Time Series Analysis

Solutions to problems in Chapter 6

IMM

Solution 6.1

Question 1.

The time series is plotted in Figure 1. The time series is not stationary as a

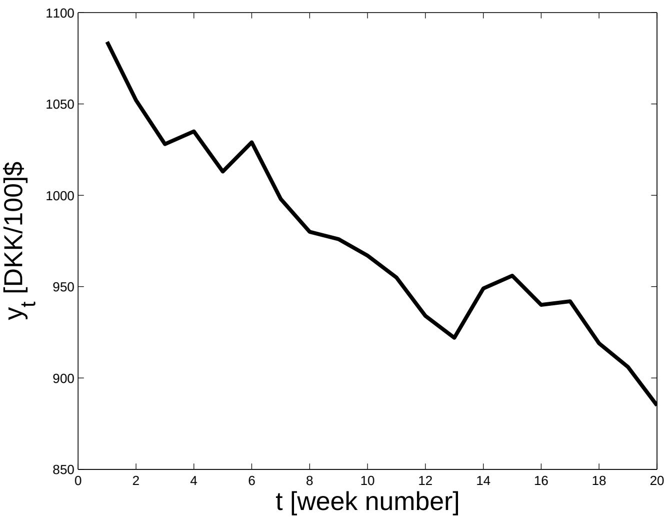

Figure 1: The time series y_t

clear trend is seen.

Question 2.

A suitable transformation from y_t to a acceptable stationary time series x_t is

$$x_t = \nabla y_t \ .$$

The time series is plotted in Figure 2.

Question 3.

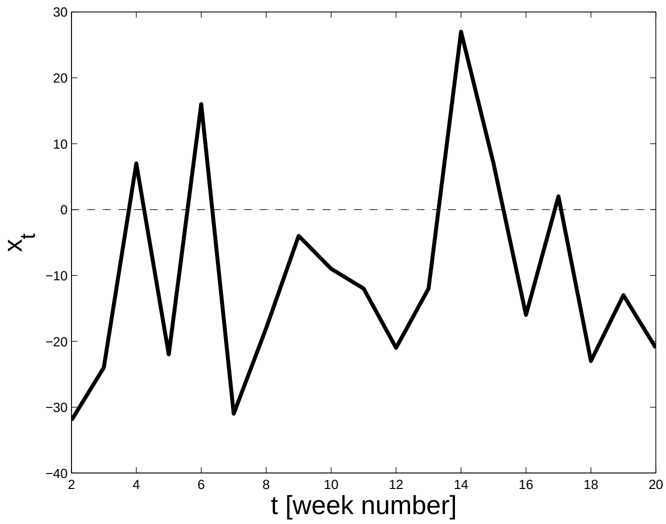

Figure 2: The time series x_t

The autocovariance function (lag ≤ 5) for $\{X_t\}$ is found by (6.1) to

$$C(k) = \frac{1}{19}\sum_{t=2}^{20-k}(x_t - \bar{x})(x_{t+k} - \bar{x}) = \begin{cases} 241.7 & \text{for k=0} \\ -27.2 & \text{for k=1} \\ -6.7 & \text{for k=2} \\ -21.1 & \text{for k=3} \\ -39.3 & \text{for k=4} \\ 37.5 & \text{for k=5} \end{cases}$$

($\bar{x} = -10.47$)
The estimated autocorrelation function is given by the estimated autocovariance function as $r_k = C(k)/C(0)$. The autocorrelation function is plotted in Figure 3.

Question 4.

If $\{x_t\}$ is white noise the estimated autocorrelation function should be approximative normal distributed with mean zero and variance 1/N. From here we get an 95% confidence interval on $[-2\sigma, 2\sigma] = [-2/\sqrt{19}, 2/\sqrt{19}]$. These limits are drawn in the plot of the autocorrelation function Figure 3. As none of the estimated autocorrelations are outside the limits we can not reject the

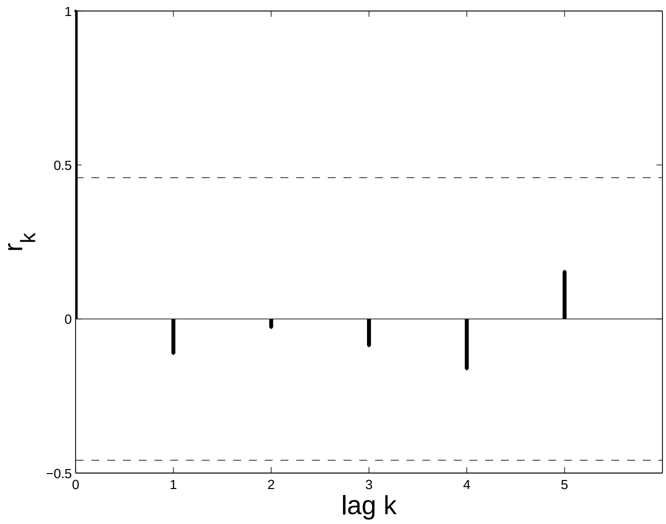

Figure 3: The estimated autocorrelation function

hypothesis that x_t is white noise.

