On the Comparison of Different Number Systems in the Implementation of Complex FIR Filters

Gian Carlo Cardarilli, Alberto Nannarelli* and Marco Re

Department of Electronics, University of Rome Tor Vergata, Rome, Italy
*Dept. of Informatics & Math. Modelling, Technical University of Denmark, Kongens Lyngby, Denmark

Abstract—**The aim of this work is to compare in terms of performance, area and power dissipation, the implementations of complex FIR filters based on the following number representations: traditional two's complement (TCS), Quadratic Residue Number System (QRNS) and radix-2j Redundant Complex Number Systems (RCNS)). The resulting implementations, designed to work at the same clock rate, show that filters implemented in QRNS outperforms in terms of area and power dissipation the filters implemented in TCS and RCNS.**

I. Introduction

In modern electronic systems, complex arithmetic computation plays an important role in the implementation of different Digital Signal Processing (DSP) and scientific computation algorithms [1], [2]. Most of the interest in complex signal processing is related to the implementation of wireless communication systems based on new concepts and architectures [3]. A very interesting tutorial paper on complex signal processing and its applications has been presented recently in [4]. In this paper, the importance of the use of complex signal processing in wireless communications systems has been shown. Regarding communication systems, one of the most critical computations to be implemented in hardware is complex FIR filtering. In fact, FIR filters are generally characterized by a high order (number of taps) to obtain sharp transition bands that, in case of high speed real time computation, require a lot of resources and have high power dissipation. In particular, for complex FIR filters, the hardware complexity is mostly determined by the number of complex multipliers (i.e. each complex multiplication is actually implemented with four scalar multiplications). Different solutions have been proposed to lower the hardware complexity of the complex multiplication either at algorithmic level (Golub Rule) [5], or by using different number systems such as the Quadratic Residue Number System (QRNS) [6] and the Quater-Imaginary Number System (QINS) [7].

The aim of this work is to compare in terms of performance, area and power dissipation, the implementations of complex FIR filters based on the traditional Two's Complement System (TCS), the QRNS and the QINS (or radix-2j) implemented in the Redundant Complex Number Systems (RCNS) [8].

Previous work was done on both the QRNS ([6], [9]) and on the radix-2j and the RCNS ([10], [11], [12]). In this paper, we compare for a specific application, the complex FIR filter, the performance and the tradeoffs of TCS, QRNS and RCNS. The results of the implementations show that the complex filter implemented in QRNS has the lowest power dissipation and the smallest area with respect to filters implemented in TCS and RCNS.

The paper is organized as follows: in Section II a background on the QRNS and the radix-2j number systems is given; the FIR filter architectures for the three number systems are described in Section III; the synthesis results and the comparisons are discussed in Section IV. Finally, the conclusions are drawn in Section V.

II. The QRNS and the radix-2j Number Systems

In this section, the basic theory regarding the QRNS and the radix-2j arithmetic is briefly recalled.

A. The QRNS Number System

A Residue Number System (RNS) is defined by a set of P relatively prime integers $\{m_1, m_2, \ldots, m_P\}$ which identify the RNS base. Its dynamic range is given by the product $M = m_1 \cdot m_2 \cdot \ldots \cdot m_P$.
Any integer $X \in \{0, 1, 2, \ldots M-1\}$ has a unique RNS representation given by:

$$X \stackrel{RNS}{\rightarrow} (\ \langle X\rangle_{m_1}, \langle X\rangle_{m_2}, \ldots, \langle X\rangle_{m_P}\ )$$

where $\langle X\rangle_{m_i}$ denotes the operation $X\ mod\ m_i$ [13]. Operations on different m_i (moduli) are done in parallel

$$Z = X\ op\ Y \quad \stackrel{RNS}{\rightarrow} \quad \begin{cases} Z_{m_1} = \langle X_{m_1}\ op\ Y_{m_1}\rangle_{m_1} \\ Z_{m_2} = \langle X_{m_2}\ op\ Y_{m_2}\rangle_{m_2} \\ \ldots \quad \ldots \quad \ldots \\ Z_{m_P} = \langle X_{m_P}\ op\ Y_{m_P}\rangle_{m_P} \end{cases} \tag{1}$$

As a consequence, operations on large wordlengths can be split into several modular operations executed in parallel and with reduced wordlength [13].

