

**Ph.D. Thesis**

# Economic Model Predictive Control for Large-Scale and Distributed Energy Systems

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# Preface

This thesis was submitted at the Department of Applied Mathematics and Computer Science (DTU Compute, formerly known as DTU Informatics) at the Technical University of Denmark in partial fulfilment of the requirements for acquiring the PhD degree in engineering. The project was funded by the Southern Denmark Growth Forum and the European Regional Development Fund in the project “Smart & Cool” (ERDFD-10-0083) started up in 2011. Smart & Cool is a co-operation between DTU Informatics and public as well as private companies: European Union, Region Syddanmark, Aalborg University (AAU), Dong Energy, Danfoss and KVCA. The advisor of the thesis are the associate professors John Bagterp Jørgensen and Niels Kjølstad Poulsen.

The thesis deals with control of the future power systems often referred to as Smart Grids. We propose Economic Model Predictive Control (MPC) as a control strategy. In addition, we tailored the control problem for the implementation of specific decomposition technique aimed to fasten the algorithm, hence, to increase its applicability in industrial applications. Along the way, we present numerical simulations alongside case studies as well as validated models in realistic scenarios.

The thesis consists of a summary report and a collection of 11 research papers and technical reports written during the period September 2011 to November 2014. Eight papers were published in international peer-reviewed scientific conferences, one is published at peer-reviewed scientific journals and two are technical reports.

*“Fatti non foste a viver come bruti, ma per seguir virtute e canoscenza.”*  
*(Ye were not made to live like unto brutes,*  
*but for pursuit of virtue and of knowledge.)*  
*Dante Alighieri, The Divine Comedy, 26.120*



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Last but not least, I thank Bonnie and my family in Italy. They have been able to constantly support me over the last three years and they have been the most important co-authors of all my works.



# Summary (in English)

In this thesis, we consider control strategies for large and distributed energy systems that are important for the implementation of smart grid technologies. An electrical grid has to ensure reliability and avoid long-term interruptions in the power supply. Moreover, the share of Renewable Energy Sources (RESs) in the smart grids is increasing. These energy sources bring uncertainty to the production due to their fluctuations. Hence, smart grids need suitable control systems that are able to continuously balance power production and consumption. We apply the Economic Model Predictive Control (EMPC) strategy to optimise the economic performances of the energy systems and to balance the power production and consumption. In the case of large-scale energy systems, the electrical grid connects a high number of power units. Because of this, the related control problem involves a high number of variables and constraints and its solution requires high computational times. Energy systems have a hierarchical control framework and the controllers have to work in the time-scale required by their hierarchy level. Dedicated optimisation techniques efficiently solve the control problem and reduce computational time. We implement the Dantzig-Wolfe decomposition technique to efficiently solve the EMPC problem.

The contributions of this thesis are primarily on:

## **Large-scale energy system**

Smart-grids connect a high number of energy units. In such a large-scale scenario the energy units are independent and dynamically decoupled. The mathematical model of the large-scale energy system embodies the decoupled dynamics of each power units. Moreover, all units of the grid contribute to the overall power production.

## **Economic Model Predictive Control (EMPC)**

This control strategy is an extension of the Model Predictive Control (MPC) strategy. Energy systems often involve stochastic variables due to the share of fluctuating Renewable Energy Sources (RESs). Moreover, the related control problems are multi variables and they are hard, or impossible, to split into single-input-single-output control systems. MPC

strategy can handle multi variables control problems and it can embody stochastic variables. The Economic MPC (EMPC) policy optimises the economic performances of the process. In this work, we apply the EMPC to energy systems and it computes the control trajectory for each energy unit. This control policy minimises production costs and ensures that the power production satisfies the customers' demand. The EMPC designs a linear control problem that has a block-angular constraints matrix and it has two sets of constraints. The independent dynamics of the energy units define the decoupling constraints sited on the diagonal. The coupling constraints represent the common goal of all power units in the energy system and this is to satisfy the customers' demand. The Dantzig-Wolfe optimisation technique applies to this structure of the constraints matrix in the view of fastening the control algorithm and increase its applicability.

### **Dantzig-Wolfe decomposition**

The Dantzig-Wolfe decomposition solves the EMPC problem through a distributed optimisation technique. The EMPC problem via Dantzig-Wolfe decomposition algorithm computes the optimal input trajectory for each energy unit and reduces the computation times. Moreover, such a control algorithm applies to large-scale energy systems and the number of energy units does not affect the performances of the controller. In this thesis, we also investigate suboptimal solutions of the EMPC problem via modified versions of the Dantzig-Wolfe decomposition algorithms. The feasibility of the suboptimal solutions suffices for stability. The goal of these modified Dantzig-Wolfe decomposition algorithms is to reduce computation time in the solution of the EMPC problem.

# Resumé

I denne afhandling, analyserer vi strategier for store og distribuerede energisystemer, der er vigtige for implementationen af smart grid-teknologier. Elnettet skal sikre pålidelighed og undgå langvarige afbrydelser i strømforsyningen. Desuden er andelen af vedvarende energikilder i de intelligente elnet stigende. Disse energikilder bringer usikkerhed for produktionen på grund af deres udsving. Derfor har de intelligente elnet brug for passende kontrolsystemer, der er i stand til at afbalancere løbende elproduktion og forbrug. Vi anvender den økonomiske modelprædiktiv reguleringsstrategi (EMPC) for at optimere de økonomiske resultater fra energisystemerne og for at skabe balance i elproduktion og -forbrug. I store energisystemer forbinder elnettet et stort antal kraftenheder. På grund af dette har det tilhørende kontrol problem et stort antal variabler og begrænsninger, og dets løsning kræver lange beregningstider. Energisystemer har en hierarkisk kontrol struktur, og regulatorerne skal arbejde på den tid skala, der kræves af deres hierarkiske niveau. Dedikerede optimeringsteknikker løser effektivt kontrol problemet og reducerer beregningstiden. Vi implementerer Dantzig-Wolfe dekompositionsteknikken til effektivt at løse EMPC problemet.

Denne afhandling har primært bidraget inden for:

## **Storskala Energisystemer**

Det intelligente elnet forbinder et stort antal energienheder. I et sådant storskala scenario er de enkelte enheder uafhængige og dynamisk afkoblede. Den matematiske model af et stort energisystem inkluderer de enkelte enheders dynamik på afkoblet vis. Desuden bidrager alle enheder på nettet til den samlede elproduktion.

## **Økonomisk Modelprædiktiv Regulering (EMPC)**

Denne kontrolstrategi er en udvidelse af den modelprædiktive reguleringsstrategi (MPC). Energisystemer involverer ofte stokastiske variable på grund af andelen af fluktuerende vedvarende energikilder. Ydermere er de tilhørende kontrolproblemer multivariable, og de er svære eller umulige at opsplutte i single-input-single-output kontrolsystemer. MPC-strategien kan håndtere multivariable kontrolproblemer, og den kan inkludere stokastiske variable. Den Økonomiske MPC strategi (EMPC)

optimerer de økonomiske resultater i processen. I denne afhandling anvender vi EMPC til energisystemer, og beregner derved styringen for hver energienhed. Denne kontrolstrategi minimerer produktionsomkostningerne og sikrer, at elproduktion opfylder kundernes efterspørgsel. Ved EMPC designes et lineært kontrolproblem, der har en blok-angulær begrænsningsmatrix, og det har to sæt begrænsninger. De uafhængige dynamikker af energienhederne definerer afkoblingsbegrænsningerne er placerede på diagonalen. Sammenkoblingsbegrænsningerne repræsenterer det fælles mål for alle kraftenheder i energisystemet, hvilket er at tilfredsstille kundernes efterspørgsel. Dantzig-Wolfe optimeringsteknikken kan anvendes på denne struktur af begrænsningsmatrixen med henblik på at gøre styringsalgoritmen hurtigere og øge dens anvendelighed.

### **Dantzig-Wolfe Dekomposition**

Dantzig-Wolfe dekompositionen løser EMPC problemet ved hjælp af en distribueret optimeringsteknik. Via Dantzig-Wolfe dekompositionsalgoritmen beregner EMPC problemet det optimale regulatorinput for hver energienhed og reducerer beregningstiden. Desuden kan en sådan en kontrolalgoritme anvendes på store energisystemer, og antallet af energienheder påvirker ikke regulatorens effektivitet. I denne afhandling undersøger vi også suboptimale løsninger af EMPC problemet via modificerede versioner af Dantzig-Wolfe dekompositionsalgoritmen. Eksistensen af suboptimale løsninger er tilstrækkeligt for at garantere stabilitet. Målet med disse modificerede Dantzig-Wolfe dekompositionsalgoritmer er at reducere beregningstiden ved løsningen af EMPC problemet.

# Contents

<b>Preface</b>	<b>iii</b>
<b>Acknowledgements</b>	<b>v</b>
<b>Summary (in English)</b>	<b>vii</b>
<b>Resumé</b>	<b>ix</b>
<b>I Introduction and Background</b>	<b>1</b>
<b>1 Introduction</b>	<b>3</b>
1.1 The energy revolution . . . . .	3
1.2 Energy systems . . . . .	4
1.3 Economic Model Predictive Control (EMPC) . . . . .	5
1.4 State-of-the-art . . . . .	5
1.5 Objective and main contributions . . . . .	7
<b>2 Future energy systems</b>	<b>9</b>
2.1 Smart grids . . . . .	9
2.2 Distributed Energy Systems (DESS) . . . . .	11
2.3 Models of energy units . . . . .	11
2.4 Power systems . . . . .	18
2.5 Control hierarchy for a power system . . . . .	22
2.6 Control for Smart Grids . . . . .	23
<b>3 Economic Model Predictive Control</b>	<b>27</b>
3.1 Choice of Economic Model Predictive Control . . . . .	27
3.2 Economic MPC formulation . . . . .	29
3.3 Stability . . . . .	31
<b>4 Dantzig-Wolfe decomposition for Linear Programs</b>	<b>33</b>
4.1 Decomposition techniques . . . . .	33
4.2 Block-angular LPs . . . . .	33
4.3 Dantzig-Wolfe Transformation . . . . .	34
4.4 Master Problem (MP) . . . . .	35
4.5 Reduced Master Problem (RMP) . . . . .	35
4.6 Subproblems . . . . .	36
4.7 Optimality . . . . .	38
4.8 Initialisation . . . . .	40
	xi

<b>II</b>	<b>Main Contributions</b>	<b>41</b>
<b>5</b>	<b>Linear Economic MPC for Power Plant Portfolio via Dantzig-Wolfe Decomposition</b>	<b>43</b>
5.1	Scientific contribution . . . . .	43
5.2	Power plant portfolio management . . . . .	44
5.3	Dantzig-Wolfe decomposition . . . . .	53
5.4	MATLAB Simulations . . . . .	56
5.5	Summary . . . . .	61
<b>6</b>	<b>Early Termination in Dantzig-Wolfe Decomposition for EMPC</b>	<b>63</b>
6.1	Early termination in Dantzig-Wolfe decomposition . . . . .	63
6.2	MATLAB simulations . . . . .	65
6.3	Summary . . . . .	66
<b>7</b>	<b>Reduced Dantzig-Wolfe Decomposition and Partial Cycling Strategies for Linear EMPC</b>	<b>69</b>
7.1	Reduced Dantzig-Wolfe decomposition . . . . .	69
7.2	Partial cycling strategies in Dantzig-Wolfe decomposition . . . . .	73
7.3	Summary . . . . .	79
<b>8</b>	<b>Conclusions and Perspectives</b>	<b>83</b>
8.1	Large and distributed energy systems . . . . .	83
8.2	Dantzig-Wolfe decomposition to solve linear EMPC . . . . .	84
8.3	How to fasten the Dantzig-Wolfe decomposition . . . . .	84
8.4	Contributions . . . . .	85
8.5	Perspectives and future works . . . . .	85
	<b>Bibliography</b>	<b>87</b>
<b>III</b>	<b>Published Papers and Technical Reports</b>	<b>95</b>
	<b>Appendices</b>	<b>97</b>
<b>A</b>	<b>Paper A - 19<sup>th</sup> IFAC World Congress, 2014</b>	<b>99</b>
<b>B</b>	<b>Paper B - Journal of Process Control, 2014</b>	<b>107</b>
<b>C</b>	<b>Paper C - 4<sup>th</sup> IEEE PES Innovative Smart Grid Technologies Europe, 2013</b>	<b>121</b>
<b>D</b>	<b>Paper D - 4<sup>th</sup> IEEE PES Innovative Smart Grid Technologies Europe, 2013</b>	<b>129</b>
<b>E</b>	<b>Paper E - DTU International Energy Conference, 2013</b>	<b>137</b>
<b>F</b>	<b>Paper F - 18<sup>th</sup> Nordic Process Control Workshop, 2013</b>	<b>141</b>
<b>G</b>	<b>Paper G - 18<sup>th</sup> Nordic Process Control Workshop, 2013</b>	<b>149</b>
<b>H</b>	<b>Paper H - 10<sup>th</sup> European Workshop on Advanced Control and Diagnosis, 2012</b>	<b>157</b>

<b>I</b>	<b>Paper I - 17<sup>th</sup> Nordic Process Control Workshop, 2012</b>	<b>167</b>
<b>J</b>	<b>Paper L - Technical report DTU, 2014</b>	<b>171</b>
<b>K</b>	<b>Paper M - Technical report DTU, 2014</b>	<b>179</b>



## Part I

# Introduction and Background





# Introduction

In this chapter, we explore and motivate this work. We define the current energy scenario, then we briefly describe its components and their organisation in an energy system. In addition, we introduce the Economic Model Predictive Control (EMPC) strategy. We present an overview of the state-of-the-art and give some references to important literature in the different fields that we address in this thesis: EMPC applied to energy systems, strategies to fasten the control algorithm and the Dantzig-Wolfe decomposition. Finally, we outline the main objectives of this thesis.