Question 5.

As $\{x_t\}$ is assumed to be white noise (which means that x_t does not contain any further information), we can summarize the model for the exchange rate as

$$\nabla Y_t = \mu + \epsilon_t \ ,$$

where $\mu = \bar{x}$ and ϵ_t is white noise with the mean value 0 and variance $\hat{\sigma}^2 = C(0)$.
To predict the exchange rate in week 21, we rewrite the model to

$$Y_{t+1} = Y_t + \mu + \epsilon_t \ .$$

Given the observation in week 20 the prediction to week 21 can be determined as

$$\hat{Y}_{t+1|t} = \mathrm{E}[Y_{t+1}|Y_t = y_t] = y_t + \mu \ .$$

i.e

$$\hat{Y}_{21|20} = 885 - 10.47 \approx \underline{\underline{875\text{kr}/100\$}}$$

Solution 6.2

Question 1.

An estimator $\hat{\theta}$ is an unbiased estimator for θ if

$$\mathrm{E}[\bar{\theta}] = \theta$$

The autocovariance at lag k for a stationary process X_t is

$$\gamma_k = \mathrm{E}[(X_t - \mu)(X_{t+k} - \mu)]$$

Ignoring the effect from μ being estimated with $\bar{X}$ we get

$$\begin{aligned}
\mathrm{E}[C_k] &= \mathrm{E}\left[\frac{1}{N}\sum_{t=1}^{N-k}(X_t - \bar{X})(X_{t+k} - \bar{X})\right] \\
&= \frac{1}{N}\sum_{t=1}^{N-k}\mathrm{E}[(X_t - \bar{X})(X_{t+k} - \bar{X})] \\
&= \frac{1}{N}(N-k)\gamma_k = \underline{\underline{\left(1 - \frac{k}{N}\right)\gamma_k}} \ ,
\end{aligned}$$

which means that the estimator is biased.
For a fixed k $\mathrm{E}[C_k] \to \gamma_k$ for $N \to \infty$.
A better estimation for $\mathrm{E}[C_k]$ can be achieved by using that

$$\begin{aligned}
&\sum_{t=1}^{N-k}(X_t - \mu)(X_{t+k} - \mu) \\
&\quad = \sum_{t=1}^{N-k}\left[(X_t - \bar{X}) + (\bar{X} - \mu)\right]\left[(X_{t+k} - \bar{X}) + (\bar{X} - \mu)\right] \\
&\quad = \sum_{t=1}^{N-k}\left[(X_t - \bar{X})(X_{t-k} - \mu) + (\bar{X} - \mu)^2\right] + \sum_{t=1}^{N-k}\left[(X_t - \bar{X})(\bar{X} - \mu) + (\bar{X} - \mu)(X_{t+k} - \bar{X})\right] \\
&\quad \approx \sum_{t=1}^{N-k}\left[(X_t - \bar{X})(X_{t-k} - \mu) + (\bar{X} - \mu)^2\right] = (N-k)(\bar{X} - \mu)^2 + \sum_{t=1}^{N-k}\left[(X_t - \bar{X})(X_{t-k} - \mu)\right]
\end{aligned}$$

as

$$\sum_{t=1}^{N-k} \big[(X_t - \bar{X})(\bar{X} - \mu)\big] \approx (\bar{X} - \mu) \sum_{t=1}^{N-k} (X_t - \bar{X}) = 0$$

Hereby a more accurate estimate for $\mathrm{E}[C_k]$ is

$$\begin{aligned} \mathrm{E}[C_k] &\approx \frac{1}{N} \sum_{t=1}^{N-k} [\mathrm{E}[(X_t - \mu)(X_{t+k} - \mu)]] - \frac{1}{N}(N-k)\mathrm{E}(\bar{X} - \mu)^2 \\ &= \underline{\underline{\left(1 - \frac{k}{N}\right)(\gamma_k - \mathrm{Var}[\bar{X}])}} \end{aligned}$$

(It is necessary to know the autocorrelation function for $\{X_t\}$ in order to calculate $\mathrm{Var}[\bar{X}]$.)