In the complex case, we can transform the imaginary term into an integer if the equation $q^2 + 1 = 0$ has two distinct roots q_1 and q_2 in the ring of integers modulo M (Z_M). A complex number $x_R + jx_I = (x_R, x_I) \in Z_M \times Z_M$, with q root of $q^2 + 1 = 0$ in Z_M, has a unique Quadratic Residue Number System representation given by

$$\begin{aligned} (x_R, x_I) \stackrel{QRNS}{\rightarrow} \quad & (X_i, \hat{X}_i) \quad i = 0, 1, \ldots, P \\ & X_i = \langle x_R + q \cdot x_I\rangle_{m_i} \\ & \hat{X}_i = \langle x_R - q \cdot x_I\rangle_{m_i} \end{aligned}$$

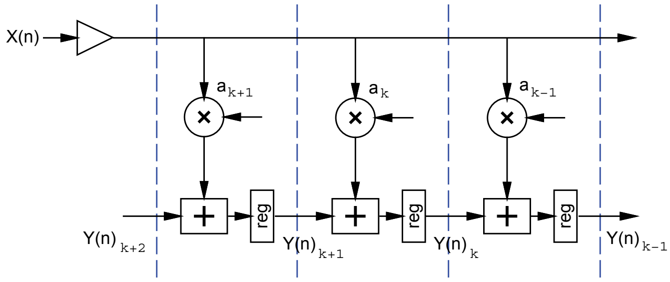

Fig. 1. Structure of FIR filter in transposed form.

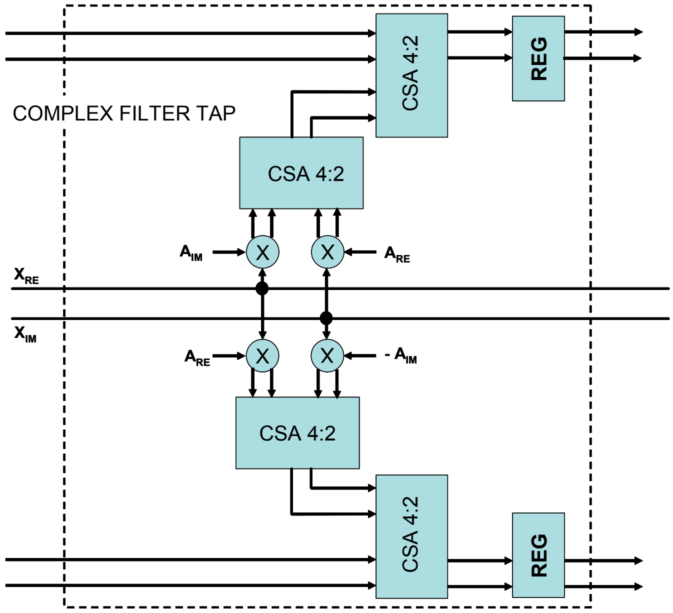

Fig. 2. Structure of tap in TCS complex FIR filter.

The inverse QRNS transformation is given by

$$x_R = \langle 2^{-1}(X_i + \hat{X}_i)\rangle_{m_i}$$
$$x_I = \langle 2^{-1} \cdot q^{-1}(X_i - \hat{X}_i)\rangle_{m_i}$$

where 2^{-1} and q^{-1} are the multiplicative inverses of 2 and q, respectively, modulo m_i:

$$\langle 2 \cdot 2^{-1}\rangle_{m_i} = 1 \qquad \text{and} \qquad \langle q \cdot q^{-1}\rangle_{m_i} = 1 \ .$$

Moreover, it can be proved that for all the prime integers which satisfy

$$p = 4k + 1 \qquad\qquad k \in N$$

the equation $q^2 + 1 = 0$ has two distinct roots q_1 and q_2.

As a consequence, the product of two complex numbers $x_R + jx_I$ and $y_R + jy_I$ is in QRNS

$$(x_R + jx_I)(y_R + jy_I) \overset{QRNS}{\rightarrow} (\langle X_i Y_i\rangle_{m_i}, \langle \hat{X}_i \hat{Y}_i\rangle_{m_i}) \qquad (2)$$

and it is realized by using two integers multiplications instead of four.

B. The Radix-2j Number System and its derivations

It is well known that an integer x can be represented by a digit-vector

$$X = (x_{n-1}, \dots, x_1, x_0)_r$$

such that

$$x = \sum_{i=0}^{n-1} x_i \cdot r^i$$

where r is the radix of the representation. By choosing $r = 2j$, we obtain a *Quater-Imaginary* Number System (QINS) [7]. Complex numbers can be represented in QINS by vectors with the non-redundant digit set $\{0, 1, 2, 3\}$. Therefore, a complex number $a + jb$ is represented in QINS as:

$$\begin{aligned} a + jb &= x_{n-1}(2j)^{n-1} + x_{n-2}(2j)^{n-2} + \ldots + \\ &\quad + x_3(-8j) + x_2(-4) + x_1(2j) + x_0(1) \\ &= (x_{n-1}, \dots, x_1, x_0)_{2j} \end{aligned}$$

The above expression, shows that the real part is represented by the digits of even weight, while the imaginary one by the digits of odd weight. Furthermore, the sign is embedded in the representation.

