s = 1 \quad \Rightarrow \quad Y_t = X_t - X_{t-1}$$

Setting $s = 1$ in the result from question 2 leads to

$$V[Y_t] = \sigma_y^2 = 2(1-\alpha)\sigma_x^2$$

I.e.

$$\sigma_y^2 < \sigma_x^2 \quad \Leftrightarrow \quad 2(1-\alpha)\sigma_x^2 < \sigma_x^2 \quad \Leftrightarrow \quad 2 - 2\alpha < 1 \quad \Leftrightarrow \quad \alpha > \frac{1}{2}$$

At the same time $|\alpha| < 1$. For

$$\frac{1}{2} < \alpha < 1$$

a smaller variance occurs for the differensed process.

Solution 5.8

Question 1.

Given the following ARMA(2,1) process

$$(1 - 1.27B + 0.81B^2)X_t = (1 - 0.3B)\epsilon_t$$

where ϵ_t is white noise. According to theorem 5.12 the process is stationary if the roots of $\phi(z^{-1})$ lie within the unit circle. For this process $\phi(z^{-1}) = 1 - 1.27z^{-1} + 0.81z^{-2}$. The roots to $z^{-2}(1 - 1.27z^{-1} + 0.81z^{-2}) = 0$ are $0.635 \pm 0.638i$, which has the modolus $(0.636^2 + 0.638^2)^{1/2} = 0.9 < 1$, i.e the given ARMA(2,1) process is stationary.

Question 2.

The impulse response function can be found by sending a '1' (a pulse) through the system, i.e. $\epsilon_0 = 1$ and $\epsilon_k = 0$ for $k \neq 0$. The results are summarized in Table 1 and the impluse response function is plotted in Figure 2.

k	< 0	0	1	2	3	4	5	6	7	8	9	10
ϵ_k	0	1	0	0	0	0	0	0	0	0	0	0
$x_k = h_k$	0	1	0.97	0.42	-0.25	-0.66	-0.63	-0.27	0.17	0.43	0.42	0.18

Table 1: Solution to question 2: The impulse response function, h_k

Question 3.

According to (4.21) the frequency response function can be determined by

$$\mathcal{H}(\omega) = \sum_{k=0}^{10} h_k e^{i\omega k} \quad - \pi \leq \omega < \pi \ ,$$

i.e.

$$\begin{aligned}\mathcal{H}(\omega) =& 1 + 1.27e^{i\omega} + 0.80e^{2i\omega} - 0.01e^{3i\omega} - 0.66e^{4i\omega} - 0.83e^{5i\omega} \\ & - 0.52e^{6i\omega} + 0.01e^{7i\omega} + 0.44e^{8i\omega} + 0.55e^{9i\omega} + 0.34e^{10i\omega}\end{aligned}$$

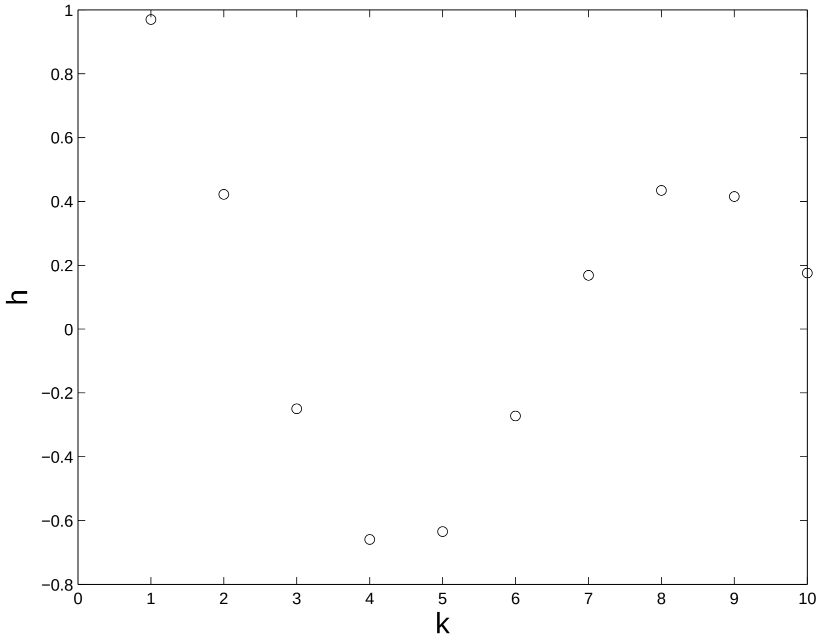

Figure 2: The impulse response function h_k

This expression is an approximation for the gain based on the 10 first values of the impulse response. The frequency response function can according to Theorem 4.10 also be obtained from the transfer function as