Solution 6.3

Question 1.

The AR(2)-process can be written as

$$(1 + \phi_1 B + \phi_2 B^2)X_t = \epsilon_t$$

or

$$\phi(B)X_t = \epsilon_t$$

where $\phi(B)$ is a second order polynomial in B. According to theorem 5.9 the process is stationary if the roots to $\phi(z^{-1}) = 0$ all lie within the unit circle. I.e. if λ_i is the i'th root it must satisfy $|\lambda_i| < 1$. From appendix A the solution is found by solving the characteristic equation

$$\lambda^2 + \phi_1 \lambda + \phi_2 = 0$$

I.e.

$$\lambda_1 = \left| \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} \right| \qquad , \quad \lambda_2 = \left| \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} \right|$$

From the above the stationary region is the triangular region satisfying

$$\begin{aligned} -\phi_1 - \phi_2 < 1 \qquad &\Leftrightarrow \qquad \phi_2 > -1 - \phi_1 \\ -\phi_1 + \phi_2 > -1 \qquad &\Leftrightarrow \qquad \phi_2 > -1 + \phi_1 \\ -\phi_2 > -1 \qquad &\Leftrightarrow \qquad \phi_2 < 1 \end{aligned}$$

In figure 4 the stationary region is shown.

Question 2.

The auto-correlation function is known to satisfy the difference equation

$$\rho(k) + \phi_1 \rho(k-1) + \phi_2 \rho(k-2) = 0 \qquad k > 0$$

The characteristic equation is

$$\lambda^2 + \phi_1 \lambda + \phi_2 = 0$$

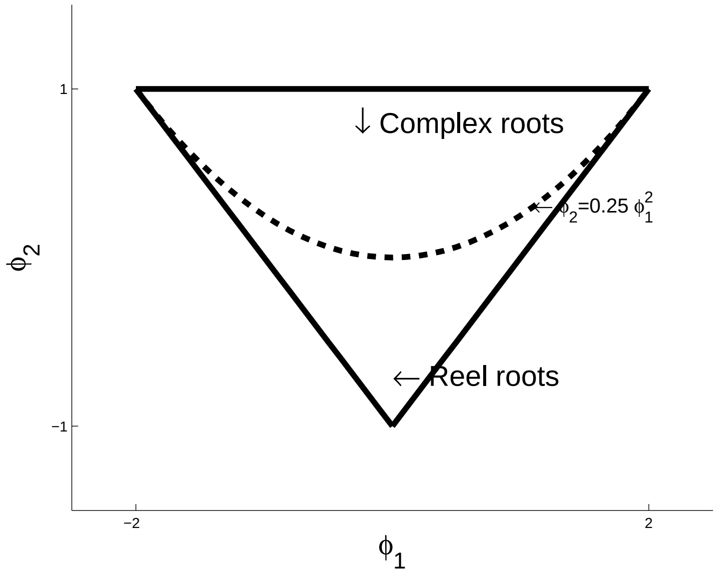

Figure 4: Parameter area for which the AR(2)-process is stationary.

According to appendix A the solution to the difference equation consist of a damped harmonic variation if the roots to the charateristic equation are complex. I.e. if

$$\phi_1^2 - 4\phi_2 < 0$$

The curve $\phi_2 = \frac{1}{4}\phi_1^2$ is sketched on figure 4.

Question 3.

The Yule-Walker equations can be used to determine the moment estimates of $\hat{\phi}_1$ and $\hat{\phi}_2$.

$$\begin{bmatrix} 1 & r_1 \\ r_1 & 1 \end{bmatrix} \begin{bmatrix} -\hat{\phi}_1 \\ -\hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \Leftrightarrow$$

$$\begin{bmatrix} -\hat{\phi}_1 \\ -\hat{\phi}_2 \end{bmatrix} = \frac{1}{1-r_1^2} \begin{bmatrix} 1 & -r_1 \\ -r_1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \Leftrightarrow$$

$$\begin{bmatrix} -\hat{\phi}_1 \\ -\hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} \frac{r_1 - r_1 r_2}{1-r_1^2} \\ \frac{r_2 - r_1^2}{1-r_1^2} \end{bmatrix} \Leftrightarrow$$

$$\begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} \frac{r_1 r_2 - r_1}{1-r_1^2} \\ \frac{r_1^2 - r_2}{1-r_1^2} \end{bmatrix}$$

Using the given values for r_1 and r_2 leads to

$$\hat{\phi}_1 = -1.031 \qquad \hat{\phi}_2 = 0.719$$

Solution 6.4

For solution see Example 6.3 in the text book.