## 1.1 The energy revolution

All of us have experienced the effects of global climate change. Trees are flowering sooner, heat waves are more intense, plant and animal ranges have shifted [1]. In the same way, there is loss of sea ice and glaciers have shrunk. Scientists believe that global temperatures are meant to rise for decades to come, largely due to greenhouse gasses produced by human activities [2]. Figure 1.1 describes that, among these activities, the generation of electricity and the heating sector represent the largest sources of emissions.

In the long term, the limited fossil fuel resources and the continuous hike in oil price cause the need of alternative energy sources. Moreover, according to the World Energy Outlook 2012, by 2035, the demand for electricity will be more than 70% higher than the current demand [2]. This is one of the reasons for including innovative energy sources in the current energy scenario.

The energy revolution, also known as third industrial revolution, is the result of renewable energy meeting Information and Communication Technologies (ICT). Renewable Energy Sources (RESs) have the potential to replace fossil fuels, but in this transition, politics and technology are fundamental.

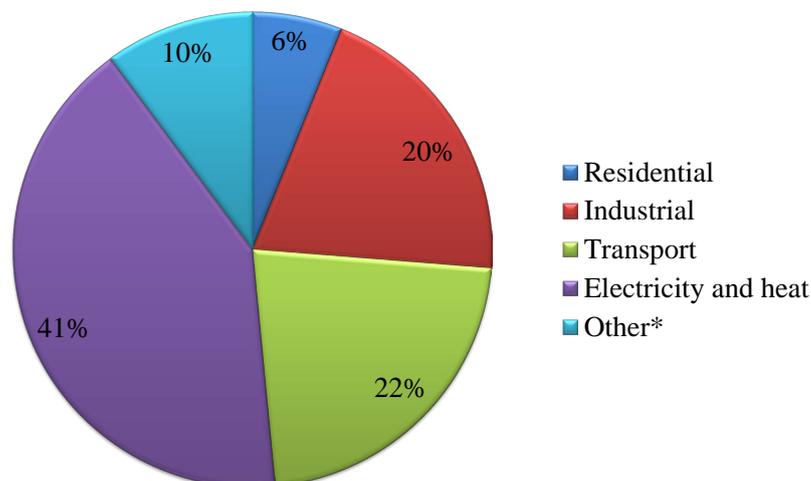


Figure 1.1: World CO<sub>2</sub> emissions by sector in 2010. \*Other includes commercial/public services, agriculture/forestry, fishing, energy industries other than electricity and heat generation, and other emissions not specified elsewhere [2].

Regarding politics, many countries are contributing in different ways [3]. Czech Republic, Germany, Netherlands, Canada, and United States have recently announced innovative energy policies [3]. Turkey, Portugal, and New Zealand are focusing on delivering energy efficiency [3]. Similarly, UK, Germany, Hungary, Australia, Greece, and Korea are seeking ways to increase customer participation and to keep prices down [3]. The increasing share of renewables is happening in countries like Sweden, Norway, Netherlands, Spain, Denmark, and Austria [3]. Moreover, Italy, Japan and Austria are reviewing their gas infrastructure, while Ireland and Norway are focusing on safe and environmentally safe production of oil [3]. Let us consider Denmark, which aims to have its energy system fully based on Renewable Energy Sources (RESs) by 2050, while by 2035, to have RESs cover the energy consumption for power and heat [4].

## 1.2 Energy systems

In the context of energy supply and consumption, the term energy usually denotes the electrical energy produced by electric power generators through conversion of primary energy sources [5]. Over the last decades, solar and wind have been considered as primary energy sources.

This inevitable transition to alternative energy sources affects the energy production, distribution, conversion, consumption and also the technologies involved.

Recently, commercial and political interests in energy systems have generated changes regarding energy structure and energy markets: the share of decentralised and distributed energy producers is increasing and electricity markets introduce liberalisation and create markets for each time scale.

Operating energy systems requires complex control techniques because these systems are responsible for generating, transmitting, and delivering the energy [6]. As a consequence, a hierarchical control structure comprises hardware and software for monitoring and controlling the energy systems [6]. However, developing suitable controllers to operate energy systems in an economical and a reliable way is a challenge.

Chapter 2 examines the current and future energy systems.

## 1.3 Economic Model Predictive Control (EMPC)

Nowadays, Model Predictive Control (MPC) is ubiquitous and its applications are multiple, e.g. chemical plants, food processing and automotive industries [7–9].

The MPC controller is implemented in a receding horizon manner and it solves an Optimal Control Problem (OCP) over a given horizon. The OCP computes the optimal control input sequence in open-loop based on the current estimated state of the system, the explicit process model and the desired reference trajectory. Only the first value of the input sequence is implemented and, at the next iteration, the new measurements build the new OCP.

One of the objectives in the synthesis of a control structure is to include economic objectives into process control objectives [10]. In general, efficiency, profitability, capacity, sustainability, and variability are the economic objectives [11]. In the Economic MPC policy the objective function of the control problem is the economic objective. As a consequence, the controller optimises the economic performances of the process, rather than tracking a setpoint [12]. Moreover, the EMPC ensures that the economic and the process performances are improved by consistently dynamic, transient or time-varying operation without forcing the process [11].

Chapter 3 examines the Economic MPC control strategy.

## 1.4 State-of-the-art

In this section we provide a brief overview of important literature related to the main topics that this thesis addresses. The following chapters focus on these topics.

### 1.4.1 Economic MPC and energy systems

In the context of energy systems, the Economic MPC has been utilised in numerous applications.

The increasing share of renewable energy sources requires flexibility in the energy consumption. The Economic MPC strategy utilises the flexibility in refrigeration systems to counteract fluctuations in the balance between power consumption and production [13–15].

Building climate control is another application of EMPC strategy. In this case, the proposed control algorithm brings flexibility and cost savings by a pre-cooling effect during off-peak period and a cooling discharge from the building thermal mass during on-peak period [16, 17]. Similarly, the EMPC strategy applies floor heating control via heat pumps [18].

Moreover, the Economic MPC applies to distributed structures, such as distributed energy resources as micro-CHP systems [19] and thermal power plants [20].

### 1.4.2 Speed-up MPC Solution

Limited computational resources often restrict the applicability of MPC-based controllers. Numerous investigations have focused on speeding-up the solution of the MPC control problem. Suboptimal solutions and warm-start techniques offer new possibilities in the applicability of MPC to real problems due to their benefits on computation times and storage [21]. Even if the solutions are not optimal, they are feasible and this suffices for stability [22].

Decomposition techniques are powerful tools to compute robust and efficient optimal control sequences by exploiting the structure of the MPC control problem [23]. Often, control problems have two specific structures: complicating constraints and complicating variables. The former structure is tailored for the Dantzig-Wolfe decomposition technique. While the latter structure is suitable for applying the Benders decomposition [23].

Due to the complicating constraints structure of our control problem, in this work we apply the Dantzig-Wolfe decomposition.

### 1.4.3 Dantzig-Wolfe decomposition

The Dantzig-Wolfe algorithm has been extensively used in many MPC applications, for instance refinery-planning [24], production optimisation in an oil field [25] and industrial process control [26]. Similarly, building temperature control problem [27] and power plant portfolio management [20] based on an  $\ell_1$ -penalty function have implemented the Dantzig-Wolfe decomposition to efficiently solve the MPC control problem.

This decomposition algorithm efficiently applies to integer problems, such as packing problems and stochastic capacity planning [28–30]. In addition, a

hybrid Dantzig-Wolfe algorithm has been implemented in mixed integer programming for security-constrained unit commitment (SCUC) problem [31].

A part of the Dantzig-Wolfe decomposition can be solved via parallel computing and this has been implemented in integer and binary problems for oil field [32] and air traffic flow problems [33].

The Dantzig-Wolfe decomposition has been applied to partition a Quadratic Programming problem (QP) into a master problem, which uses Lemke's method and a subproblem to solve alternately [34]. Moreover, an interior-point method has been implemented in the Dantzig-Wolfe decomposition to compute the optimal trajectory [35]. An alternative version of the Dantzig-Wolfe algorithm distributes the master problem to local agents that exchange only primal information [36].

Computational strategies, i.e. partial cycles and multi-proposal generation, demonstrate that these strategies accelerate the Dantzig-Wolfe algorithm for LPs [37]. An extension of this decomposition technique proposes an alternative master problem other than the standard one in which there is a variable for each of the extreme points and extreme rays of the corresponding polyhedron [38]. This strategy improves computational performances of the decomposition technique. Finally, a chronological and critical survey of computational experience with Dantzig-Wolfe decomposition is crucial for further advances [39].

## 1.5 Objective and main contributions

The approach we adopt in this work aims to develop controller algorithms for future energy systems, which are commonly known as smart grids. We consider these energy systems as comprising independent and dynamically decoupled power consumers and producers. We address the Economic MPC strategy aimed to coordinate and control the power units and optimise the economic objective of the energy system. Our main research contributions formulate, model and rephrase theoretic results and methods in the view of extending those to control real-life problems:

- We model large and distributed energy systems as comprising multiple and independent energy units that are dynamically decoupled. These energy units represent power consumers and producers connected to the electrical grid. All units in the energy system cooperate to satisfy customers' demand. The electrical grid embodies renewable energy sources that cause the need of flexible consumption.
- We propose an Economic MPC (EMPC) strategy to balance power production and consumption in large and distributed energy systems. Linear models of energy units, linear constraints and linear cost function define a linear control problem. Additionally, the decoupled dynamics

of the units and their cooperation affect the constraints matrix that has a block-angular structure.

- Operating large ad distributed energy systems involves a high number of variables and constraints. Hence, the limits on computation times might prevent the solution of such a control problem. In order to reduce computation times, we apply the Dantzig-Wolfe decomposition to solve the linear EMPC problem and compute the optimal input trajectories for each energy unit.
- We investigate strategies aimed to fasten the Dantzig-Wolfe decomposition. We consider that their suboptimal solutions ensure feasibility which suffices for stability. Moreover, we inspect the effects of these strategies such as reduction in computation times and deterioration in objective functions optimal values.

# Future energy systems

This chapter outlines the future energy systems also known as smart grids. Firstly, we compare the current electrical energy system to the future one. Then we describe Distributed Energy Sources (DEs) and the key role that they are predicted to play in the future energy systems. Subsequently, we illustrate the power systems structure and the models of some of the energy units involved. Finally, we present the control hierarchy of the power systems.

## 2.1 Smart grids

Smart grids represent the combination of Information and Communication Technology (ICT) and energy systems. Digital communication, control technologies, electricity generators, consumers, and grid operators are connected in one large grid. This innovative structure has multiple benefits; among those, reduced environmental impact, cheaper supply of electricity and greater system reliability [40]. In the future systems a proportion of the electricity generated by large conventional power plants will be displaced by RESs (Renewable Energy Sources) and DERs (Distributed Energy Resources) [41].

This revolution in the current energy scenario will introduce the demand-response strategy in the energy management system aimed to balance supply and demand. The demand-response enables consumers to reduce or shift their electricity usage during peak periods in response to time-based rates or other forms of financial incentives. This strategy requires continuous, fast, and reliable communications in order to regularly satisfy customers' demand. Therefore, ICT are fundamental in such a scenario.

Clearly, smart grids embody a wide variety of power generation options, e.g. central, distributed, and mobile. The pervasive control and the intelligence of the grids must reside across all geographies, components, and functions of the

Existing Grid	Intelligent Grid
Electromechanical	Digital
One-Way Communication	Two-Way Communication
Centralized Generation	Distributed Generation
Hierarchical	Network
Few Sensors	Sensors Throughout
Blind	Self-Monitoring
Manual Restoration	Self-Healing
Failures and Blackouts	Adaptive and Islanding
Manual Check/Test	Remote Check/Test
Limited Control	Pervasive Control
Few Customer Choices	Many Customer Choices

Figure 2.1: Differences in the current grid and the future intelligent grid [42].

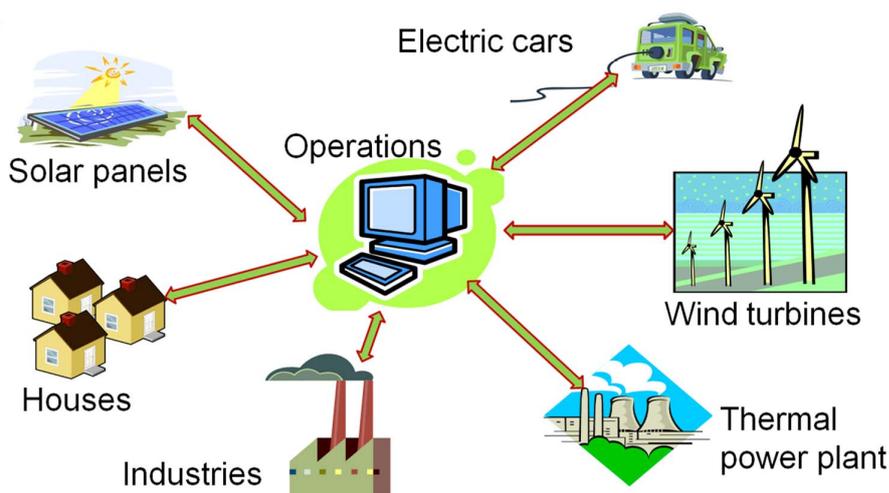


Figure 2.2: Distributed Energy System (DES).

system. The need of fast and reliable communications usually limits the applicability of centralised control strategies. Instead, distributed control strategies are commonly preferred because of they provide fast control signals across the entire grid [42]. Consequently, future energy networks are and will be large-scale, complex and seen as interconnected systems that require controllers and coordination all the way from the production to the consumption [4]. Figure 2.1 compares the main features of the existing grid and the intelligent grid.