$$\mathcal{H}(\omega) = H(e^{i\omega}) = \frac{1 - 0.3e^{-i\omega}}{1 - 1.27e^{-i\omega} + 0.81e^{-i2\omega}} \ ,$$

which leads to the following exact amplitude function

$$\begin{aligned} G(\omega)^2 &= \mathcal{H}(\omega)\overline{\mathcal{H}(\omega)} = \frac{1 - 0.3e^{-i\omega}}{1 - 1.27e^{-i\omega} + 0.81e^{-i2\omega}} \frac{1 - 0.3e^{i\omega}}{1 - 1.27e^{i\omega} + 0.81e^{i2\omega}} \\ &= \frac{1.09 - 0.3e^{-i\omega} - 0.3e^{i\omega}}{1 - 1.27e^{i\omega} + 0.81e^{i2\omega} - 1.27e^{-i\omega} + 1.613 - 1.03e^{i\omega} + 0.81e^{-i2\omega} - 1.03e^{-i\omega} + 0.66} \\ &= \frac{1.09 - 0.6\cos(\omega)}{3.273 - 4.6\cos(\omega) + 1.62\cos(2\omega)} \Rightarrow \\ G(\omega) &= \sqrt{\frac{1.09 - 0.6\cos(\omega)}{3.273 - 4.6\cos(\omega) + 1.62\cos(2\omega)}} \end{aligned}$$

The two solutions for the amplitude function are plotted in 3. The high gain

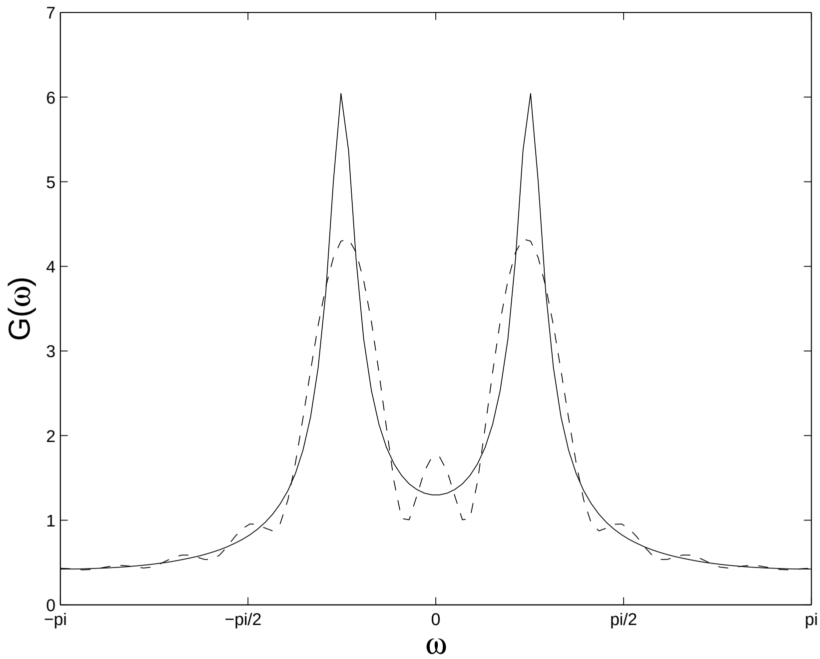

Figure 3: The amplitude function calculated from the impulse response function h_k for $k \leq 10$ (dotted line) and from the transfer function $H(z)$ (full line)

for $\omega = \pi/4$ is connected to the argument of the complex conjugated roots, which is $\pi/4$, and to the fact that the roots are close to the unit circle. The closer the roots lie to the unit circle, the higher is the gain.

Solution 5.9

Question 1.

$$\text{Impulse response function:} \qquad h_k = \begin{cases} 0 & k < 0 \\ 1 & k = 0 \\ 0 & k = 1,2,3 \\ -1 & k = 4 \\ 0 & k > 4 \end{cases}$$

$$\text{Step response function:} \qquad S_k = \begin{cases} 0 & k < 0 \\ 1 & k = 0,1,2,3 \\ 0 & k > 3 \end{cases}$$

$$\text{Frequency response function:} \qquad \mathcal{H}(\omega) = 1 - e^{-i4\omega}$$

$$\begin{aligned} G^2(\omega) &= \mathcal{H}(\omega)\overline{\mathcal{H}(\omega)} = (1 - e^{-i4\omega})(1 - e^{i4\omega}) \\ &= 2 - 2\cos 4\omega = 2(1 - (1 - 2\sin^2 2\omega)) = 4\sin^2 2\omega \\ &\Rightarrow \text{ Amplitude:} \quad G(\omega) = |2 \sin 2\omega| \end{aligned}$$

The amplitude function is sketched in figure 4.