Solution 6.5

From Example 5.9 in Section 5.5.3 the auto-correlation function of an ARMA(1,1)-process is given by

$$\rho(1) = \frac{(1-\phi_1\theta_1)(\theta_1-\phi_1)}{1+\theta_1^2-2\theta_1\phi_1} \qquad (1)$$

$$\rho(k) = (-\phi_1)^{k-1}\rho(1) \qquad k \geq 2 \qquad (2)$$

From (2) for $k = 2$

$$\phi_1 = \frac{\rho(2)}{\rho(1)}$$

I.e. the moment estimate is

$$\hat{\phi}_1 = \frac{r_2}{r_1} = \frac{0.50}{0.57} = 0.88$$

From (1) follows

$$\begin{aligned}
&\rho(1)(1+\theta_1^2-2\theta_1\phi_1) = \phi_1-\phi_1^2\theta_1-\phi_1+\phi_1\theta_1^2 \qquad \Leftrightarrow \\
&(\rho-\phi_1)\theta_1^2+(1-2\phi_1\rho(1)+\phi_1^2)\theta_1+\rho(1)-\phi_1 = 0 \qquad \Leftrightarrow \\
&\theta_1 = \frac{2\phi_1\rho(1)-1-\phi_1^2 \pm \sqrt{(2\phi_1\rho(1)-1-\phi_1^2)^2-4(\rho(1)-\phi_1)^2}}{2(\rho(1)-\phi_1)}
\end{aligned}$$

The momement estimate is calculated by inserting $r_1 = 0.57$ and $\hat{\phi}_1 = 0.88$. I.e.

$$\hat{\theta}_1 = \begin{cases} 1.98 \\ 0.50 \end{cases}$$

The requirement of invertibility leads to $\hat{\theta}_1 = 0.50$.

Solution 6.6

For an AR(p)-process holds

$$V[\hat{\phi}_{kk}] = \frac{1}{N} \quad \text{and} \quad E[\hat{\phi}_{kk}] \simeq 0 \qquad k > p$$

where N is the number of observations. Furthermore $\hat{\phi}_{kk}$ is approximately normal distributed and an approximated 95% confidence interval can therefore be constructed

$$\left(-2 \cdot \frac{1}{\sqrt{N}}, 2 \cdot \frac{1}{\sqrt{N}}\right) = (-0.24, 0.24)$$

It is observed that the hypothesis for $p = 1$, i.e. and AR(1)-process, cannot be rejected since none of the values of $\hat{\phi_{kk}}$ for $k = 2, 3, \ldots$ are outside the interval. Because of this an AR(1)-process is assumed to be a suitable model.

For an AR(1) model the following is given

$$\rho(1) = -\alpha_1$$

and

$$\phi_{11} = \rho(1)$$

From above follows that a momentestimate of α_1 is

$$\hat{\alpha}_1 = -\hat{\phi}_{11} = \underline{\underline{0.40}}$$

Solution 6.7

Question 1.

Given the following ARMA(1,1) process

$$
\begin{aligned}
&(1-0.9B)X_t = (1+0.8B)\epsilon_t \Rightarrow \\
&\epsilon_t = \frac{1-0.9B}{1+0.8B}X_t = \left(1+\frac{-1.7B}{1+0.8B}\right)X_t \ ,
\end{aligned}
$$

i.e

$$
\begin{aligned}
\epsilon_t &= X_t - 1.7\sum_{k=1\infty}(-0.8)^{k-1}X_{t-k} \Rightarrow \\
X_t &= 1.7\sum_{k=1}^{\infty}(-0.8)^{k-1}X_{t-k} + \epsilon_t
\end{aligned}
$$

From where we can calculate the one-step prediction

$$
X_{t+1} = 1.7\sum_{k=1}^{\infty}(-0.8)^{k-1}X_{t-k} + \epsilon_{t+1} \tag{3}
$$

e.i.

$$
\begin{aligned}
\hat{X}_{t+1|t} &= \mathrm{E}[X_{t-1}|X_t, X_{t-1}, ...] \\
&= \underline{\underline{1.7\sum_{k=0}^{\infty}(-0.8)^{k}X_{t-k}}} \qquad (4)
\end{aligned}
$$

The prediction error is $e_{t+1} = X_{t+\ell} - \hat{X}_{t+1|t}$. Subtracting (4) from (3) we get ϵ_{t+1}, i.e. the variance of the prediction error is σ^2.