The implementation of the basic arithmetic operators in radix-$2j$ can take advantage of the Signed-Digit (SD) representation [14], which allows carry free addition. The combination of radix-$2j$ and SD representation, resulted in the Redundant Complex Number Systems (RCNS), which is described in [8], [10], [11], [12] and [15].

We now briefly recall the characteristics of the RCNS. The RCNS is a redundant positional number system based on the radix rj where its digits can assume the $2\alpha + 1$ values: $A_\alpha = \{\overline{\alpha}, \cdots, \overline{1}, 0, 1, \cdots, \ \alpha\}$ where $\overline{\alpha} = -\alpha$. .
In the case of the radix $2j$, two possible RCNSs [10] are:

1) RCNS 2j, 2 with digit set $A_2 = \{\overline{2}, \overline{1}, 0, 1, 2\}$
2) RCNS 2j, 3 with digit set $A_3 = \{\overline{3}, \overline{2}, \overline{1}, 0, 1, 2, 3\}$

In this paper, RCNS 2j, 2 is used to recode the multiplier, and RCNS 2j, 3 is used for the signed-digit additions, as illustrated next.

III. FIR Filter Implementations

A complex FIR filter of order N is expressed by

$$\underline{y}(n) = \sum_{k=0}^{N-1} \underline{a}_k \underline{x}(n-k) \qquad (3)$$

where $\underline{x}$, $\underline{y}$ and $\underline{a}_k$ denote complex numbers. We consider the implementation of a FIR filter in transposed form because its structure is more regular with respect to the filter order N and it does not require a tree of adders. The filter in transposed form can be regarded as the sequence of groups, often referred as *taps*, composed of:

- a multiplier
- an adder
- a register

as highlighted in Fig. 1.

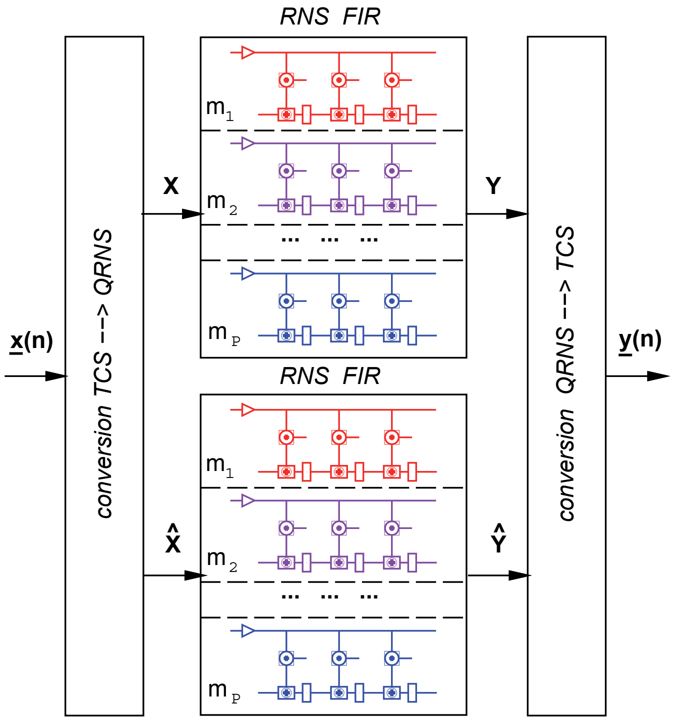

Fig. 3. QRNS FIR Filter architecture

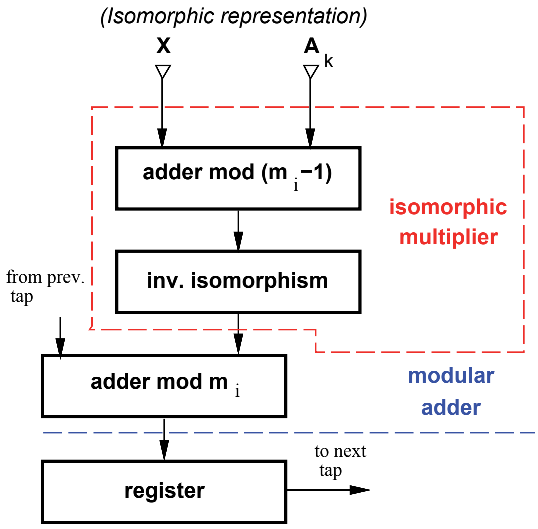

Fig. 4. Structure of RNS tap for filter in transposed form.