Question 2.

$$Y_t = SX_t = \nabla^{-1}X_t = \frac{1}{1-B}X_t$$

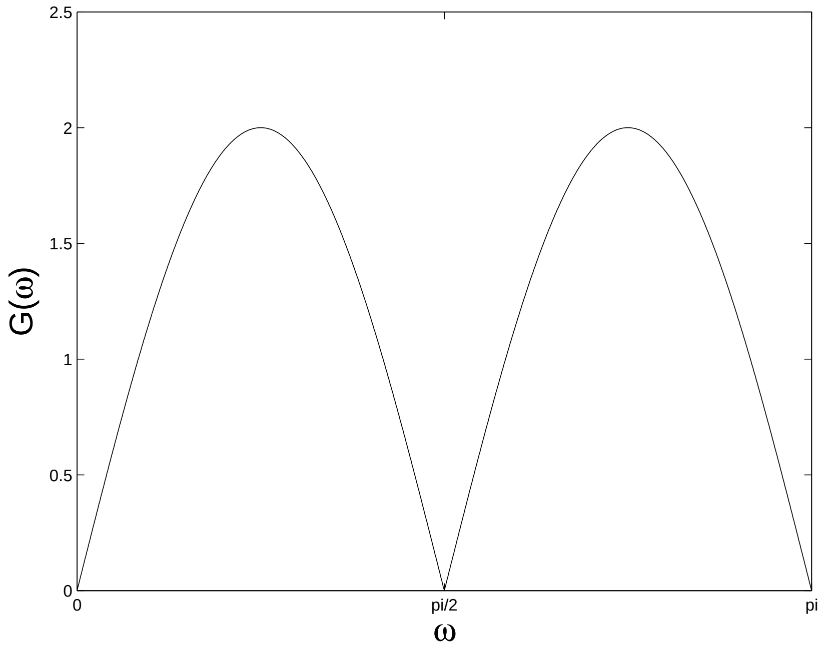

Figure 4: $G(\omega)$ Question 1.

$$\begin{array}{lrcl}
\text{Impulse response function:} & h_k & = & \begin{cases} 0 & k < 0 \\ 1 & k \geq 0 \end{cases} \\
\text{Step response function:} & S_k & = & \begin{cases} 0 & k < 0 \\ k+1 & k \geq 0 \end{cases} \\
\text{Frequency response function:} & \mathcal{H}(\omega) & = & \frac{1}{1-e^{-i\omega}}
\end{array}$$

$$G^2(\omega) = \mathcal{H}(\omega)\overline{\mathcal{H}(\omega)} = \frac{1}{(1-e^{-i\omega})}\frac{1}{(1-e^{i\omega})} = \frac{1}{2(1-\cos\omega)}$$

Notice that $G^2(\omega) \to \infty$ for $\omega \to 0$. For Y_t, determined by $Y_t = SX_t$, to be a stationary process one has to require that $f_y(\omega) < \infty$. This requirement can be satisfied when

$$\exists \quad K : f_x(\omega) \leq K(1-\cos\omega) \text{ for all } \omega$$

Question 3.

$$Y_t = (1-\alpha)Y_{t-1} + \alpha X_t \quad \Leftrightarrow \quad Y_t = \frac{\alpha}{(1-(1-\alpha)B)} X_t$$

$$\begin{aligned} G^2(\omega) &= \mathcal{H}(\omega)\overline{\mathcal{H}(\omega)} = \frac{\alpha}{(1-(1-\alpha)e^{-i\omega})} \frac{\alpha}{(1-(1-\alpha)e^{i\omega})} \\ &= \frac{\alpha^2}{2+\alpha^2-2\alpha-2(1-\alpha)\cos\omega} \end{aligned}$$

$\Rightarrow$ Amplitude:

$$G(\omega) = \frac{\alpha}{(2+\alpha^2-2\alpha-2(1-\alpha)\cos\omega)^{1/2}}$$

For $\alpha = 0.1$:

$$G(\omega) = \frac{\alpha}{(1.81-1.8\cos\omega)^{1/2}}$$

In figure 5 the amplitude function is plotted for $\alpha = 0.1$.