Question 2.

Calculation the k-step prediction

$$\begin{aligned}
(1-0.9B)X_t =& (1+0.8B)\epsilon_t \Rightarrow \\
X_{t+k} - 0.9X_{t+k-1} =& \epsilon_{t+k} + 0.8\epsilon_{t+k-1} \Rightarrow \\
\mathrm{E}[X_{t+k}|X_t, X_{t-1}, ...] =& 0.9\mathrm{E}[X_{t+k-1}|X_t, X_{t-1}, ...] + \mathrm{E}[\epsilon_{t+k}|X_t, X_{t-1}, ...] \\
& + 0.8\mathrm{E}[\epsilon_{t+k-1}|X_t, X_{t-1}, ...] \\
=& 0.9\hat{X}_{t+k-1|t} \text{ for } k \geq 2 \ .
\end{aligned}$$

I.e. the k-step prediction is

$$\hat{X}_{t+k|t} = \underline{\underline{0.9^{k-1}\hat{X}_{t+1|t}}} \text{ for } k \geq 2$$

Rewriting the process to MA-form

$$\begin{aligned}
X_t &= \frac{1+.08B}{1-0.9B}\epsilon_t = \left(1 + \frac{1.7B}{1-0.9B}\right)\epsilon_t \\
&= \epsilon_t + 1.7\sum_{k=1}^{\infty} 0.9^{k-1}\epsilon_{t-k}
\end{aligned}$$

Thus, the variance of the k-step prediction error is

$$\mathrm{Var}[X_{t+k} - \hat{X}_{t+k|t}] = \underline{\underline{\sigma^2\left(1 + 1.7^2\sum_{j=1}^{k-1} 0.81^{j-1}\right)}}$$

Solution 6.8

Question 1.

The times series ∇Z_t has the smallest variance. Furthermore the values of $\hat{\rho}_k$ will quickly become small for ∇Z_t, but not for Z_t. It can therefore be concluded that $d = 1$.

From the time series ∇Z_t it is observed that $\hat{\rho}_1$ is positive while $\hat{\rho}_k$ is small for $k \geq 2$. Due to this fact it is reasonable to check if ∇Z_t can be described by a MA(1)-process. We investigate the hypothesis: $\rho_k = 0$ for $k \geq 2$. Theorem 6.4 in section 6.3.2 leads to

$$V(\hat{\rho}_k) = \frac{1}{N}(1 + 2\hat{\rho}_1^2) = 0.059^2 \qquad , \qquad k \geq 2$$

Since none of the values of $\hat{\rho}$ for $k \geq 2$ is outside $\pm 2 \cdot 0.059$ we assume that ∇Z_t can be described by a MA(1)-process. I.e. overall the IMA(1,1)-process:

$$Z_t - Z_{t-1} = e_t + \theta e_{t-1}$$

The moment estimate of θ can be determined from (4.71) to

$$\hat{\rho}_1 = \frac{\hat{\theta}}{1 + \hat{\theta}^2} \quad \Rightarrow \quad \hat{\theta} = \frac{1}{2\hat{\rho}_1} \pm \sqrt{\left(\frac{1}{2\rho_1}\right)^2 - 1} = \begin{cases} 0.14 \\ 7 \end{cases}$$

The requirement of invertibility leads to $\hat{\theta} = 0.14$. ($|\hat{\theta}| < 1$).
The variance is found from the variance $\gamma(0)$ of the MA(1) process (4.70)

$$\sigma^2_{\nabla Z_t} = (1 + \hat{\theta}^2)\hat{\sigma}_e^2 \quad \Rightarrow \quad \hat{\sigma}_e^2 = \frac{52.5}{1 + 0.14^2} = 51.5$$

Question 2.

$$\begin{aligned} Z_t &= Z_{t-1} + e_t + \theta e_{t-1} \quad \Rightarrow \\ Z_{t+1} &= Z_t + e_{t+1} + \theta e_t \quad \Rightarrow \\ \hat{Z}_{t+1|t} &= Z_t + \theta e_t & (5) \\ Z_{t+k} &= Z_{t+k-1} + e_{t+k} + \theta e_{t+k-1} \quad \Rightarrow \\ \hat{Z}_{t+k|t} &= \hat{Z}_{t+k-1|t} \quad \text{for} \quad k \geq 2 & (6) \end{aligned}$$