## 2.2 Distributed Energy Systems (DESS)

A Distributed Energy System (DES) consists of small and local power suppliers and consumers that are connected to the electrical grid [43]. Figure 2.2 shows a DES that involves localised energy producers as well as Renewable Energy Sources (RESs) connected to one operations centre.

A Smart Grid has the potential to embody DESs and this has multiple benefits [44, 45]. First, one of the benefits is environmental due to the share of RESs. Local power suppliers require shorter transport pipelines for district heating and there is less energy loss. Furthermore, small and local producers need shorter plant implementation times. Finally, such a structure causes flexibility in the production given by the small and local power generators.

Over the past few years, Distributed Energy Resources (DERs) have attracted much attention and they have become constituents of DESs. Clearly, many technical issues need to be investigated in order to guarantee the efficiency of such an innovative energy system [46].

One of the most important technological aspects in the context of future energy systems is to design suitable control frameworks that continuously ensure system performances and take advantage of the inherent scalability and robustness benefits of DERs [47]. Commonly, the control strategies can have decentralised or centralised framework. While decentralised control structure computes the optimal control signal locally with each energy unit, these local optima do not guarantee the best solution for the overall energy system. However, the distributed control framework increases the scalability and the flexibility of the energy system. Additionally, including power sources into the energy systems to meet the increasing demand is easier in a decentralised structure than in the traditional centralised control framework [48].

## 2.3 Models of energy units

DESS comprise independent and dynamically decoupled energy units. All these units cooperate to achieve one common goal which is to satisfy customers' consumption of electricity. Furthermore, the dynamic of one unit does not affect the dynamics of the other energy systems. The energy units are multiple and they are characterised by different time scales.

### 2.3.1 Power plant

Thermal power plants can be modelled as a third order system [49, 50]

$$Z_p(s) = G_p(s)(U_p(s) + D_p(s)) \quad G_p(s) = \frac{1}{(\tau_p s + 1)^3} \quad (2.1)$$

where  $z_p(t)$  denotes the power produced by the power plant  $p$ ,  $u_p(t)$  is the corresponding reference signal and  $d_p(t)$  represents the non-controllable input

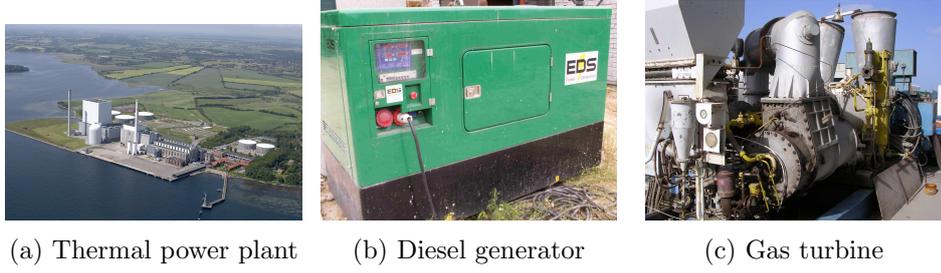


Figure 2.3: Power plant models.

as a disturbance. The cost of operating the unit  $p$  at time  $t$  is  $c_p(t)$ . By varying the time constant value  $\tau_p$ , the model (2.1) designs three kinds of power units: central thermal power plants, diesel generators and gas turbines in Figure 2.3. Furthermore, the discrete reference signal,  $u_{p,k}$ , is subject to input and rate of movement constraints (2.2) that are hard constraints

$$u_{min,k,p} \leq u_{k,p} \leq u_{max,k,p} \quad (2.2a)$$

$$\Delta u_{min,k,p} \leq \Delta u_{k,p} \leq \Delta u_{max,k,p} \quad (2.2b)$$

### 2.3.2 Wind farm

A wind farm consists of multiple wind turbines  $w$  and it produces the power  $z_w(t)$  based only on the wind speed  $d_w(t)$  and the reference signal  $u_w(t)$  [49, 50]

$$Z_w(s) = H_w(s)(U_w(s) + D_w(s)) \quad (2.3a)$$

$$H_w(s) = \frac{1}{\tau_w s + 1} \quad (2.3b)$$

$z_w(t)$  is the produced power. The manipulated variable,  $u_w(t)$ , is subject to the following hard constraints

$$-d_w(t) \leq u_w(t) \leq 0 \quad (2.4)$$

$$\Delta u_{min,w} \leq \Delta u_w \leq \Delta u_{max,w} \quad (2.5)$$

Likewise, the power produced is subject to soft constraints related to its mechanics and grid specifications

$$0 \leq z_w(t) \leq z_{max,w} \quad (2.6)$$

$$\Delta z_{min,w} \leq \Delta z_w \leq \Delta z_{max,w} \quad (2.7)$$

### 2.3.3 Refrigeration system

Cooling systems are attracting much attention as they can be used to balance power consumption and production due to their inner cooling capacity. The food temperature can vary within an interval and this may be used in balancing supply and demand of power in electrical systems Figure 2.4 shows the three main types of cold room modelled: milk cooler, vertical display and frost room [51–56]. Additionally, one unique model fits all these three types and parameters have different values.

The energy balance for a cold room in a refrigeration system is

$$m_{food}c_{p,food}\dot{T}_{food} = Q_{food-air} \quad (2.8)$$

$m_{food}$  is the mass of food stored in the cold room,  $c_{p,food}$  denotes the constant specific heat capacity of the foods and  $T_{food}$  is the temperature of goods.  $Q_{food-air}$  represents the energy flows from the air in the cold room to the food stored. The cold room dynamics include also

$$m_{air}c_{p,air}\dot{T}_{air} = Q_{load} - Q_{food-air} - Q_e \quad (2.9)$$

where  $m_{air}$  is the mass of air in the cold room and  $c_{p,air}$  is the constant specific heat capacity of air.  $T_{air}$  is the temperature of the air in the cold room,  $Q_{load}$  represents the heat transfer from the surrounding to the air, and  $Q_e$  is the energy absorbed in the evaporator.

The heat transfer from the food to the air and from the cold room to the supermarket is described by

$$Q_{food-air} = k_{food-air}(T_{air} - T_{food}) \quad (2.10a)$$

$$Q_{load} = k_{amb-cr}(T_{amb} - T_{air}) + Q_{dist} \quad (2.10b)$$

$$Q_e = K_{evap}(T_{air} - T_e) \quad (2.10c)$$

$Q_{dist}$  is the injection of heat into the cold room and it can be considered as a disturbance to the load.  $k_{food-air}$  and  $k_{amb-cr}$  represent the constant overall heat transfer coefficient between two media, while  $k_{evap}$  is the heat transfer coefficient of the evaporator. The work done in the compressor is  $W_c$  and this represents the power consumption in the refrigeration system

$$W_c = \eta Q_e \quad (2.11)$$

where  $\eta$  is the coefficient of performance.

This model has constraints related to the refrigerant used (2.12a), the energy consumption of the compressor (2.12b) and the temperature of food stored in the cold room (2.12c)

$$0 \leq Q_e \leq k_{evap,max}(T_{air} - T_e) \quad (2.12a)$$

$$0 \leq W_c \leq W_{c,max} \quad (2.12b)$$

$$T_{food,min} \leq T_{food} \leq T_{food,max} \quad (2.12c)$$

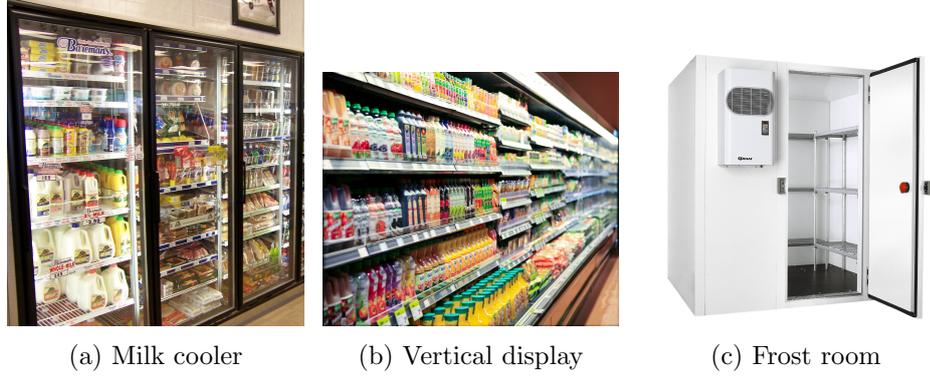


Figure 2.4: Refrigeration systems.

where  $k_{evap,max}$  is the constant overall heat transfer coefficient from the refrigerant to the air when the evaporator is completely full and  $W_{c,max}$  represents the constant limit on the energy consumption of the compressor.

Supermarkets usually comprise multiple cold rooms with one common compressor rack and one condensing unit. Figure 2.5 shows a refrigeration system with  $R$  cold rooms, one compressor rack and one common condenser unit. Accordingly, the overall power consumption is given by

$$W_c = \eta \sum_i^R Q_{e,i} \quad (2.13)$$

$W_c$  denotes the manipulated variable. Its contribution to the overall power balance is given by  $z(t) = -u(t) = -W_c(t)$ . To summarise, a refrigeration system output  $z_p(t) = -W_c$  absorbs power from the electrical grid that it is connected to.

### 2.3.4 Heat pump and floor heating system

Heat pumps supply heating to buildings. Their thermal capacities effect the power consumption in a flexible way. Figure 2.6 shows a house floor heating system connected to a ground source based heat pump. Energy balances and heat dynamics, both in the room and in the heat pump, yield to the model [18]

$$C_{p,r} \dot{T}_r = (UA)_{fr}(T_f - T_r) - (UA)_{ra}(T_r - T_a) + \phi_s \quad (2.14a)$$

$$C_{p,f} \dot{T}_f = (UA)_{wf}(T_w - T_f) - (UA)_{fr}(T_f - T_r) \quad (2.14b)$$

$$C_{p,w} \dot{T}_w = \eta W_c - (UA)_{wf}(T_w - T_f) \quad (2.14c)$$

where  $C_p$  defines the heat capacity,  $T_r$  is the temperature of the air in the room,  $T_f$  denotes the floor temperature.  $T_w$  is the temperature of the water



ing constraints [18]

$$0 \leq W_c \leq W_{c,max} \quad (2.15a)$$

$$T_{r,min} \leq T_r \leq T_{r,max} \quad (2.15b)$$

$$T_{f,min} \leq T_f \leq T_{f,max} \quad (2.15c)$$

$$T_{w,min} \leq T_w \leq T_{w,max} \quad (2.15d)$$

The interval (2.15b) for the room temperature  $T_r$  has time varying set-points because consumers can define their desirable temperature.

The manipulated variable is  $u(t) = W_c(t)$ .

### 2.3.5 Smart solar tank

The combination of solar heated roof-top collectors and storage tank provides a heating building system. A smart solar tank system consists of solar heated roof-top collectors and electrical heating in combination with a storage tank. Figure 2.7 depicts the heat storage tank.

The heat dynamics of the smart solar tank can be described as a simple first order differential equation [57]

$$C_t \dot{T}_t = Q_e + Q_s - Q_c - UA(T_t - T_a) \quad (2.16)$$

where  $C_t$  represents the specific heat capacity of the tank and  $Q_e$  is the heat provided to the tank by conversion of power  $W_c$  with efficiency  $\eta$ .  $Q_s$  is the energy contribution from the solar collector and  $Q_c$  denotes the house consumption.

The ambient temperature is  $T_a$  while  $T_t$  denotes the tank temperature. The system consisting of a smart solar tank is subject to the following constraints

$$0 \leq W_c \leq W_{c,max} \quad (2.17a)$$

$$T_{t,min} \leq T_t \leq T_{t,max} \quad (2.17b)$$

$$0 \leq Q_e \leq Q_{e,max} \quad (2.17c)$$

where the constraint (2.17a) refers to the power necessary to heat the tank; the tank temperature is subject to the soft constraint (2.17b). The heat transfer from the tank to the heating pipes is subject to the hard constraint (2.17c).

The manipulated variable for the heat tank system is  $u(t) = W_c(t)$ . Accordingly, the contribution of this system to the overall power balance is  $z(t) = -u(t) = -W_c$ .

### 2.3.6 Electric Vehicle (EV)

Electric vehicles can be seen as flexible storage resources capable of exchanging power with the electrical grid. Figure 2.8 shows an electric vehicle that is



Figure 2.7: Heat storage tank in a smart solar tank system connected to solar thermal collector on building roof.

charging in a dedicated parking area. Its battery is modelled by considering the State Of Charge (SOC)  $\zeta \in [0; 1]$  as a state variable [58, 59]

$$\dot{\zeta} = \frac{1}{Q_n}(\eta^+ P_c^+ - \frac{1}{\eta^-} P_c^-) \quad (2.18)$$

$P_c^+$  represents the power flowing into the battery during the charging while  $P_c^-$  is the power transferred during the discharging. Accordingly,  $\eta^+$  and  $\eta^-$  are the efficiency of charging and discharging. Usually,  $\eta^+ \leq \eta^-$  and the vehicle can exchange power to the grid only when it is plugged in.

The SOC of the battery is subject to soft constraints (2.19c), similarly, the power transferred in and out of the battery is limited by hard constraints (2.19a)-(2.19b)

$$0 \leq u^+(t) \leq P_{c,max}^+ \quad (2.19a)$$

$$0 \leq u^-(t) \leq P_{c,max}^- \quad (2.19b)$$

$$\eta_{min} \leq \eta \leq \eta_{max} \quad (2.19c)$$



Figure 2.8: Electric car charging stations.

The manipulated variables for the battery are the charging and discharging power,  $u = [u^+; u^-]$ . As a result, the contribution of the battery operation to the power balance is  $z(t) = [-1 \quad 1]u(t) = -u^+(t) + u^-(t)$ .