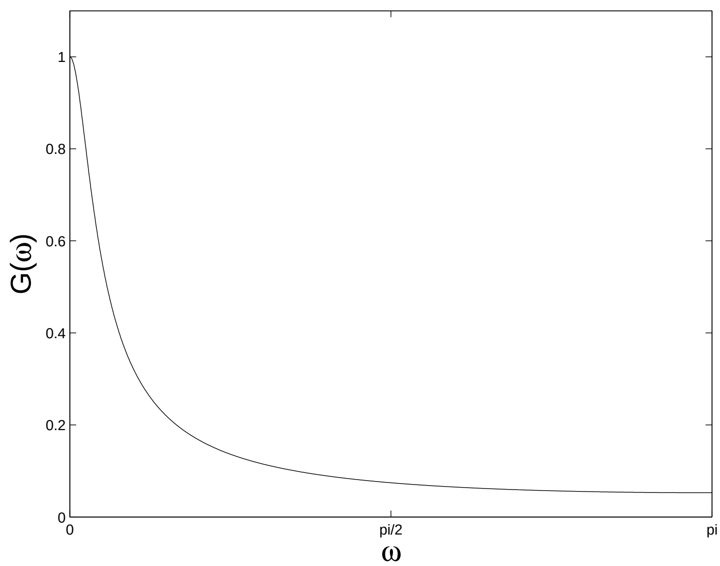

Figure 5: $G(\omega)$ Question 3.

Question 4.

The two timeseries Z_t and Y_t are defined as

$$\begin{aligned} Y_t &= \frac{\alpha}{1-(1-\alpha)B} X_t \\ Z_t &= X_t - Y_t \end{aligned}$$

I.e.

$$Z_t = (1 - \frac{\alpha}{1-(1-\alpha)B}) X_t = \frac{(1-\alpha)(1-B)}{1-(1-\alpha)B} X_t$$

$$\begin{aligned} G^2(\omega) &= \mathcal{H}(\omega)\overline{\mathcal{H}(\omega)} = \frac{1-\alpha)^2(1-e^{-i\omega})(1-e^{i\omega})}{(1-(1-\alpha)e^{-i\omega})(1-(1-\alpha)e^{i\omega}} \\ &= \frac{2(1-\alpha)^2(1-\cos\omega)}{2+\alpha^2-2\alpha-2(1-\alpha)\cos\omega} \end{aligned}$$

$\Rightarrow$ Amplitude:

$$G(\omega) = \left(\frac{2(1-\alpha)^2(1-\cos\omega)}{(2+\alpha^2-2\alpha-2(1-\alpha)\cos\omega)\omega} \right)^{1/2}$$

In figure 6 the amplitude function is plotted for $\alpha = 0.1$.

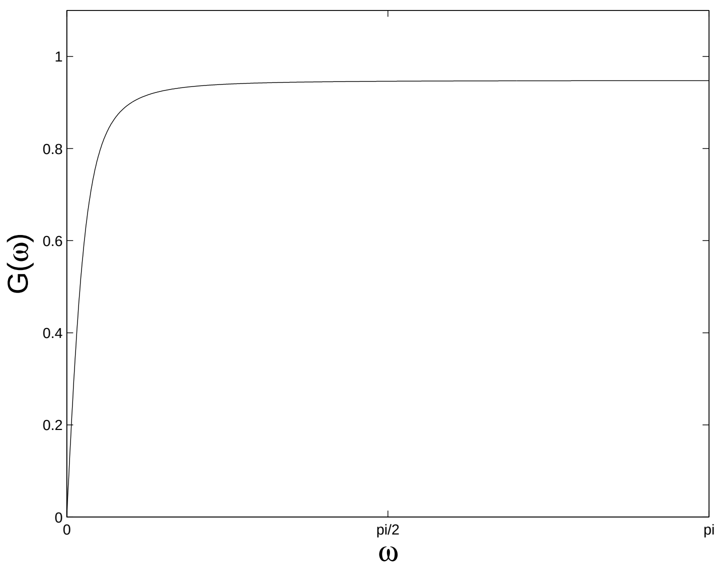

Figure 6: $G(\omega)$ Question 4.

Solution 5.10

Question 1.

Given a process X_t defined by

$$(1 - B + 0.5B^2)X_t = (1 + 0.5B)\epsilon_t$$

where ϵ_t is white noise with $\mathrm{E}[\epsilon_t] = 0$ and $\mathrm{Var}[\epsilon_t] = 1$. The transfer function of the process is

$$H(z) = \frac{\theta(z^{-1})}{\phi(z^{-1})} = \frac{1 + 0.5z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

The roots of $z^2(1 - z^{-1} + 0.5z^{-2}) = 0$ are $0.5 \pm 0.5i$, which have the modulus $(0.5^2 + 0.5^2)^{1/2} = 0.707 < 1$. As the roots of $\phi(z^{-1})$ lie within the unit circle the process is stationary. The root to $\theta(z^{-1})$ is -0.5, and thus the process is invertible.
X_t is an stationary, invertible ARMA(2,1) process.