The value of e_{10} is found by using (5) from e.g. $t = 8$ and put $e_8 = 0$. (Since θ is very small we only need to start a few steps back).

$$\hat{Z}_{9|8} = Z_8 + \theta \cdot 0 = 206 \quad \Rightarrow \quad e_9 = Z_9 - \hat{Z}_{9|8} = -11$$
$$\hat{Z}_{10|9} = Z_9 + \theta \cdot e_9 = 193.5 \quad \Rightarrow \quad e_{10} = Z_{10} - \hat{Z}_{10|9} = -14.5$$
$$\hat{Z}_{11|10} = Z_{10} + \theta \cdot e_{10} = 179 + 0.14 \cdot (-14.5) = 177$$

From (6)

$$\hat{Z}_{13|10} = \hat{Z}_{11|10} = 177$$

Question 3.

Updating:

$$\hat{Z}_{13|11} = \psi_2 e_{11} + \hat{Z}_{13|10}$$

We write the model on MA-form:

$$Z_t = e_t + (\theta + 1)e_{t-1} + (\theta + 1)e_{t-2} + (\theta + 1)e_{t-3} + \dots$$

I.e. $\psi_2 = (\theta + 1)$ which results in

$$\hat{Z}_{13|11} = 1.14 \cdot 7 + 177 = 185$$

where $e_{11} = 184 - 177 = 7$.

Similarly

$$\hat{Z}_{12|11} = \hat{Z}_{13|11} = 185 \qquad \text{(from (6))}$$

I.e. $e_{12} = Z_{12} - \hat{Z}_{12|11} = 196 - 185 = 11$ and

$$\hat{Z}_{11+2|11+1} = \psi_1 \cdot e_{12} + \hat{Z}_{11+2|11} = 1.14 \cdot 11 + 185 = 197.5$$

Question 4.

The variance on the k-step prediction is

$$\sigma_k^2 = (1 + \psi_1^2 + \dots + \psi_{k-1}^2)\sigma_e^2$$

I.e.

$$\sigma_1^2 = 51.5 = 7.2^2$$
$$\sigma_2^2 = (1 + 1.14^2) \cdot 51.5 = 10.9^2$$
$$\sigma_3^2 = (1 + 1.14^2 + 1.14^2) \cdot 51.5 = 13.6^2$$

and the following 95%-confidence interval

$$\begin{aligned} Z_{13|10} &: 177 \pm 27.2 \\ Z_{13|11} &: 185 \pm 21.8 \\ Z_{13|12} &: 197.5 \pm 14.2 \end{aligned}$$

Notice that all the confidence intervals contains the realized value. Furthermore the confidence interval narrows down when predicting less steps.

Solution 6.9

Question 1.

The auto-correlations

$$\hat{\rho}_1 = \frac{1.58}{2.25} = 0.70 \qquad \hat{\rho}_2 = \frac{1.13}{2.25} = 0.50 \qquad \hat{\rho}_3 = 0.40$$

The partial auto-correlations

$$\hat{\phi}_{33} = \frac{\begin{vmatrix} 1 & 0.70 & 0.70 \\ 0.70 & 1 & 0.50 \\ 0.50 & 0.70 & 0.40 \end{vmatrix}}{\begin{vmatrix} 1 & 0.70 & 0.50 \\ 0.70 & 1 & 0.70 \\ 0.50 & 0.70 & 1 \end{vmatrix}} = \frac{0.022}{0.260} = 0.0846$$

$$\hat{\phi}_{22} = \frac{\begin{vmatrix} 1 & 0.70 \\ 0.70 & 0.50 \end{vmatrix}}{\begin{vmatrix} 1 & 0.70 \\ 0.70 & 1 \end{vmatrix}} = \frac{0.01}{0.51} = 0.0196$$

$$\hat{\phi}_{11} = \hat{\rho}_1 = 0.70$$

It is appearent that the process is an AR(1)-process, but to be sure the relevant tests are carried out

$$V[\hat{\phi}_{kk}] \simeq \frac{1}{N} \qquad k \geq p+1 \text{ in an AR(p)-process}$$

$$V[\hat{\rho}_{kk}] \simeq \frac{1}{N}\left(1 + 2\left(\hat{\rho}_1^2 + \cdots + \hat{\rho}_q\right)\right) \qquad k \geq q+1 \text{ in an MA(q)-process}$$