## 2.4 Power systems

The current power systems present inefficiencies. For instance, about 8% of the electrical power produced is lost during transmission. Moreover, the current electrical grid has a hierarchical structure, which can bring cascading failures and faults [42]. An example of the domino-effects is the Italian and Swiss blackout in 2003. In this case, a failure in the Swiss distribution system propagated and, ultimately, caused the separation of the Italian system from the rest of the European grid [60].

We consider a power system as a variety of power producers and consumers connect to the same electrical grid.

In the view of assessing power systems performance, we consider four main criteria [61]. The first concerns reliability because power systems operation is supposed to be ensured with few interruptions over a long time period and the electric service provided is on a nearly continuous basis. The second criterion is about minimising production costs, i.e. economy. Quality represents the third criterion as the differences of current and voltage waveform from the regular sinusoidal shape define the quality of the power supplied. Lastly, the power systems operation should be eco-friendly reducing pollution, noise and radiation.

### 2.4.1 Structure of power systems

Typically, the structure of the power systems is characterised by two groups of components: primary equipment and secondary equipment. Primary equipment is responsible for transferring energy. In comparison, the secondary

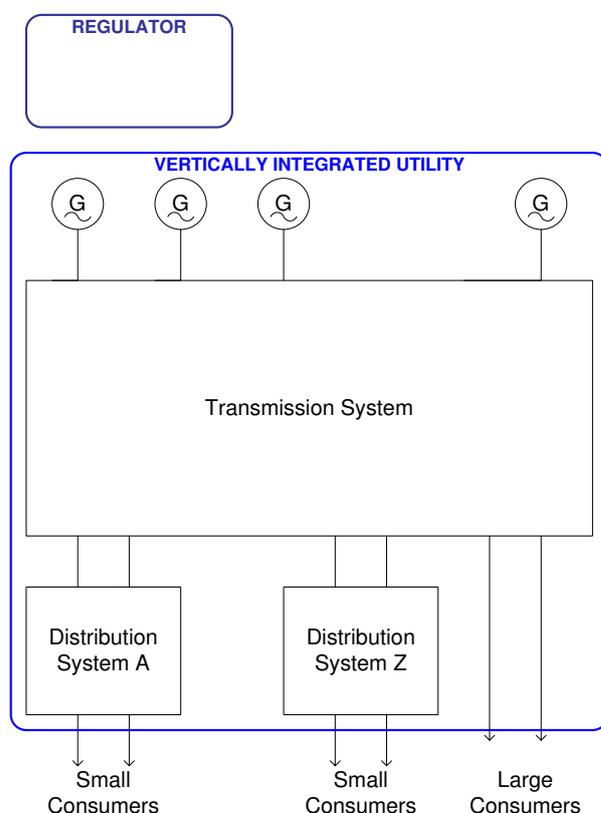


Figure 2.9: Power system seen as a vertical integrated utility [61].

equipment consists of those components and systems for monitoring, protection and control [61].

Figure 2.9 illustrates the organisational structure of power systems, which has been traditionally seen as a vertical integrated utility which includes production, transmission and distribution. The company that administers the entire power system determines the final price of the electricity and it is approved by the regulator. Usually, the regulator is an entity controlled by the government.

Figure 2.10 illustrates the actual organisational structure of power systems where generation, transmission and distribution are three different units. In this structure the regulator has to guarantee the balance in the electricity market.

Traders represent participants in a power trading or providers of financial services. Usually, traders have the power to increase competition in the electricity market and they might be subject to large economical risks [62]. The split in the organisational structure of power systems has implied the need of multiple suppliers in the view of introducing competition [61].

The generation unit consists of GENCO (GENERATION COMPANY) [61]. In

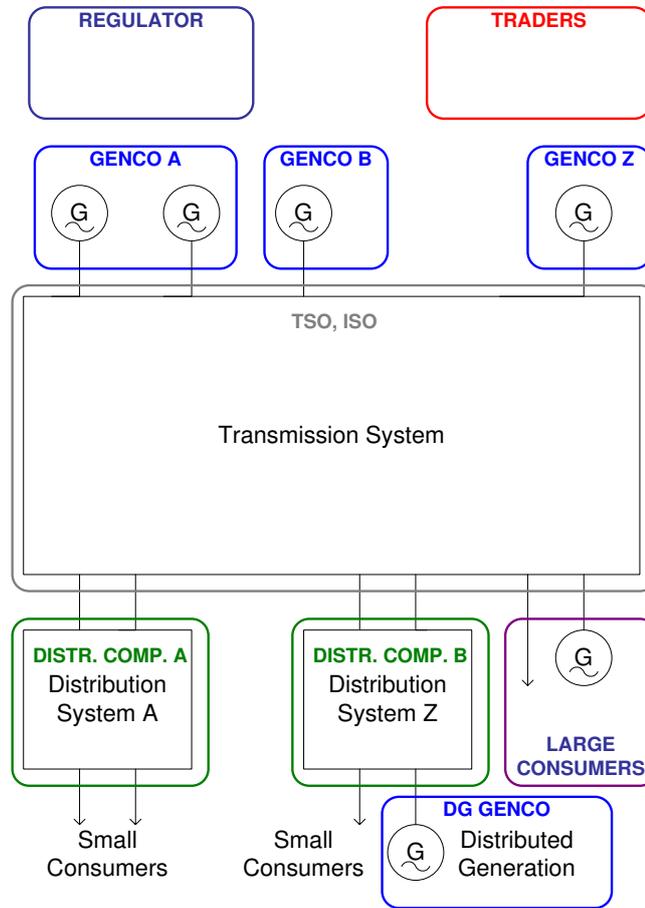


Figure 2.10: Recent organisational structure of power systems [61].

this part of the structure, conventional producers like power plants and wind turbines are connected through a high voltage transmission system [59].

In the distribution part, low voltage distribution grids transport the power produced to the consumers. Distribution companies ensure the supply of electricity to the consumers through contracts [59]. The distribution companies buy electricity either at a power exchange market or from a producer or from a third party trader [59]. In this thesis, we assume that we have a forecast of the electricity price available.

In addition, the transmission unit consists of the TSO (Transmission System Operator) or the ISO (Independent System Operator). This unit operates transmission assets and it can have different names as the electricity markets might have different layouts [61]. The TSO owns the high-voltage grid and it is responsible for operating and guaranteeing the supply of electricity in the country. Due to its role, the TSO must be a non-commercial organisation,

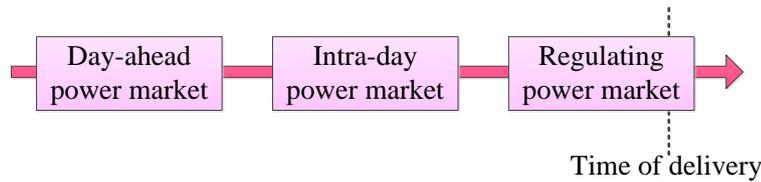


Figure 2.11: Electricity markets and time scales [59].

neutral and independent of commercial players.

## 2.4.2 Electricity market

In the organisational structure of power systems, the electricity market involves transmission, distribution, retail activities and generation [59]. Such a structure of power systems requires new electricity markets. The grid must balance continuously power consumption and production and this involves fast dynamics in real-time. As a result, electricity market involves trading performed in trading periods [63]. Figure 2.11 shows that each time scale of operation corresponds to a related market [59]. Daily transactions belong to a day-ahead market. Intra-day and regulating power markets are responsible for adjustments in energy needs [59]. The power can be sold or purchased in the view of balancing the system [63]. Additionally, the increasing share of renewable energy systems creates a new business of selling flexibility to the grid. Therefore, a suitable market strategy is an opportunity for making profit and reducing costs [56].

### 2.4.2.1 Regulating power

The standard frequency in Europe's electricity grid is 50 Hz and Figure 2.12 shows the regulating power services to balance the power grid in West Denmark [56, 59].

- Primary
 

This regulation is automatic and it applies to frequency deviations of between 20 and 200 mHz from the 50 Hz. Its time scale is about 15-30 seconds and the regulation lasts 15 minutes. This primary service has to be active until the secondary takes over.
- Secondary
 

The TSO activates the secondary service and then it releases the primary service. The secondary reserve has to restore any imbalances caused by operational disturbances. The time scale is around 15 minutes.
- Manual
 

This service restore power grid balance in the long term in order to

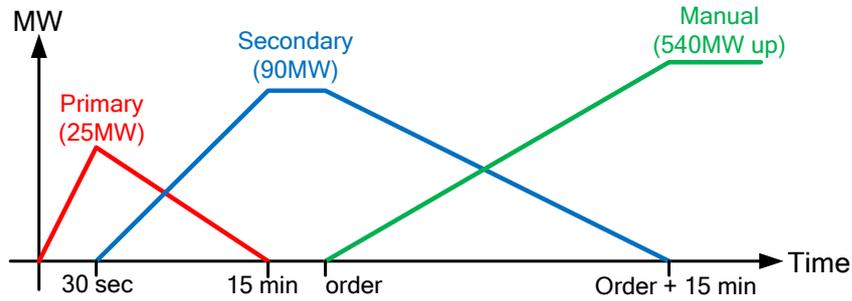


Figure 2.12: Power balance in West Denmark [56, 59].

release the primary and the secondary reserves. The TSO manually activates this service and orders suppliers to do upward and downward regulation. The manual, or tertiary, service requires 15 minutes for activation.

## 2.5 Control hierarchy for a power system

The current control hierarchy of the power systems can be seen in Figure 2.13. The lower the position of the block is in the hierarchy, the faster is the execution. Furthermore, the control framework can be divided in three main blocks:

- \* *Business planning.* This part of the control system has the longest time horizon that is from months to years. For example, this section of the controller includes production planning and maintenance scheduling.
- \* *Energy management.* In this block of the control system the time scale is from days to hours as it ensures the power supply on a daily and hourly basis. Energy management is responsible for planning the production processes in the most economical way, satisfying energy consumption. The day-ahead or hour-ahead predictions of the future demand for the markets in Figure (2.11) are processed by this unit.
- \* *Power management.* In this unit of the control system, there are two time horizons. Optimal Power Flow (OPF) works on the order of minutes. It optimises the operation of the electric power systems and the power transmission and distribution. In addition, voltage and frequency controls work on the order of seconds as they need to stabilise continuously the power in the electrical grid.

In this thesis, we address control strategies that work on hour-minute scale. Accordingly to Figure 2.13, such a controller is applicable to power and energy

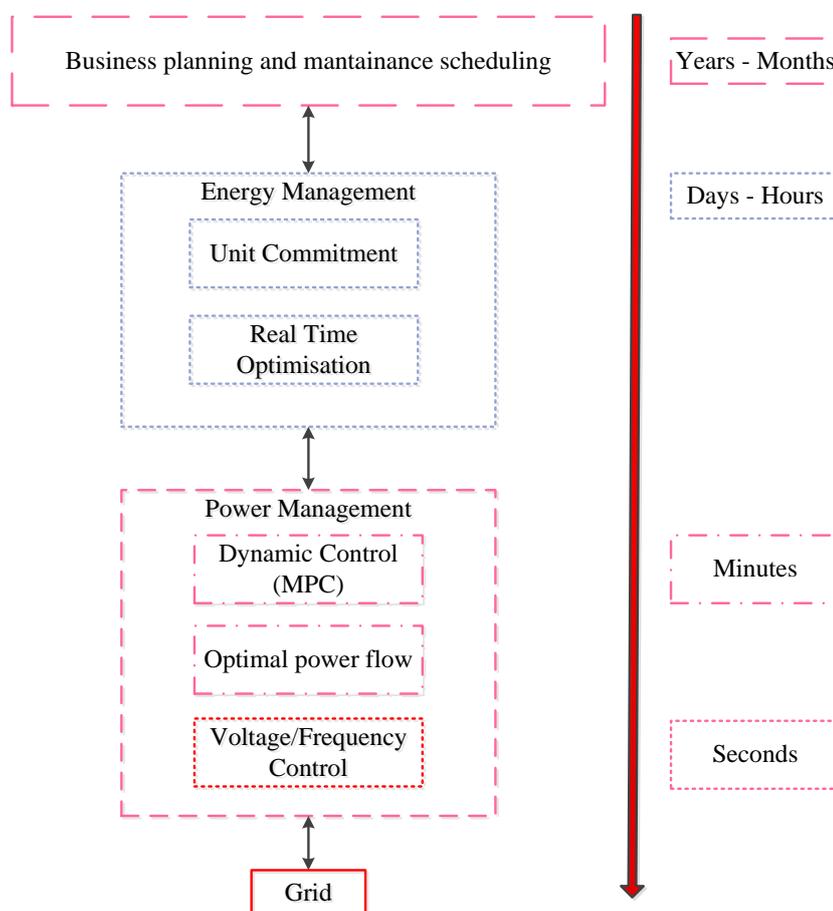


Figure 2.13: Control framework of the current power system.

scheduling. Moreover, in this work we assume sufficient capacity and disregard both frequency and voltage control. Figure 2.14 shows the control framework that we consider. Let us consider an electrical grid that comprises  $P$  energy units. The unit commitment provides a reference for the Model Predictive Control (MPC), which computes the optimal input signal for each energy unit in the grid. The measured portfolio power production is the feedback signal. Chapter 5 describes such a control framework.