Question 2.

$$X_t - X_{t-1} + 0.5X_{t-2} = \epsilon_t + 0.5\epsilon_{t-1}$$

By using the results from section 5.5.3 we determine $\gamma_{x\epsilon}(k)$:

$$\begin{aligned} k = 0: &\quad \gamma_{x\epsilon}(0) = 1 \\ k = 1: &\quad \gamma_{x\epsilon}(1) - \gamma_{x\epsilon}(0) = 0.5 \quad \Rightarrow \quad \gamma_{x\epsilon}(1) = 1.5 \end{aligned}$$

Determining of $\gamma(k)$ from (5.99) and (5.100):

$$k = 0: \quad \gamma(0) - \gamma(1) + 0.5\gamma(2) = \gamma_{x\epsilon}(0) + 0.5\gamma_{x\epsilon}(1) \quad \Rightarrow \tag{2}$$
$$\gamma(0) - \gamma(1) + 0.5\gamma(2) = 1.75$$
$$k = 1: \quad \gamma(1) - \gamma(0) + 0.5\gamma(1) = 0.5\gamma_{xz}(0) \quad \Rightarrow \tag{3}$$
$$\gamma(1) - \gamma(0) + 0.5\gamma(1) = 0.5$$
$$k \geq 2: \quad \gamma(k) - \gamma(k-1) + 0.5\gamma(k-2) = 0 \tag{4}$$

One can now use two different methods to determine $\gamma(k), k = 0, 1, \ldots$.

In the first method you write equation (4) for $k = 2$ and then solve the 3 equations (2), (3) and (4) (with $k = 2$) with respect to $\gamma(0), \gamma(1)$ and $\gamma(2)$. The auto-covariance at higher lags can then be found recursive by repeated use of equation (4).

The second method involves solving the difference equation (4) and then use equation (2) and (3) to determine the actual solution. This method results in a direct expression of $\gamma(k)$. This method will now be applied (though it is the most difficult one).

The characteristic equation for (4) is

$$y^2 - y + 0.5 = 0$$

where the corresponding roots are

$$y = \frac{1 \pm \sqrt{1-2}}{2} = \frac{1}{2}(1 \pm i) = \begin{cases} \frac{\sqrt{2}}{2} e^{i\frac{\pi}{4}} \\ \frac{\sqrt{2}}{2} e^{-i\frac{\pi}{4}} \end{cases}$$

I.e. the complete solution to (4) is

$$\gamma(k) = A_1 \left(\frac{\sqrt{2}}{2} e^{i\frac{\pi}{4}} \right)^k + A_2 \left(\frac{\sqrt{2}}{2} e^{-i\frac{\pi}{4}} \right)^k = A_1 \left(\frac{1}{2} \right)^{\frac{k}{2}} e^{ik\frac{\pi}{4}} + A_2 \left(\frac{1}{2} \right)^{\frac{k}{2}} e^{-ik\frac{\pi}{4}}$$

The constants A_1 and A_2 can be determined on the basis of (2) and (3)

$$\begin{aligned}
\gamma(0) =& A_1 + A_2 \\
\gamma(1) =& A_1 \frac{\sqrt{2}}{1} e^{i\frac{\pi}{4}} + A_1 \frac{\sqrt{2}}{1} e^{-i\frac{\pi}{4}} = \frac{1}{2}(A_1 + A_2) + \frac{i}{2}(A_1 - A_2) \\
\gamma(2) =& A_1 \frac{1}{2} e^{i\frac{\pi}{2}} + A_2 \frac{1}{2} e^{-i\frac{\pi}{2}} = \frac{i}{2}(A_1 - A_2)
\end{aligned}$$

A_1 and A_2 can now be determined using these expressions for $\gamma(0), \gamma(1)$ and $\gamma(2)$ in (2) and (3). After some calculations the following results are found

$$A_1 = \frac{1}{10}(23 - 11i) \qquad ; \qquad A_2 = \frac{1}{10}(23 + 11i)$$

I.e. the auto-covariance is

$$
\begin{aligned}
\gamma(k) =& \frac{1}{10}\left((23-11i)\left(\frac{1}{2}\right)^{\frac{k}{2}} e^{ik\frac{\pi}{4}} + (23+11i)\left(\frac{1}{2}\right)^{\frac{k}{2}} e^{-ik\frac{\pi}{4}}\right) \\
=& \frac{1}{5}\left(\frac{1}{2}\right)^{\frac{k}{2}}\left(23\cos\frac{k\pi}{4} + 11\sin\frac{k\pi}{4}\right) \\
=& \frac{1}{5}\left(\frac{1}{2}\right)^{\frac{k}{2}}\left(25\cos\left(\frac{k\pi}{4} - 0.4461\right)\right)
\end{aligned}
$$