First we consider the test for a MA-process

$$\frac{1}{N}\left(1 + 2\hat{\rho}_1^2\right) = 0.0198 = 0.14^2$$

$$\frac{1}{N}\left(1 + 2\left(\hat{\rho}_1^2 + \hat{\rho}_2^2\right)\right) = 0.0248 = 0.16^2$$

Since $\hat{\rho}_2 > 2 \cdot 0.14$ and $\hat{\rho}_3 > 2 \cdot 0.16$ there is no basis for assuming that the auto-correlation is zero from a certain step. On the other hand

$$\frac{1}{N} = \frac{1}{100} = 0.1^2$$

and therefore ϕ_{33} and ϕ_{22} can be assumed to be zero. For that reason an AR(1)-model is suggested

$$(1 + \phi_1 B)Z_t = \epsilon_t$$

where ϵ_t is a white noise process with variance σ_ϵ^2

Question 2.

The Yule-Walker equations degenerate to

$$\rho_1 = -\phi_1 \qquad \Rightarrow \qquad \hat{\phi}_1 = \underline{\underline{-0.70}}$$

From the variance of $\{Z_t\}$ we get

$$\begin{aligned} \sigma_Z^2 &= \frac{1}{(1-\phi_1^2)}\sigma_\epsilon^2 \Rightarrow \\ \sigma_\epsilon^2 &= \sigma_Z^2(1-\phi_1^2) \\ &= 2.25 \cdot (1 - 0.7^2) = 1.1475 = \underline{\underline{1.07^2}} \end{aligned}$$

Question 3.

We first define a new stochastic process $\{X_t\}$ by $X_t = Z_t - \bar{z}$, where $\bar{z}$ is the mean value of the 5 observations, $\bar{z} = 76$, i.e. we have the new time series

t	1	2	3	4	5
X_t	2	-2	-3	0	3

The one-step prediction equations are from (6.52)

$$\begin{aligned} \hat{X}_{6|5} &= -\phi \cdot X_5 = 0.70 \cdot 3 = 2.1 \\ \hat{X}_{7|5} &= -\phi \cdot \hat{X}_{6|5} = 0.70^2 \cdot 3 = 1.47 \\ \hat{X}_{8|5} &= -\phi \cdot \hat{X}_{7|5} = 0.70^2 \cdot 3 = 1.03 \end{aligned}$$

whereby we get the following one-step predictions for Z_t

$$\begin{aligned} \hat{Z}_{6|5} &= \bar{z} + \hat{X}_{6|5} = 77.01 \\ \hat{Z}_{7|5} &= \bar{z} + \hat{X}_{7|5} = 77.47 \\ \hat{Z}_{8|5} &= \bar{z} + \hat{X}_{8|5} = 77.03 \end{aligned}$$

Rewriting the process into MA- form we get

$$Z_t = \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + ...$$

i.e.

$$\psi_0 = 1$$
$$\psi_1 = \phi_1 = 0.70$$
$$\psi_2 = \phi_1^2 = 0.49$$

which from (5.151) leads to the 95% confidence intervals

$$77.8 \pm 1.96 \cdot 1.07 = 77.10 \pm 2.1$$
$$77.0 \pm 1.96 \cdot 1.07 \cdot \sqrt{1 + 0.7^2} = 77.47 \pm 2.6$$
$$76.4 \pm 1.96 \cdot 1.07 \cdot \sqrt{1 + 0.7^2 + 0.49^2} = 77.03 \pm 2.8$$

The observations, the predictions and the 95% confidence intervals are shown in figure 5.

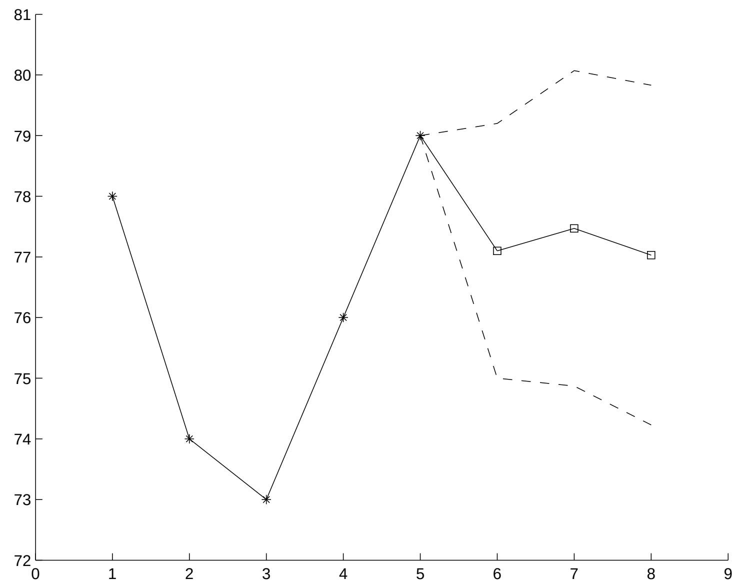

Figure 5: Plot of observations, predictions and the 95% confidence intervals.