## 2.6 Control for Smart Grids

Future energy systems require dedicated control systems in the view of enhancing the potential of the smart grid technology. Smart grids require innovative control systems characterised by the following features:

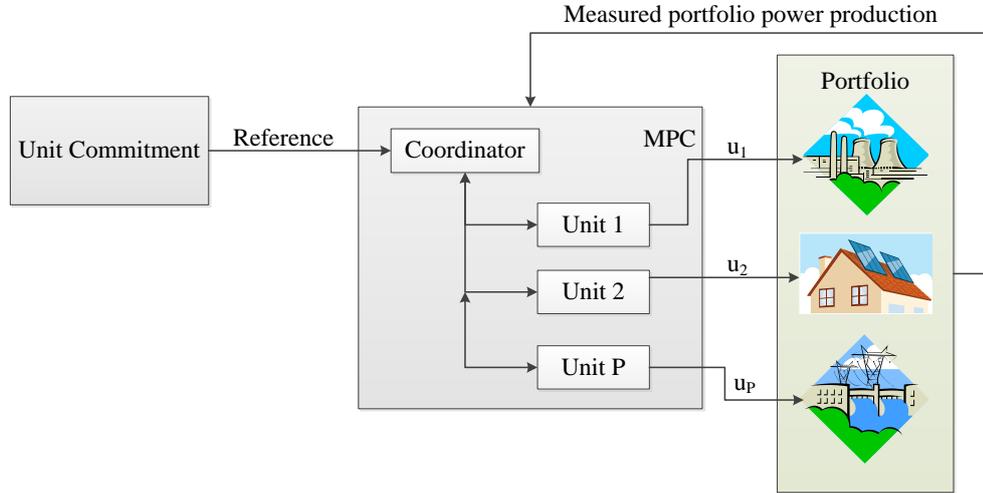


Figure 2.14: Control framework considered in this thesis.

### Scalability

Smart grids require control systems that are able to coordinate and control power generators of various types, such as wind farms, thermal power plants, heat pumps, solar tanks, EVs. All these units have different dynamics and the electricity grid must be able to efficiently include these power sources. Therefore, the new controllers must be scalable in terms of computational complexity because they are supposed to coordinate and control multiple energy units [20].

### Balance production & consumption

The introduction of renewables into the grids brings a rise of prediction uncertainty and this yields to the need of regulating power reserves. One of the control task in a power system is to maintain the balance between the power produced by generators and the power consumed by loads, including network losses, at all-time instants [64].

### Satisfy quality criteria

Voltage magnitude, frequency and wave shape must be maintained within specified limits in order to guarantee reliability of the power system [64].

### Information and Communication Technology (ICT)

Future smart-grids require modern communication and information technologies to ensure that the IT infrastructure efficiently coordinates, monitors and controls the grid. Communication needs to be bilateral and nearly instantaneous among all devices of the grids. At the same time, this has to be secure and to ensure privacy, thus, this involves modern cybersecurity [65].

### 2.6.1 Control and Optimisation for Smart Grids

Recently, optimisation and control theory have attracted much attention in smart grid applications [66]. Residential-level energy management, smart metering, power markets, and portfolio management are problems that require control and optimisation theory in order to be solved efficiently [67].

In this thesis, we implement an optimisation based control strategy in order to efficiently balance power production and consumption.



# Economic Model Predictive Control

In this chapter, we discuss the Economic Model Predictive Control (EMPC) policy. We outline this control strategy as a special case of the Model Predictive Control (MPC) policy. Moreover, we introduce the general formulation of the Economic Model Predictive Control and we describe its stability.

## 3.1 Choice of Economic Model Predictive Control

In the previous sections we introduce energy systems and their mathematical models. In the synthesis of control structures for energy systems we need to consider that:

- \* Energy system models have constraints due to physical and mechanical limitations. In addition, the desired control performances define constraints for the system variables. Accordingly, the control systems are multi variables and they are hard, or impossible, to split into single-input-single-output control problems.
- \* Energy systems might require predictions and forecasts related, for instance, to electricity prices and weather, e.g. wind speed and sun radiation. Therefore, the control systems must incorporate these variables.

The Model Predictive Control (MPC) strategy overcomes these two issues. MPC policy applies to multi variable constrained optimal control problems and it can easily incorporate forecasts in the control formulation [56].

In addition, in the synthesis of control structures for energy systems, we need to define control objectives. One category of control objectives is related

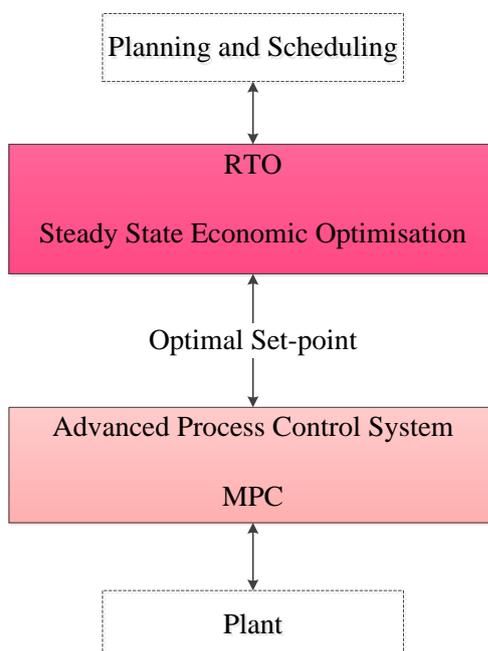


Figure 3.1: Two-layers structure framework for economic optimisation and control of industrial processes [12].

to operational feasibility [10]. In this case, the goal of the controller is to track certain set-points or trajectories [56]. The second category of control objectives are economic [10]. The standard practice is to deploy a hierarchical control structure in the view of achieving overall economic objectives [11]. Typically, this structure consists of two layers [12] and Figure 3.1 illustrates such a control framework.

#### **First layer: RTO**

The first layer consists of Real-Time Optimisation (RTO). It performs a steady-state economic optimisation of the plant's variables. Its timescale is relatively long, typically, hours or days. This optimisation is static as it determines the set-points with the minimal costs among all feasible steady-state plant operating conditions [68]. The RTO computes the optimal set-points and it sends the solution to the second layer.

#### **Second layer: Process control**

This layer computes the desired dynamic control actions. These actions guide the plant's operation to the desired set-points computed by the RTO in an optimal way. The Model Predictive Control (MPC) is located in this layer where this controller accounts for process constraints and coupling of variables. Furthermore, the MPC needs measurements for

the predicted deviation of the error of the states and the inputs from the corresponding steady-state values sent by the RTO.

In such a two-layers structure, offline economic objectives lead to an optimal trajectory, which correspond to the tracking controller's goal.

The Economic MPC (EMPC) formulation merges this hierarchical structure into only one layer. The EMPC performs a dynamic economic optimisation of the process performances directly, without reference to any steady-state [12]. Consequently, the controller forces the process to the economical optimal steady-state. Moreover, such a control system can include stochastic elements of the system due to its closed-loop optimisation [56].

## 3.2 Economic MPC formulation

The EMPC is a special case of MPC, which is the receding horizon technique that Figure 3.2 illustrates [11]. The Economic MPC embodies economic objectives into the control system [12, 69, 70].

In general, we consider the discrete system for  $k = 0, \dots, N - 1$

$$x_{k+1} = f(x_k, u_k, d_k) \quad (3.1)$$

where  $x_k$  denotes the dynamical state of the system,  $u_k$  the manipulated variable, and  $d_k$  is a predictable disturbance. Moreover, the system is subject to a set of constraints

$$h(x_k, u_k, d_k) \leq 0 \quad (3.2)$$

The system dynamics  $f$  and the set of constraints  $h$  might be either linear or non-linear. Furthermore, the EMPC presents an economic stage cost function that is different from the function  $\phi_{T,k}$  in MPC tracking problems

$$\phi_{T,k} = \frac{1}{2} \sum_{k=0}^{N-1} \|x_k - x_{sp}\|_Q^2 + \|u_k - u_{sp}\|_R^2 \quad (3.3)$$

where  $(x_{sp}, u_{sp})$  denote the set-points and  $Q$  and  $R$  are constant weights. In particular,  $R$  defines the regularisation term that penalises the control action. In the EMPC formulation, the cost function is economic and it can be either linear or non-linear.

In this work, we consider a linear economic stage cost function; additionally, both functions  $f$  and  $h$  are linear. Because of this, our formulation of the EMPC problem is linear.

### 3. ECONOMIC MODEL PREDICTIVE CONTROL

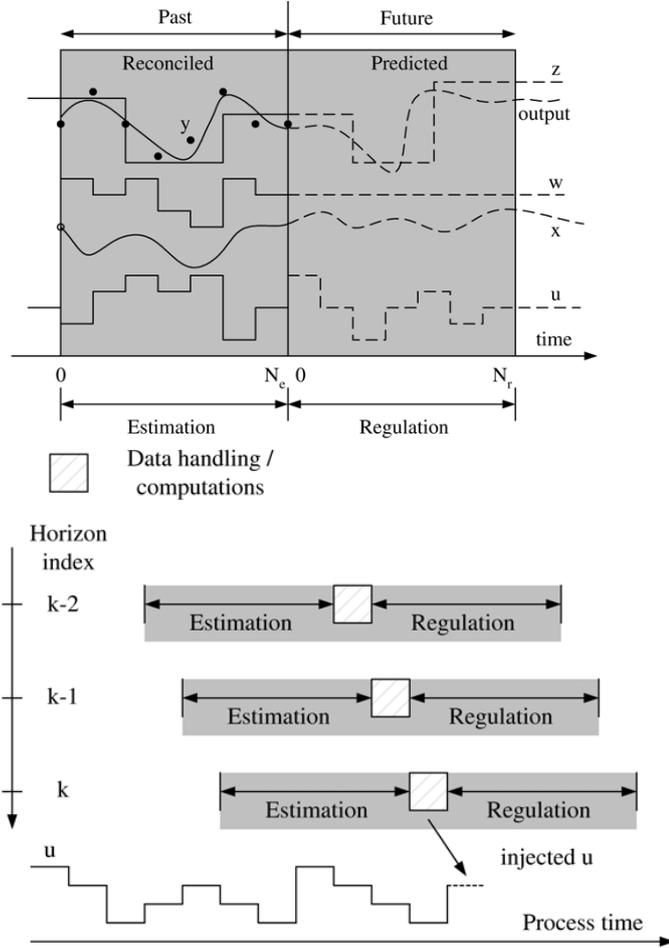


Figure 3.2: Receding horizon technique.

#### 3.2.1 Linear EMPC formulation

Let us consider a linear discrete system

$$x_{k+1} = Ax_k + Bu_k + Ed_k \quad (3.4a)$$

$$y_k = Cx_k \quad (3.4b)$$

$x_k$  denotes the state variable and  $d_k$  represents a process noise given by external forecasting systems. The output,  $y_k$ , and the manipulated variable,  $u_k$ , are subject to the following constraints due to the dynamics of the system and the control performances

$$y_k^{min} \leq y_k \leq y_k^{max} \quad (3.5a)$$

$$u_k^{min} \leq u_k \leq u_k^{max} \quad (3.5b)$$

$$\Delta u_k^{min} \leq \Delta u_k \leq \Delta u_k^{max}, \Delta u_k = u_{k-1} - u_k \quad (3.5c)$$

The Economic MPC performs a direct, dynamic and economic optimisation of the process performances. Its formulation for the discrete and constrained system (3.4)-(3.5) is

$$\min_{\{u_k, x_k\}_{k=0}^{N-1}} \phi = \sum_{k=0}^{N-1} c_k^T u_k \quad (3.6a)$$

$$s.t. \quad x_{k+1} = Ax_k + Bu_k + Ed_k \quad (3.6b)$$

$$y_k = Cx_k \quad (3.6c)$$

$$y_k^{min} \leq y_k \leq y_k^{max} \quad (3.6d)$$

$$u_k^{min} \leq u_k \leq u_k^{max} \quad (3.6e)$$

$$\Delta u_k^{min} \leq \Delta u_k \leq \Delta u_k^{max} \quad (3.6f)$$

The linear objective function (3.6a) includes the cost  $c_k$  for operating the process following the optimal trajectory  $u_k$ . The constraints (3.6b)-(3.6c) represent the state-space representation of the system. In addition, the system variables have to satisfy the constraints (3.5) that correspond to the constraint (3.6d)-(3.6f).

In this thesis, we add the slack variables,  $s_k$ , to the EMPC formulation (3.6). Accordingly, the objective function is

$$\phi = \sum_{k=0}^{N-1} c_k^T u_k + \rho_k^T s_k \quad (3.7)$$

$\rho_k$  is the penalty to pay every time the slack variable  $s_k$  is non-zero. Usually, in energy system applications the manipulated variable,  $u_k$ , is subject to hard constraints while the constraints for the output variable,  $y_k$ , (3.6d) can include the slack variable

$$y_k^{min} - s_k \leq y_k \leq y_k^{max} + s_k \quad (3.8)$$

$$s_k \geq 0 \quad (3.9)$$

### 3.3 Stability

EMPC formulation usually includes terminal conditions, i.e. terminal constraints and terminal costs in order to ensure stability and average performances [11, 12, 68, 71]. At each sampling time, the receding horizon problem solves the finite horizon optimal control problem that implements the optimal equilibrium of the infinite horizon problem as a terminal constraint. It is worth noticing that adding terminal conditions to the control problem causes disadvantages as it limits the operating regions [72]. Similarly, the Lyapunov techniques apply to the EMPC in order to prove steady-state stability and the performances of such a controller [68, 71].

### 3. ECONOMIC MODEL PREDICTIVE CONTROL

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EMPC formulation, without terminal conditions, reduces the computation time because it does not require the optimal solution for an initial analysis. In addition, the feasible regions increase and this yields to a larger operating region of the controller. However, removing terminal conditions causes the need of stronger assumptions on the finite horizon problem in order to assure asymptotic stability and transient optimality for the infinite horizon [72, 73]. In this work, we do not consider the steady-states or the target set-points because we aim at minimising production costs. Moreover, we implement long prediction and control horizon in order to overcome stability issues in our formulation [56].

# Dantzig-Wolfe decomposition for Linear Programs

The central theme of this chapter is the Dantzig-Wolfe decomposition technique. We introduce the block-angular Linear Programming problems (LPs) and we consider these particular LPs to describe the Dantzig-Wolfe decomposition. Finally, we outline how to initialise the proposed Dantzig-Wolfe algorithm.