In table 2 the values of the auto-covariance and auto-correlation function are shown for $k = 0, 1, \ldots, 9$. $(\rho(k) = \frac{\gamma(k)}{\gamma(0)})$

The auto-correlation function is plotted in figure 7.

k	0	1	2	3	4	5	6	7	8	9
$\gamma(k)$	4.60	3.40	1.10	-0.60	-1.15	-0.85	-0.27	0.15	0.30	0.21
$\rho(k)$	1	0.74	0.24	-0.13	-0.25	-0.18	-0.06	0.03	0.06	0.05

Table 2: The auto-covariance and auto-correlation function.

Question 3.

The partial autocorrelation function is calculated by means of the recursive method, which is described in appendix B.

$$
\begin{aligned}
\phi_{k+1,j} =& \phi_{k,j} - \phi_{k+1,k+1}\phi_{k,k+1-j} \qquad , j = 1, \ldots, k \\
\phi_{k+1,k+1} =& \frac{\rho_{k+1} - \sum_{j=1}^{k} \phi_{k,j}\rho_{k+1-j}}{1 - \sum_{j=1}^{k} \phi_{k,j}\rho_j}
\end{aligned}
$$

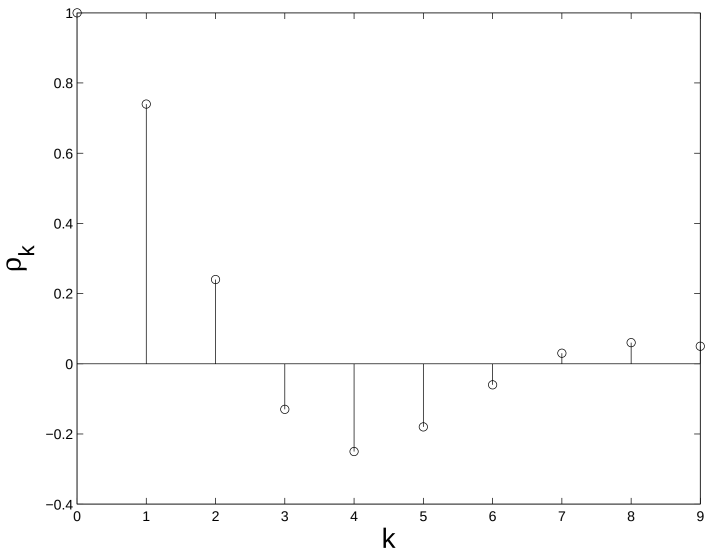

Figure 7: Auto-correlation function.

with start value $\phi_{11} = \rho_1$. I.e.

$$
\begin{aligned}
\phi_{1,1} &= \rho_1 = 0.74 \\
\phi_{2,2} &= \frac{\rho_2 - \phi_{1,1}\rho_1}{1 - \phi_{1,1}\rho} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = -0.6799 && k = 1 \\
\phi_{3,3} &= \frac{\rho_3 - \sum_{j=1}^{2} \phi_{,j}\rho_{3-j}}{1 - \sum_{j=1}^{2} \phi_{2,j}\rho_j} = \frac{\rho_3 - \phi_{2,1}\rho_2 - \phi_{2,2}\rho_1}{1 - \phi_{2,1}\rho_1 - \phi_{2,2}\rho_2} = 0.3075 && k = 2 \\
\text{Since :} \quad & \phi_{2,1} = \phi_{1,1} - \phi_{2,2}\phi_{1,1} = 1.2431 \\
\phi_{4,4} &= \frac{\rho_4 - \sum_{j=1}^{3} \phi_{3,j}\rho_{4-j}}{1 - \sum_{j=1}^{3} \phi_{3,j}\rho_j} \\
&= \frac{\rho_4 - \phi_{3,1}\rho_3 - \phi_{3,2}\rho_2 - \phi_{3,3}\rho_1}{1 - \phi_{3,1}\rho - \phi_{3,2}\rho_2 - \phi_{3,3}\rho_3} = -0.1539 && k = 3 \\
\text{Since :} \quad & \phi_{3,1} = \phi_{2,1} - \phi_{3,3}\phi_{2,2} = 1.4521 \\
& \phi_{3,2} = \phi_{2,2} - \phi_{3,3}\phi_{2,1} = -1.0620
\end{aligned}
$$

The partial auto-correlation function is plotted in figure 8.