Solution 6.10

Question 1.

We find the difference operator

$$\begin{aligned}&(1-0.8B)(1-0.2B^6)(1-B)\\&=(1-0.2B^6-0.8B+0.16B^7)(1-B)\\&=(1-0.2B^6-0.8B+0.16B^7-B+0.2B^7+0.8B^2-0.16B^8\\&=1-1.8B+0.8B^2-0.2B^6+0.36B^7-0.16B^8\end{aligned}$$

The process written on difference equation form is then

$$Y_t=1.8Y_{t-1}-0.8Y_{t-2}+0.2Y_{t-6}-0.36Y_{t-7}+0.16Y_{t-8}+\epsilon_t$$

The predictions are

$$\begin{aligned}\hat{Y}_{t+1|t}&=1.8Y_t-0.8Y_{t-1}+0.2Y_{t-5}-0.36Y_{t-6}+0.16Y_{t-7}\\\hat{Y}_{t+2|t}&=1.8\hat{Y}_{t+1|t}-0.8Y_t+0.2Y_{t-4}-0.36Y_{t-5}+0.16Y_{t-6}\end{aligned}$$

We find

$$\begin{aligned}\hat{Y}_{11|10}&=1.8\cdot(-3)-0.8\cdot 0+0.2\cdot(-3)-0.36\cdot(-2)+0.16\cdot(-1)\\&=-5.4-0.6+0.72-0.16\\&=-5.44\\\hat{Y}_{12|10}&=1.8\cdot(-5.44)-0.8\cdot(-3)+0.2\cdot 1-0.36\cdot(-3)+0.16\cdot(-2)\\&=-9.792+2.4-0.2+1.08-0.32\\&=-6.43\end{aligned}$$

Question 2.

In order to determine the 95% confidence interval ψ_1 must be found. This is most easily done by sending a unit pulse through the system as described in Remark 5.5 on page 136. We get

$$\begin{aligned}\psi_0&=\epsilon_0=1\\\psi_1&=\phi_1=1.8\end{aligned}$$

I.e.

$$\hat{Y}_{12|10} \pm 1.96 \cdot \sqrt{0.31} \cdot \sqrt{1 + 1.8^2} = \hat{Y}_{12|10} \pm 2.26 = [-8.68, -4.18]$$

The confidence interval of $\hat{Y}_{11|10}$ is

$$\hat{Y}_{11|10} \pm 1.96\sqrt{0.31} = \hat{Y}_{11|10} \pm 1.10 = [-6.54, -4.34]$$

The observations, the predictions and the 95% confidence intervals are shown in figure 6.

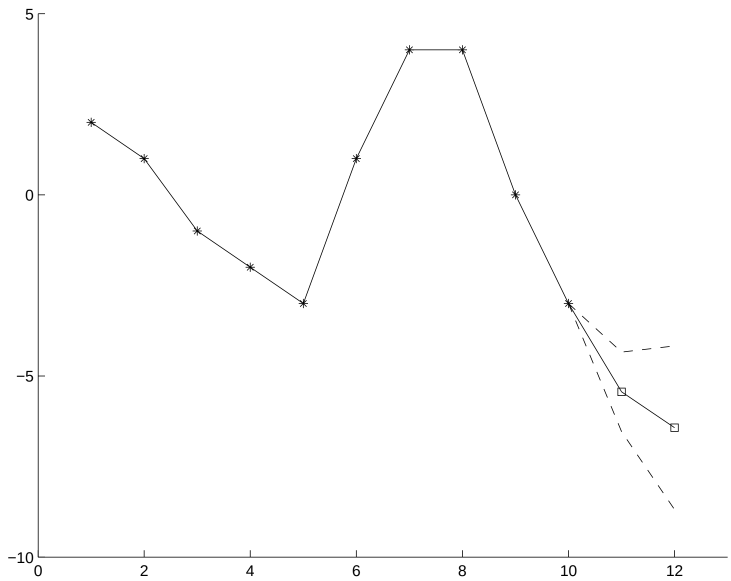

Figure 6: Plot of observations, predictions and the 95% confidence intervals.