## 4.1 Decomposition techniques

Nowadays many engineering applications define optimisation problems that involve a high number of variables and constraints. Decomposition techniques efficiently solve these problems, especially if they have appropriate structures. In practice, two specific structures can be often seen in real-world problems: complicating constraints and complicating variables. In both of these two structures either a block of constraints or variables complicates and prevents a straightforward solution of the problem [23].

George Dantzig and Philip Wolfe introduced this decomposition technique in 1960-1961 for those systems with complicating constraints [74]-[75]. This structure drastically complicates the solution of the problem and prevents its solution by blocks.

## 4.2 Block-angular LPs

A block-angular system is a Linear Programming problem (LP) characterised by two types of constraints: coupling and decoupling constraints. Thus, this

class of LPs presents complicating constraints that correspond to the above-mentioned coupling constraints.

Let us consider the block-angular LP (4.1), where  $i \in \mathcal{M} = \{1, 2, \dots, M\}$

$$\min_{x_i} \phi = (c_1)^T x_1 + (c_2)^T x_2 + \dots + (c_M)^T x_M \quad (4.1a)$$

$$s.t. \quad G_1 x_1 + G_2 x_2 + \dots + G_M x_M = g \quad (4.1b)$$

$$F_1 x_1 = f_1 \quad (4.1c)$$

$$F_2 x_2 = f_2 \quad (4.1d)$$

$$\ddots = \vdots \quad (4.1e)$$

$$F_M x_M = f_M \quad (4.1f)$$

$$x_i \geq 0 \quad (4.1g)$$

$x_i \in \mathbb{R}^n$  denotes the optimisation variable and  $c_i \in \mathbb{R}^n$  represents the objective cost coefficient.  $G_i \in \mathbb{R}^{n_g \times n}$  and  $g \in \mathbb{R}^{n_g}$  define the set of coupling or complicating constraints (4.1b). In the same way,  $F_i \in \mathbb{R}^{n_f \times n}$  and  $f \in \mathbb{R}^{n_f}$  denote the set of decoupling constraints (4.1c)-(4.1f).

Potential applications of the block-angular systems are in those areas where multiple units are independent and, at the same time, these units cooperate in achieving a common goal [76].

### 4.3 Dantzig-Wolfe Transformation

The Dantzig-Wolfe transformation represents the core of the Dantzig-Wolfe decomposition technique.

**Theorem 4.3.1 (Dantzig-Wolfe transformation).** *Let us consider the block-angular LP (4.1). Given a convex polyhedral set in  $\mathbb{R}^n$*

$$\mathcal{X}_i = \{x_i | F_i x_i = f_i\}, \quad i \in \mathcal{M} = \{1, \dots, M\} \quad (4.2)$$

*every point  $x_i$  can be expressed as a convex linear combination of the  $V$  extreme points  $v_i^j$  and a non-negative linear combination of the  $Z$  extreme rays  $z_i^k$*

$$x_i = \sum_{j=1}^V \alpha_i^j v_i^j + \sum_{k=1}^Z \beta_i^k z_i^k, \quad \sum_{j=1}^V \alpha_i^j = 1 \quad (4.3)$$

*with  $\alpha_i^j, \beta_i^k \geq 0$  and  $i \in \mathcal{M}$ .*

*Proof.* See [76]. □

The extreme points  $v_i^j$  and the extreme rays  $z_i^k$  in Theorem 4.3.1 represent, respectively, the basic feasible solutions and the normalised extreme homogeneous solutions of the block-angular problem (4.1).

The Dantzig-Wolfe transformation provides an equivalent formulation of the optimisation variable  $x_i$  (4.3) and this forms the basis of the Dantzig-Wolfe decomposition technique.

#### 4.4 Master Problem (MP)

Substituting the formulation of the optimisation variable  $x_i$  (4.3) into the block-angular system (4.1) yields to a LP named Master Problem (MP) or full extremal problem (4.4) [76]

$$\min_{\alpha_i^j, \beta_i^k} \quad \phi = \sum_{i=1}^M \left[ \sum_{j=1}^V p_i^j \alpha_i^j + \sum_{k=1}^Z q_i^k \beta_i^k \right] \quad (4.4a)$$

$$\text{s.t.} \quad \sum_{i=1}^M \left[ \sum_{j=1}^V h_i^j \alpha_i^j + \sum_{k=1}^Z r_i^k \beta_i^k \right] = g \quad (4.4b)$$

$$\sum_{j=1}^V \alpha_i^j = 1 \quad i = 1, \dots, M \quad (4.4c)$$

$$\alpha_i^j \geq 0 \quad i = 1, \dots, M, j = 1, \dots, V \quad (4.4d)$$

$$\beta_i^k \geq 0 \quad i = 1, \dots, M, k = 1, \dots, Z \quad (4.4e)$$

Where the coefficients  $p_i^j$ ,  $q_i^k$  and  $r_i^k$  are defined as

$$p_i^j = c_i v_i^j \quad h_i^j = F_i v_i^j \quad i = 1, \dots, M, j = 1, \dots, V \quad (4.5a)$$

$$q_i^k = c_i z_i^k \quad r_i^k = F_i z_i^k \quad i = 1, \dots, M, k = 1, \dots, Z \quad (4.5b)$$

The master problem (4.4) is equivalent to the original block-angular problem (4.1). Moreover, the MP has fewer number of constraints but it includes more variables due to the extreme points  $v_i^j$  and rays  $z_i^k$  of each polyhedron  $\mathcal{X}_i$  (4.2), with  $i \in \mathcal{M}$ .

#### 4.5 Reduced Master Problem (RMP)

In practice, the full MP is never implemented because solving it usually requires high computational time due to the large number of optimisation variables. Consequently, the Dantzig-Wolfe decomposition introduces a Reduced Master Problem (RMP) that includes a reduced number of variables.

A subset  $\bar{V}$  and  $\bar{Z}$  of the, respectively,  $V$  and  $Z$  columns associated to the decision variables  $\alpha_i^j$  and  $\beta_i^k$  of the MP (4.4) designs a linear programming

problem named Reduced, or Restricted, Master Problem (RMP)

$$\min_{\alpha_i^j, \beta_i^k} \phi = \sum_{i=1}^M \left[ \sum_{j=1}^{\bar{V}} p_i^j \alpha_i^j + \sum_{k=1}^{\bar{Z}} q_i^k \beta_i^k \right] \quad (4.6a)$$

$$s.t. \quad \sum_{i=1}^M \left[ \sum_{j=1}^{\bar{V}} h_i^j \alpha_i^j + \sum_{k=1}^{\bar{Z}} r_i^k \beta_i^k \right] = g \quad (4.6b)$$

$$\sum_{j=1}^{\bar{V}} \alpha_i^j = 1 \quad i = 1, \dots, M \quad (4.6c)$$

$$\alpha_i^j \geq 0 \quad i = 1, \dots, M, j = 1, \dots, \bar{V} \quad (4.6d)$$

$$\beta_i^k \geq 0 \quad i = 1, \dots, M, k = 1, \dots, \bar{Z} \quad (4.6e)$$

The coefficients of the RMP  $p_i^j$ ,  $q_i^k$ ,  $h_i^j$  and  $r_i^k$  are obtained as expressed in (4.5) with  $j = 1, \dots, \bar{V}$  and  $k = 1, \dots, \bar{Z}$ .

Simplex multipliers  $\pi$  obtained from the constraints (4.6b) and  $\gamma_i$  from the convexity constraints (4.6c) are crucial to the Dantzig-Wolfe decomposition.

## 4.6 Subproblems

The Dantzig-Wolfe technique populates the two subsets  $\bar{V}$  and  $\bar{Z}$  of the RMP (4.6) at each iteration of the Simplex algorithm by adding just those columns of the MP (4.4) selected. The columns of the RMP correspond to a basic feasible solution of the full MP and the Dantzig-Wolfe algorithm includes into basis only those columns that have the most negative reduced cost.

Let us consider the simplex multipliers  $\pi$  and  $\gamma_i$ . By definition, these are the multiples of their set of constraints such that the simplex multipliers are multiplied by their respective equations and subtracted from the initial objective function [77]. Therefore, for the RMP (4.6) the simplex multipliers satisfy, for  $i = 1, \dots, M$

$$\gamma_i + (h_i^j)^T \pi = p_i^j \quad j = 1, \dots, \bar{V} \quad (4.7a)$$

$$(r_i^k)^T \pi = q_i^k \quad k = 1, \dots, \bar{Z} \quad (4.7b)$$

Assuming that  $j^*$  and  $k^*$  define the indices at which the minima is achieved with  $i \in \mathcal{M}$

$$p_i^{j^*} - \gamma_i^{j^*} - (h_i^{j^*})^T \pi = \min_{j=1, \dots, \bar{V}} \left\{ p_i^j - (h_i^j)^T \pi \right\} - \gamma_i^{j^*} \quad (4.8a)$$

$$q_i^{k^*} - (r_i^{k^*})^T \pi = \min_{k=1, \dots, \bar{Z}} \left\{ q_i^k - (r_i^k)^T \pi \right\} \quad (4.8b)$$

The optimal solution of (4.8) defines the indices  $j$  and  $k$  of the columns to introduce into basis for  $i = 1, \dots, M$ . Accordingly, let us reformulate (4.7a) and (4.8a) for the extreme points  $v_i^j$

$$\min_{j=1, \dots, \bar{V}} \left\{ p_i^j - (h_i^j)^T \pi \right\} - \gamma_i^j \quad (4.9a)$$

$$= \min_{j=1, \dots, \bar{V}} \left\{ (c_i)^T v_i^j - (F_i^T v_i^j)^T \pi \right\} - \gamma_i^j \quad (4.9b)$$

$$= \min_{j=1, \dots, \bar{V}} \left\{ (c_i - F_i^T \pi)^T v_i^j \right\} - \gamma_i^j \quad (4.9c)$$

$$= \min_{j=1, \dots, \bar{V}} \left\{ \rho_i^T v_i^j \right\} - \gamma_i^j \quad (4.9d)$$

Complementary to the optimisation problem in (4.9) we reformulate (4.7b) and (4.8b) for the extreme rays  $z_i^k$

$$\min_{k=1, \dots, \bar{Z}} \left\{ q_i^k - (r_i^k)^T \pi \right\} \quad (4.10a)$$

$$= \min_{k=1, \dots, \bar{Z}} \left\{ (c_i)^T z_i^k - (F_i^T z_i^k)^T \pi \right\} \quad (4.10b)$$

$$= \min_{k=1, \dots, \bar{Z}} \left\{ (c_i - F_i^T \pi)^T z_i^k \right\} \quad (4.10c)$$

$$= \min_{k=1, \dots, \bar{Z}} \left\{ \rho_i^T z_i^k \right\} \quad (4.10d)$$

In the optimisation problems (4.9) and (4.10)  $\rho_i$  represents the adjust costs with  $i \in \mathcal{M}$ . Let us focus on the vertices  $v_i^j$ . Solving the optimisation problem (4.9) involves all the  $\bar{V}$  extreme points in order to evaluate which prices out negative. Instead, the Dantzig-Wolfe decomposition selects only the extreme points that give the most negative reduced costs by solving the following optimisation problem at iteration  $j$

$$\psi_i = \min_{x_i} \rho_i^T x_i + \gamma_i \quad (4.11a)$$

$$s.t. \quad G_i x_i = g_i \quad (4.11b)$$

$$x_i \geq 0 \quad (4.11c)$$

It is evident that the subproblems (4.11) are independent and decoupled. Hence, these can be solved via parallel computing.

$\psi_i^{j*}$  and  $v_i^{j*}$  are the optimal objective function value and the basic feasible solution of the subproblem (4.11). If

$$\psi_i^{j*} - \gamma_i \geq 0 \quad \forall i \in \mathcal{M} \quad (4.12)$$

then all the reduced costs for the MP are non-negative. Hence, the Dantzig-Wolfe algorithm has an optimal solution to the MP and, consequently, to the original block-angular problem (4.1) through the convex combination (4.3).

If the optimality condition (4.12) is not satisfied then the Dantzig-Wolfe decomposition augments the columns of the RMP by

$$\begin{pmatrix} h_i^{j+1} \\ p_i^{j+1} \\ 1 \end{pmatrix} = \begin{pmatrix} F_i v_i^{j*} \\ c^T v_i^{j*} \\ 1 \end{pmatrix} \quad (4.13)$$

where  $v_i^{j*} = x_i^{j*}$  is the optimal basic feasible solution of (4.11) at iteration  $j$ . In comparison, let us assume that the optimisation problem (4.11) provides an extreme homogeneous solution  $x_i^{k*} = z_i^k$ . The Dantzig-Wolfe decomposition requires the best normalised extreme homogeneous solution, thus, the algorithm does not evaluate all the extreme rays that prices out negative in the optimisation problem (4.10). Instead, the best extreme homogeneous solution  $x_i^{k*} = z_i^{k*}$  is the solution of the problem (4.14)

$$\varphi_i = \min_{x_i} \rho_i^T x_i \quad (4.14a)$$

$$s.t. \quad G_i x_i = 0 \quad (4.14b)$$

$$e^T x = 1 \quad (4.14c)$$

$$x_i \geq 0 \quad (4.14d)$$

where  $e = (1, 1, \dots, 1)^T$  and  $\varphi_i^{k*}$  is the optimal objective function value associated to the normalised homogeneous solution at iteration  $k$ . In practice the optimisation problem (4.14) is never solved and the Dantzig-Wolfe algorithm accepts any homogeneous solution of the problem (4.11) without normalising it. Consequently, the columns of the RMP are augmented by

$$\begin{pmatrix} r_i^{k+1} \\ q_i^{k+1} \\ 0 \end{pmatrix} = \begin{pmatrix} F_i z_i^{k*} \\ c^T z_i^{k*} \\ 0 \end{pmatrix} \quad (4.15)$$

where  $x_i^{k*} = z_i^k$ . As a result, the RMP includes into basis new columns given by (4.13)-(4.15) and then it is re-optimised.