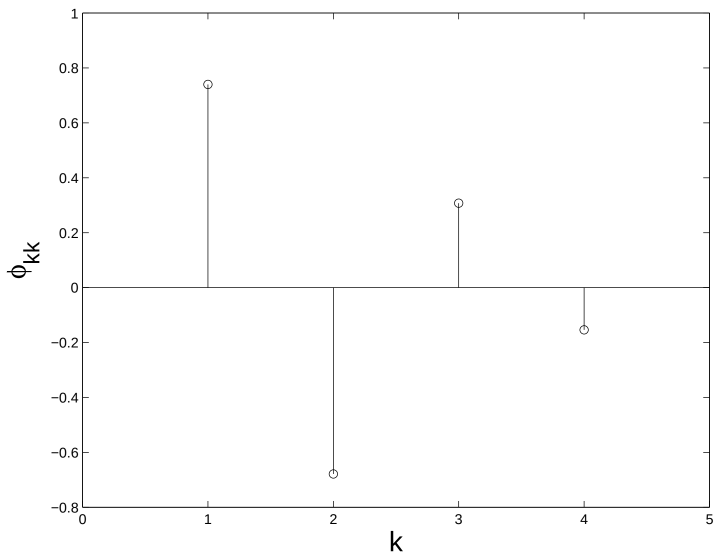

Figure 8: Partial auto-correlation function.

Solution 5.11

Question 1.

Given the seasonal model

$$X_t = (1 - 0.6B^{12})\epsilon_t$$

where ϵ_t is white noise. This is a pure seasonal model of the form (0,0,0)x(0,0,1)$_{12}$. The autocovariane of the process is

$$\gamma_k = \text{Cov}[X_t, X_{t+k}] = \text{Cov}[\epsilon_t - 0.6\epsilon_{t-12}, \epsilon_{t+k} - 0.6\epsilon_{t-12+k}] ,$$

from where we get

$$\begin{aligned}
\gamma_0 &= (1 + 0.6^2)\sigma_\epsilon^2 = 1.36\sigma_\epsilon^2 \\
\gamma_1 &= \gamma(2) = .. = \gamma(11) = 0 \\
\gamma_{12} &= -0.6\sigma_\epsilon^2 \\
\gamma_k &= 0 \text{ for } k > 12
\end{aligned}$$

The autocorrelation function becomes

$$\rho_k = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k = 1, 2, ..11 \\ -\frac{0.6}{1.36} = -0.44 & \text{for } k = 12 \\ 0 & \text{for } k > 12 \end{cases}$$

A sketch of the autocorrelation function is shown in Figure 9.

Question 2.

Given the seasonal model

$$(1 - 0.6B^{12})X_t = \epsilon_t$$

where ϵ_t is white noise. This is a pure seasonal model of the form (0,0,0)x(1,0,0)$_{12}$. First we rewrite the process to MA-form

$$\begin{aligned}
X_t &= (1 - 0.6B^{12})^{-1}\epsilon_t = \sum_{n=0}^{\infty}(0.6B^{12})^n\epsilon_t \\
X_t &= (1 + 0.6B^{12} + 0.6^2B^{24} + ...)\epsilon_t
\end{aligned}$$

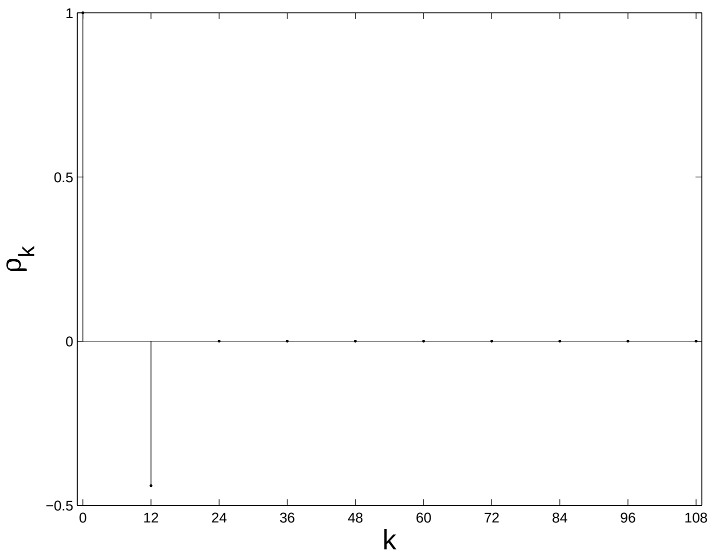

Figure 9: Autocorrelation function for the seasonal model in question 1.