We perform our design space exploration for programmable N-tap complex FIR filters with input and coefficients size of 10 bits for both the real part and imaginary parts. The 20 bit dynamic range of the filter guarantees error free operations[1].

A. TCS FIR Filter

A single tap of the The programmable N-tap TCS complex FIR filter is realized as sketched in Fig. 2. The real and imaginary products are both realized with two Booth multipliers each, and the resulting partial products are accumulated in a Wallace's tree structure which produces a carry-save (CS) representation of the product in each side of the filter. We convert the CS representation of y_{Re} and y_{Im} with two carry-propagate adders at the filter output.

B. QRNS FIR Filter

The architecture of the QRNS filter, is a direct consequence of (1), (2) and (3), and it can be realized by two RNS filters in parallel as shown in Fig. 3. Each RNS filter is then decomposed into P filters working in parallel, where P is the number of moduli used in the RNS representation. In addition, the RNS filter requires both binary to QRNS and QRNS to binary converters. In order to have a dynamic range of 20 bits, as required by the specifications, we chose the following set of moduli:

$$m_i = \{5, 13, 17, 29, 41\}$$

[1]These wordlengths are derived from the specification of an actual digital filter for satellite TV broadcasting.

such that

$$log_2(5 \cdot 13 \cdot 17 \cdot 29 \cdot 41) > 20 \ .$$

For each path mod m_i, we have to build a FIR filter with a structure similar to that of Fig. 1. Therefore, we need to implement modular multiplication and addition. By using the isomorphism technique, the product of the two residues is transformed into the sum of their indices which are obtained by an isomorphic transformation [16]. As a result, in each tap, the modular multiplication is reduced to a modular addition followed by an access to table (inverse isomorphism) as depicted in Fig. 4 (see [9] for more detail).

C. Radix-2j Filter (RCNS)

Because of the radix-2j representation, the filter tap is simply implemented with a multiplier and an adder. We implement the multiplier as described in [10]. The complex $\underline{x}$ and $\underline{a}_k$ are converted in non-redundant QINS and then $\underline{a}_k$ is recoded into RCNS 2j, 2. The partial products (PPs) are then accumulated by a tree of arrays of signed-digit full-adders (SDFA) which operates in RCNS 2j, 3. An extra array of SDFAs adds the product $\underline{x} \cdot \underline{a}_k$ to the partial sum coming from the previous tap. The implementation of the RCNS tap is sketched in Fig. 5.

IV. Synthesis and Comparisons

The filters are implemented in the 90 nm STM library of standard cells and they have been synthesized by Synopsys Design Compiler. All the filters can be clocked at $f_{max} = 300\ MHz$. By interpolating the results obtained by synthesis on filters of different order (number of taps), we obtain the trends shown in Fig. 6 for the area and Fig. 7 for the power. The values of area and power dissipation for the single tap (Fig. 2, Fig. 4 and Fig. 5) determine the slopes of the curves in the figures. The conversions from the TCS to the other number systems (and vice versa) are a constant contribution

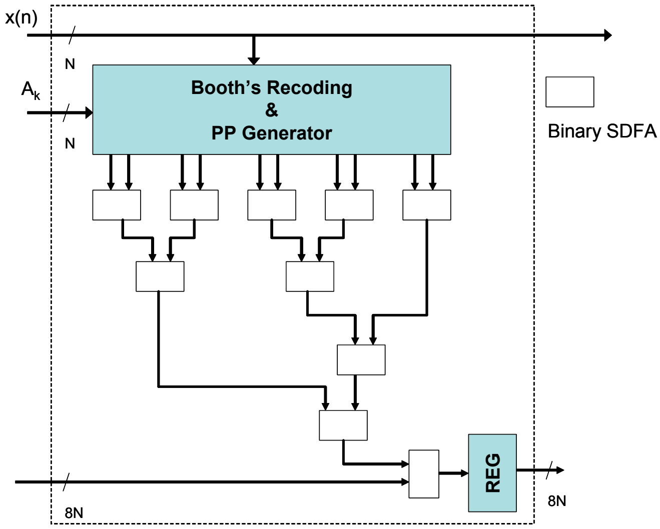

Fig. 5. Structure of RCNS tap.