Figure 4.1 shows a schematic representation of the Dantzig-Wolfe decomposition algorithm.

## 4.7 Optimality

The Dantzig-Wolfe decomposition efficiently solves the block-angular LP (4.1) and computes the optimal solution that satisfies the following properties.

**Theorem 4.7.1** (Feasible and Optimal Solution). *Any  $\alpha_i^j$  and  $\beta_i^k$  solutions of the MP (4.4), determine an  $x$  as feasible solution to the block-angular problem (4.1) by convex combination (4.3). Likewise, if  $\phi^*$  is the minimum of MP (4.4) for  $\alpha_i^{j*}$  and  $\beta_i^{k*}$ , then (4.3) provides an optimal feasible solution  $x^*$  to the original LP (4.1).*

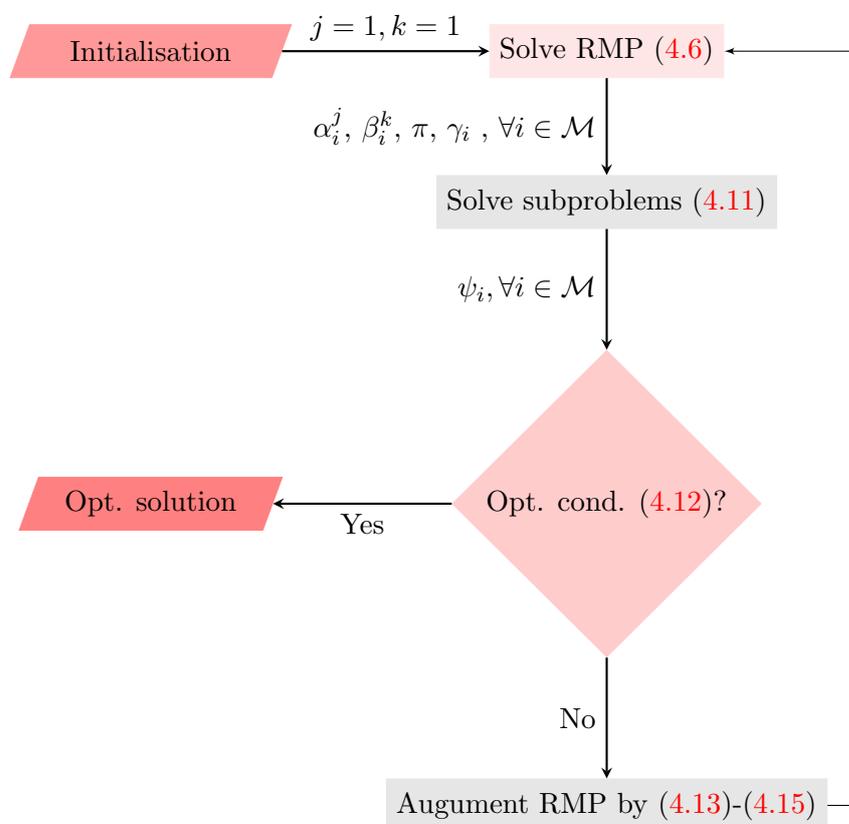


Figure 4.1: Dantzig-Wolfe algorithm described as a flowchart.

*Proof.* See [76]. □

**Theorem 4.7.2** (Optimality and Finiteness under Nondegeneracy). *An optimal basic feasible solution to the RMP (4.6) is also optimal for the MP (4.4) if*

$$\psi_i = \gamma_i \quad \forall i \in \mathcal{M} \quad (4.16)$$

*In case of nondegenerate RMP the algorithm computes such an optimum in a finite number of iterations.*

*Proof.* See [76]. □

**Corollary 1** (Optimality and Finiteness under degeneracy). *Theorem 4.7.2 holds if the RMP is degenerate in the case of some anti cycling scheme is used.*

## 4.8 Initialisation

The Reduced Master Problem (RMP) requires an initial feasible solution to solve the RMP coefficients. Commonly, the Phase I procedure computes an initial feasible solution by including the artificial variables  $\kappa_i, \forall i \in \mathcal{M}$  in the RMP

$$\min_{\alpha_i, \kappa_i} \sum_{i=1}^M \kappa_i \quad (4.17a)$$

$$s.t. \quad \sum_{i=1}^M F_i v_i \alpha_i \pm e_i \kappa_i = f_i \quad (4.17b)$$

$$\alpha_i = 1 \quad \forall i = 1, \dots, M \quad (4.17c)$$

$$\alpha_i \geq 0, \kappa_i \geq 0, \quad \forall i = 1, \dots, M \quad (4.17d)$$

where  $e_i$  is the column of an identity matrix with  $i \in \mathcal{M}$ . In addition, feasibility is guaranteed with

$$f_i - F_i \geq 0 \quad \Rightarrow \quad +e_i \quad (4.18a)$$

$$f_i - F_i < 0 \quad \Rightarrow \quad -e_i \quad (4.18b)$$

The variables  $\alpha_i, \kappa_i$  for  $i \in \mathcal{M}$  constitute a basic set of variables for the optimisation problem (4.17). If Phase I computes a feasible solution, then all nonbasic artificial variables are dropped and the basic artificial variables are set equal to 0. Alternatively, if no feasible solution is computed, then the problem is infeasible and the Dantzig-Wolfe cannot be applied to solve the block-angular problem (4.1) [76].

Clearly, this way of initialising the RMP is equivalent to solving a block-angular LP and this requires high computation times. In this thesis, we introduce an alternative initialisation strategy for the Dantzig-Wolfe decomposition when it is used in a MPC controller. The proposed strategy does not solve any block-angular LP in order to find an initial feasible solution for the RMP.

**Part II**

**Main Contributions**



# Linear Economic MPC for Power Plant Portfolio via Dantzig-Wolfe Decomposition

In this Chapter, we highlight the scientific contributions related to the implementation of the Dantzig-Wolfe decomposition algorithm to solve the Economic Model Predictive Control (EMPC) problem. Firstly, we illustrate a linear formulation of the EMPC strategy for power plant portfolio management. Secondly, we apply the Dantzig-Wolfe decomposition to such a linear control problem. Thirdly, we explore the warm-start strategy to initialise the Dantzig-Wolfe algorithm. Finally, we test the proposed controller in simulations and we consider a scenario that consists of multiple and dynamically decoupled power plants.

## 5.1 Scientific contribution

The work presented in Papers [B](#), [D](#), [G](#), [H](#), [I](#) and [K](#) investigates the Economic MPC controller to operate power units in large and distributed energy systems. In addition, in these papers, we implement the Dantzig-Wolfe decomposition to efficiently compute the optimal input signals for each power unit.

Power system operation includes energy and power management. The former works on a day-minute scale, the latter on a minute-second scale. The energy management includes the Unit Commitment problem (UC) that is usually defined as a Mixed Integer Linear Programming problem (MILP). In this thesis we propose a controller for power units that have already been committed, however, we reassign more precise and updated forecasts related to the system variables. The Economic MPC recomputes the power production plan more

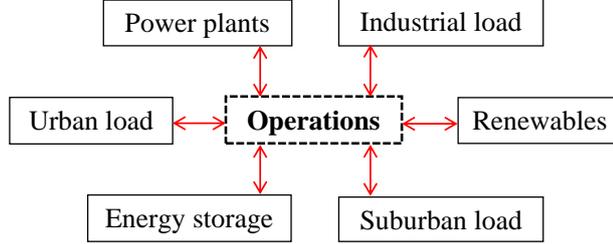


Figure 5.1: Future energy systems.

frequently than the UC and based on updated prognosis. In order to increase the applicability of such a controller, we formulate the EMPC problem as a LP and we solve it via the Dantzig-Wolfe decomposition. It is worth noticing that we address neither frequency nor voltage control.

## 5.2 Power plant portfolio management

In this work, we formulate the EMPC problem for a power plant portfolio. The EMPC provides the optimal power production set-point for each power plant. Moreover, we assume that the running power plants have already been committed by the Unit Commitment (UC) problem. The EMPC strategy minimises production costs and satisfies customers' demand of electricity.

### 5.2.1 Energy Units

Figure 5.1 shows an energy system where multiple independent power units are connected to one common operation centre. In this work, we consider such a scenario. These power units generate and consume electricity and they can include Renewable Energy Systems (RESs). The operation centre must coordinate and control all these power units in a way such that the production costs are minimised and the portfolio production of electricity satisfies customers' demand.

An energy unit is assumed to be represented by a linear stochastic discrete time state-space model

$$x_{k+1} = Ax_k + Bu_k + Ed_k \quad (5.1a)$$

$$y_k = Cx_k + v_k \quad (5.1b)$$

$$z_k = C_z x_k + D_z u_k + F_z d_k \quad (5.1c)$$

The initial state is distributed as  $x_0 \sim N(\hat{x}_{0|-1}, P_{0|-1})$ , the measurement noise is  $v_k \sim N_{iid}(0, R_{vv})$ , and  $d_k \sim N(\bar{d}_k, R_{dd,k})$  is predicted by an external

prognosis system [78]. In addition,  $x_k$  denotes the state,  $u_k$  the manipulated variable,  $y_k$  represents the measurement used for feedback, and  $z_k$  is the output variable. We assume that the process disturbance,  $d_k$ , can be predicted by some realization  $I_k^d$  of a stochastic information vector  $\mathcal{I}_k^d$ . We assume that the conditional variable has the distribution

$$d_{k+j|k} = (d_{k+j} | \mathcal{I}_k^d = I_k^d) \sim N(\hat{d}_{k+j|k}, R_{dd,k+j|k}) \quad (5.2)$$

In the future energy systems, the disturbance  $d_k$  might represent wind speed, outdoor temperature or sun radiation. Consequently, the forecast  $d_{k+j|k}$  is the result of a weather prognosis and the mean value is given by

$$\mathcal{D}_k = \{\hat{d}_{k+j|k}\}_{j=0}^{N-1} \quad (5.3)$$

The manipulated variable,  $u_k$ , is a stochastic variable. This implies that  $u_k : \Omega \mapsto \mathbb{R}^{n_u}$ , i.e.  $u_k = u_k(\omega)$  for  $\omega \in \Omega$  and  $(\Omega, \mathcal{G}, P)$  is an associated probability field [79]. The manipulated variable,  $u_k$ , is subject to the following hard constraints

$$u_k^{\min} \leq u_k \leq u_k^{\max} \quad (5.4a)$$

$$\Delta u_k^{\min} \leq \Delta u_k \leq \Delta u_k^{\max} \quad (5.4b)$$

The system output,  $z_k$ , must be within an interval  $[r_k^{\min}, r_k^{\max}]$  where  $r_k^{\min} \sim \mathbb{F}(\bar{r}_k^{\min}, R_{(r^{\min}, r^{\min}), k})$  and  $r_k^{\max} \sim \mathbb{F}(\bar{r}_k^{\max}, R_{(r^{\max}, r^{\max}), k})$  are stochastic variables stemming from some distribution. The forecasts,  $\mathcal{R}_k$ , of the interval  $[r_k^{\min}, r_k^{\max}]$  are available and used by the controller. Let

$$r_{k+j|k}^{\min} = (r_{k+j}^{\min} | \mathcal{I}_k^r = I_k^r) \sim \mathbb{F}(\hat{r}_{k+j|k}^{\min}, R_{(r^{\min}, r^{\min}), k+j|k}) \quad (5.5a)$$

$$r_{k+j|k}^{\max} = (r_{k+j}^{\max} | \mathcal{I}_k^r = I_k^r) \sim \mathbb{F}(\hat{r}_{k+j|k}^{\max}, R_{(r^{\max}, r^{\max}), k+j|k}) \quad (5.5b)$$

such that the mean of the forecast,  $\mathcal{R}_k$ , may be denoted as

$$\mathcal{R}_k = \{\hat{r}_{k+j|k}^{\min}, \hat{r}_{k+j|k}^{\max}\}_{j=1}^N \quad (5.6)$$

This interval is a reference zone and it may represent the electricity demand forecast, the indoor temperature in a building, the temperature in a refrigeration system or the state-of-charge of a battery. However, due to some disturbances or in some scenarios, it may be impossible to obtain and maintain  $z_k$  within the defined interval. Therefore, the constraints on the output variable include the slack variables,  $s_k$ , which represent violations of temperature limits, or violations of state-of-charge limits. Every time  $s_k$  is non-zero, a penalty cost, i.e. the cost of buying or selling power on the short-term market must be paid. Accordingly, the output variable,  $z_k$ , is subject to

$$r_k^{\min} - s_k \leq z_k \leq r_k^{\max} + s_k \quad (5.7a)$$

$$s_k \geq 0 \quad (5.7b)$$

## 5. LINEAR ECONOMIC MPC FOR POWER PLANT PORTFOLIO VIA DANTZIG-WOLFE DECOMPOSITION

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The average cost of operating the system in a period  $[0, \dots, N]$  is the stochastic variable

$$\psi = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^N c'_k u_k + \rho'_k s_k \quad (5.8)$$

with  $c_k \sim \mathbb{F}(\bar{c}_k, R_{cc,k})$  and  $\rho_k \sim \mathbb{F}(\bar{\rho}_k, R_{\rho\rho,k})$  being unit costs. The unit price forecasts are the conditional stochastic variables

$$c_{k+j|k} = (c_{k+j} | \mathcal{I}_k^c = I_k^c) \sim N(\hat{c}_{k+j|k}, R_{cc,k+j|k}) \quad (5.9)$$

$$\rho_{k+j|k} = (\rho_{k+j} | \mathcal{I}_k^\rho = I_k^\rho) \sim N(\hat{\rho}_{k+j|k}, R_{\rho\rho,k+j|k}) \quad (5.10)$$