The autocovariane of the process is

$\gamma_k = \mathrm{Cov}[X_t, X_{t+k}] = [\epsilon_t + 0.6\epsilon_{t-12} + 0.6^2\epsilon_{t-24}, \epsilon_{t+k} + 0.6\epsilon_{t-12+k} + 0.6^2\epsilon_{t-24+k}]$,

from where we get

$$\begin{aligned} \gamma_0 &= \frac{1}{1-0.6^2}\sigma_\epsilon^2 \\ \gamma_1 &= \gamma(2) = .. = \gamma(11) = 0 \\ \gamma_k &= 0.6\gamma_{k-12} \text{ for } k \geq 12 \end{aligned}$$

The autocorrelation function becomes

$$\rho_k = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k = 1, 2, ..11 \\ 0.6\rho_{k-12} & \text{for } k \geq 12 \end{cases}$$

A sketch of the autocorrelation function is shown in Figure 10. *Question 3.*

Given the seasonal model

$$(1 - 0.6B^{12})X_t = (1 + 0.5B)\epsilon_t$$

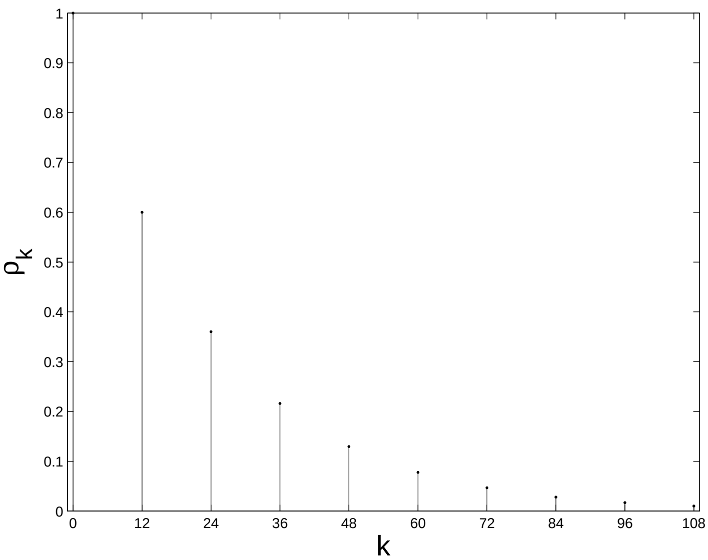

Figure 10: Autocorrelation function for the seasonal model in question 2.

where ϵ_t is white noise. This is a multiplicative seasonal model of the form (0,0,0)x$(1,0,1)_{12}$. The autocorrelation function can be found as in Example 5.10. We get

$$\rho_k = \begin{cases} 1 & \text{for } k = 0 \\ \frac{0.5}{1+0.5^2} = 0.4 & \text{for } k = 1 \\ 0 & \text{for } k = 2, 3..10 \\ \frac{0.5 \cdot 0.6}{1+0.5^2} = 0.24 & \text{for } k = 11 \\ 0.6\rho_{k-12} & \text{for } k \geq 12 \end{cases}$$

A sketch of the autocorrelation function is shown in Figure 11.

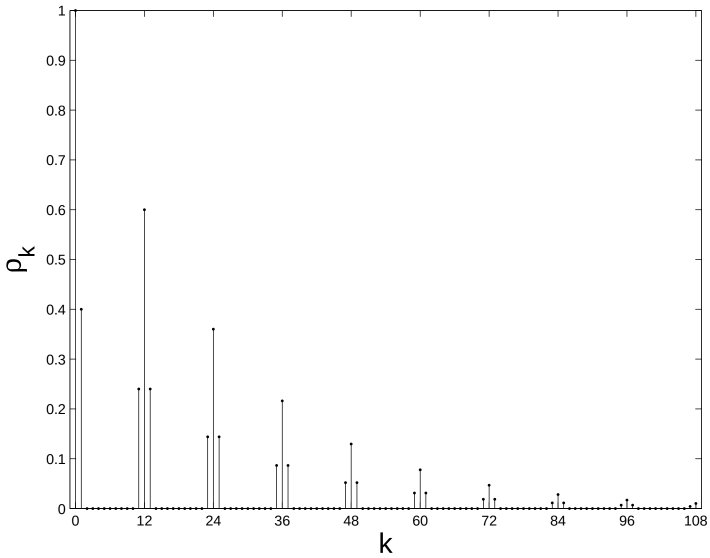

Figure 11: Autocorrelation function for the seasonal model in question 3.