	Area		P at 100 MHz	
	tap	conv.	tap	conv.
TCS	21.8K	2.0K	1.00	0.10
QRNS	9.6K	12.0K	0.25	1.20
RCNS	23.9K	8.0K	1.05	0.30
	$[\mu m^2]$		$[mW]$	

TABLE I
VALUES OF AREA AND POWER DISSIPATION.

that does not depend on the number of taps, but only on the dynamic range of the filters. Table I reports the data for tap and conversion contribution for the three number systems.

The results show that complex filters implemented in QRNS consume significantly less power than the corresponding ones in TCS and RCNS. The expression for the power dissipated dynamically [17] in a system composed of n cells is

$$P_{dyn} = V_{DD}^2 f \cdot \sum_{i=1}^{n} C_{Li} a_i \tag{4}$$

where

- V_{DD} is the power supply voltage;
- f is the clock frequency;
- C_{Li} is the load connected to the i-th cell (both active load and interconnections);
- a_i is the activity factor of the i-th cell, which is the measure of how many transitions occur at its output. The activity factor is normally related to the clock $a_i \in [0, 1]$.

The lower power dissipation in the QRNS filter is due to the combination of two factors:

1) As clearly shown in Fig. 6, the smaller area results in a global reduced capacitance $\sum_{i=1}^{n} C_{Li}$ (including shorter interconnections).
2) The work in [18] showed that the number of transitions, i.e. the switching activity, for vectors of the same number

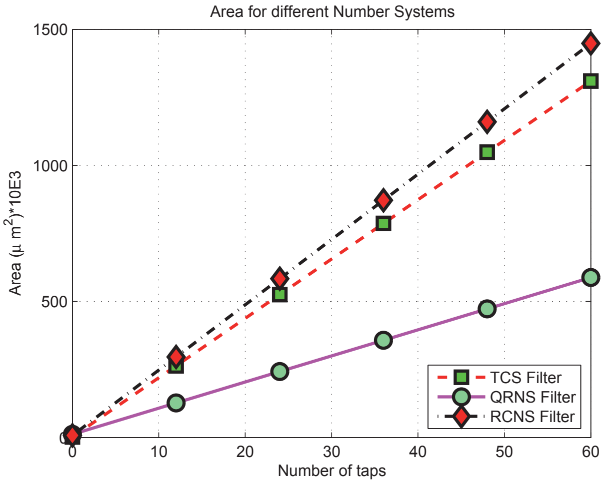

Fig. 6. Trends in area for increasing N.

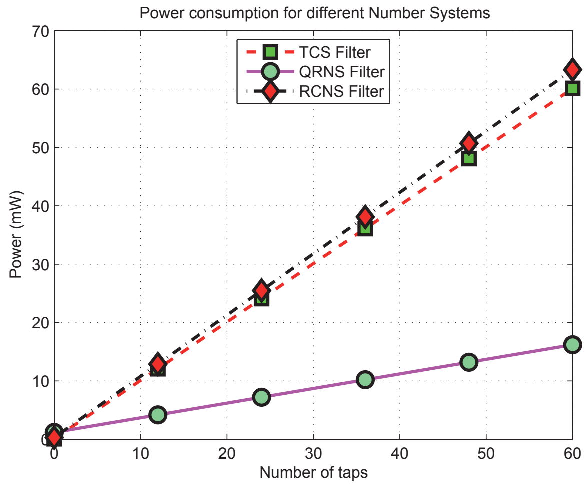

Fig. 7. Trends in power dissipation (at 100 MHz) for increasing N.

of bits k, in RNS is lower than in TCS

$$\left(\sum_{i=1}^{k} a_i\right)_{RNS} < \left(\sum_{i=1}^{k} a_i\right)_{TCS}$$

Therefore, the switched capacitance $\sum_{i=1}^{n} C_{Li} a_i$, and by (4 the power consumption, in QRNS is smaller than in TCS and RCNS.

V. Conclusion

In this work, the use of different number representations for the implementation of complex FIR filters has been investigated.

Complex multipliers determine the performance, area and power dissipation of complex filters. Previously in [10], complex multipliers in TCS and RCNS were evaluated, while in [9], complex filters in QRNS and TCS were compared. Here we extended the comparison to complex filters implemented in TCS, QRNS and RCNS.

The experimental results on complex filters with 20 bit dynamic range show that for the TCS and the RCNS the area and power dissipation are similar and confirms the findings of [10]. As for the QRNS, the results presented here, confirm those of [9], based on the implementation of TCS and QRNS complex filters in a 0.35 μm technology.

To summarize, this work shows that for complex high order FIR filters implementations based on QRNS offer significant advantages in area and power dissipation without any performance degradation.

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