The mean of the forecasts,  $\mathcal{F}_k$ , is denoted as

$$\mathcal{F}_k = \{\hat{c}_{k+j|k}, \hat{\rho}_{k+j|k}\}_{j=0}^N \quad (5.11)$$

### 5.2.2 Filtering, Prediction and Certainty Equivalence

We introduce a Kalman filter in order to estimate the state variable,  $x_k$ , of (5.1) [80–83]

$$\hat{x}_{k|k} = E\{x_k | \mathcal{Y}_k = Y_k\} \quad (5.12)$$

and the innovation,  $e_k$ , is given by

$$e_k = y_k - \hat{y}_{k|k-1} = y_k - C\hat{x}_{k|k-1} \quad (5.13)$$

The standard Kalman filter values, for instance, the innovation covariance,  $R_{e,k}$ , the filter gain,  $K_{fx,k}$ , and the filtered state covariance,  $P_{k|k}$ , are computed as

$$R_{e,k} = R_{vv} + CP_{k|k-1}C' \quad (5.14a)$$

$$K_{fx,k} = P_{k|k-1}C'R_{e,k}^{-1} \quad (5.14b)$$

$$P_{k|k} = P_{k|k-1} - K_{fx,k}R_{e,k}K'_{fx,k} \quad (5.14c)$$

while the filtered state can be computed by

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{fx,k}e_k \quad (5.15)$$

Given the conditional predictions of the external disturbance,  $\hat{d}_{k+j|k}$ , and the manipulated variables,  $\hat{u}_{k+j|k}$ , the conditional predictions of the states and the outputs are

$$\hat{x}_{k+1+j|k} = A\hat{x}_{k+j|k} + B\hat{u}_{k+j|k} + E\hat{d}_{k+j|k} \quad (5.16a)$$

$$\hat{y}_{k+j+1|k} = C\hat{x}_{k+1+j|k} \quad (5.16b)$$

$$\hat{z}_{k+j+1|k} = C_z\hat{x}_{k+1+j|k} + D_z\hat{u}_{k+j|k} + F_z\hat{d}_{k+j|k} \quad (5.16c)$$

for  $j = 0, 1, \dots, N - 1$  and all  $k \geq 0$ . The corresponding covariances of the predicted states are

$$P_{k+j+1|k} = AP_{k+j|k}A' + GR_{ww,k+j}G' + ER_{dd,k+j|k}E' \quad (5.17)$$

The Kalman filter minimises the errors from measurements noise, process noise and model mismatch [84].

### 5.2.3 Portfolio Energy Production

In this work, the scenario is a large and distributed energy system consisting of multiple power units, each of these is modelled as a stochastic linear state-space representation (5.1). The Economic MPC strategy aims to minimise production costs and satisfy customers' demand. For this purpose, the overall portfolio energy production needs to be included in the control problem formulation.

Let us consider an energy system consisting of  $M$  power units modelled as (5.1). Accordingly, the overall energy production is defined as

$$\tilde{y}_k = \sum_{i=1}^M C_i x_{i,k} + v_k \quad (5.18a)$$

$$\tilde{z}_k = \sum_{i=1}^M C_{z,i} x_{i,k} + D_{z,i} u_{i,k} \quad (5.18b)$$

(5.18) defines a set of coupling constraints connecting the produced and the consumed power from all  $M$  units indexed  $i$ .

Therefore, considering the output constraints (5.7), the portfolio energy production,  $\tilde{z}_k$ , must be within the interval

$$\hat{r}_{k+j+1|k}^{\min} - \hat{s}_{k+j+1|k} \leq \tilde{z}_{k+j+1|k} \leq \hat{r}_{k+j+1|k}^{\max} + \hat{s}_{k+j+1|k} \quad (5.19a)$$

$$\hat{s}_{k+j+1|k} \geq 0 \quad (5.19b)$$

where the bounds  $\hat{r}_{\min}$  and  $\hat{r}_{\max}$  are the mean value of the customers' demand forecasts  $\mathcal{R}_k$  given by external forecasting systems

$$\mathcal{R}_k = \{\hat{r}_{k+j+1|k}^{\min}, \hat{r}_{k+j+1|k}^{\max}\}_{j=1}^N \quad (5.20)$$

$\tilde{s}_{k+j+1|k}$  is the slack-variable for the overall power portfolio production.

Then, the cost of operating the power portfolio is

$$\phi_k = \sum_{i=1}^M \psi_{i,k} + \sum_{j=0}^{N-1} \hat{\rho}'_{k+j+1|k} \hat{s}_{k+j+1|k} \quad (5.21)$$

and

$$\psi_{i,k} = \sum_{j=0}^{N-1} (\hat{c}'_{i,k+j|k} \hat{u}_{i,k+j|k} + \hat{\rho}'_{i,k+j+1|k} \hat{s}_{i,k+j+1|k}) \quad (5.22)$$

### 5.2.4 Certainty Equivalence Economic MPC

The certainty equivalence principle allows the substitution of all stochastic variables with their mean value predictions  $\mathcal{D}_k$ ,  $\mathcal{R}_k$ , and  $\mathcal{F}_k$ . Therefore, given the mean value of the forecasts, i.e.  $\mathcal{D}_k$  (5.3),  $\mathcal{F}_k$  (5.11) and  $\mathcal{R}_k$  (5.20), the filtered state,  $\hat{x}_{k|k}$ , from (5.15), the previous input,  $u_{k-1}$ , as well as the predictions (5.16) and the objective function (5.21), we compute the optimal trajectories of the predicted manipulated variables,  $\hat{u}_{i,k+j|k}$ , and the slack variables,  $\hat{s}_{i,k+j+1|k}$  and  $\hat{z}_{k+j+1|k}$ , by the solution of the linear program

$$\min_{\hat{u}_i, \hat{s}_i, \hat{z}} \phi = \phi(\{\hat{u}_{i,k+j|k}, \hat{s}_{i,k+j+1|k}, \hat{z}_{k+j+1|k}\}_{j=0}^{N-1}) \quad (5.23a)$$

$$s.t. \quad \hat{x}_{i,k+1+j|k} = A_i \hat{x}_{i,k+j|k} + B_i \hat{u}_{i,k+j|k} + E_i \hat{d}_{i,k+j|k} \quad (5.23b)$$

$$\hat{y}_{i,k+j+1|k} = C_i \hat{x}_{i,k+1+j|k} \quad (5.23c)$$

$$\hat{z}_{i,k+j+1|k} = C_{i,z} \hat{x}_{i,k+1+j|k} + D_{i,z} \hat{u}_{i,k+j|k} + F_z \hat{d}_{i,k+j|k} \quad (5.23d)$$

$$u_{i,k+j+1|k}^{\min} \leq \hat{u}_{i,k+j|k} \leq u_{i,k+j+1|k}^{\max} \quad (5.23e)$$

$$\Delta u_{i,k+j+1|k}^{\min} \leq \Delta \hat{u}_{i,k+j|k} \leq \Delta u_{i,k+j+1|k}^{\max} \quad (5.23f)$$

$$\hat{z}_{i,k+j+1|k} + \hat{s}_{i,k+j+1|k} \geq \hat{r}_{i,k+j+1|k}^{\min} \quad (5.23g)$$

$$\hat{z}_{i,k+j+1|k} - \hat{s}_{i,k+j+1|k} \leq \hat{r}_{i,k+j+1|k}^{\max} \quad (5.23h)$$

$$\hat{s}_{i,k+j+1|k} \geq 0 \quad (5.23i)$$

$$\hat{z}_{k+j+1|k} + \hat{s}_{k+j+1|k} \geq \hat{r}_{k+j+1|k}^{\min} \quad (5.23j)$$

$$\hat{z}_{k+j+1|k} - \hat{s}_{k+j+1|k} \leq \hat{r}_{k+j+1|k}^{\max} \quad (5.23k)$$

$$\hat{s}_{k+j+1|k} \geq 0 \quad (5.23l)$$

$N$  is the prediction horizon, which is normally chosen quite large in order to capture the dominating dynamics of the systems. The linear program (5.23) is based on the certainty equivalence assumption. Furthermore, due to the receding horizon strategy, only the first input  $\hat{u}_{k|k}$  of this control sequence is implemented. The function involving the solution of (5.23) and selecting  $\hat{u}_{k|k}$  is denoted as

$$u_k = \hat{u}_{k|k} = \mu(\hat{x}_{k|k}, u_{k-1}, \mathcal{D}_k, \mathcal{R}_k, \mathcal{F}_k) \quad (5.24)$$

Figure 5.2 shows the overall control framework. The Economic MPC unit consists of the Kalman filter and the regulator that is the Optimal Control Problem (OCP).

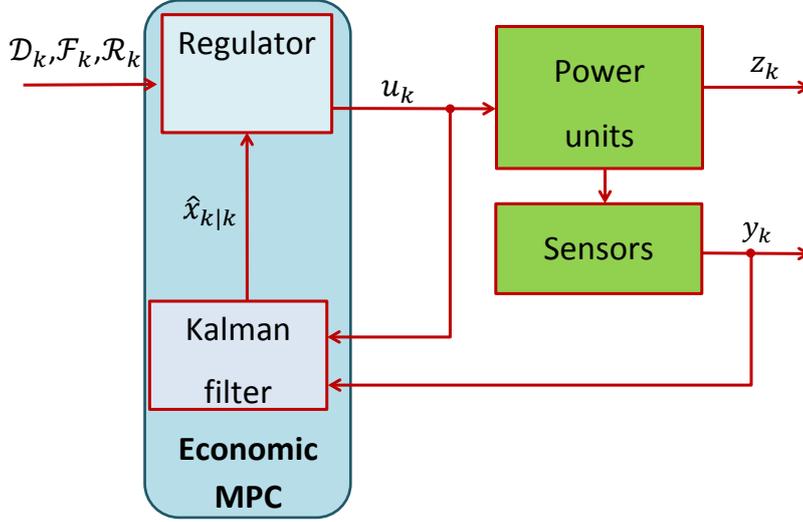


Figure 5.2: Forecast based Certainty Equivalent MPC Algorithm.

Algorithm 1 outlines the certainty equivalent Economic MPC developed. It computes the manipulated variable,  $u_k$ , based on the current measurement,  $y_k$ , the previous input,  $u_{k-1}$ , the forecasts,  $(\mathcal{D}_k, \mathcal{R}_k, \mathcal{F}_k)$ , and the smoothed mean-covariance estimate,  $(\hat{d}_{k-1|k}, R_{dd,k-1|k})$ . The smoothed estimate,  $(\hat{d}_{k-1|k}, R_{dd,k-1|k})$ , is needed because we do the one-step prediction of the states,  $\hat{x}_{k|k-1} = E\{x_k | \mathcal{Y}_{k-1} = Y_{k-1}\}$ , at the time  $k$  when the information vector  $\mathcal{I}_k^d = I_k^d$  has been realized and is known. These information availability considerations are the reason that the one-step predictions in Algorithm 1 must be expressed as (5.26a) and (5.28a).

The main computational load in Algorithm 1 is the solution of the linear program (5.23).

In this work, we omit the local output constraints (5.23g)-(5.23i) and the local slack variables  $s_{i,k}$ . This is due to the simulations scenario that consists of power plants described in Chapter 2, (2.1). Moreover, our goal is to balance the overall portfolio power production.

### 5.2.5 Condensing and Economic MPC as LP

In this work, we implement the state elimination strategy by condensing the state space to a Finite Impulse Response (FIR) model [85–87]. Consequently, considering an energy system consisting of  $M$  energy units modelled as (5.1), the output variable,  $z_{i,k}$  of each unit  $i$  with  $i = 1, 2, \dots, M$ , can be expressed in the matrix form

$$Z_i = \Gamma_{u,i} U_i + \Phi_i x_{0,i} + \Gamma_{d,i} D_{z,i} \quad (5.30)$$

5. LINEAR ECONOMIC MPC FOR POWER PLANT PORTFOLIO VIA  
DANTZIG-WOLFE DECOMPOSITION

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**Algorithm 1** Certainty equivalent Economic MPC with external forecasts

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**Require:**  $y_k, u_{k-1}, \hat{x}_{k-1|k-1}, P_{k-1|k-1}$  and forecasts:

$$\mathcal{D}_k = \{\hat{d}_{k+j|k}\}_{j=0}^{N-1} \quad (5.25a)$$

$$\mathcal{R}_k = \{\hat{r}_{\min, k+j+1|k}, \hat{r}_{\max, k+j+1|k}\}_{j=0}^{N-1} \quad (5.25b)$$

$$\mathcal{F}_k = \{\hat{c}_{k+j|k}, \hat{\rho}_{k+j+1|k}\}_{j=0}^{N-1} \quad (5.25c)$$

One-step predictor and filter

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + Bu_{k-1} + E\hat{d}_{k-1|k} \quad (5.26a)$$

$$\hat{y}_{k|k-1} = C\hat{x}_{k|k-1} \quad (5.26b)$$

$$\hat{z}_{k+j+1|k} = C_z\hat{x}_{k+1+j|k} + D_z\hat{u}_{k+j|k} + F_z\hat{d}_{k+j|k} \quad (5.26c)$$

Compute the innovation

$$e_k = y_k - \hat{y}_{k|k-1} \quad (5.27)$$

Compute

$$P_{k|k-1} = AP_{k-1|k-1}A' + GR_{ww, k-1}G' + ER_{dd, k-1|k}E' \quad (5.28a)$$

$$R_{e, k} = R_{vv} + CP_{k|k-1}C' \quad (5.28b)$$

$$K_{fx, k} = P_{k|k-1}C'R_{e, k}^{-1} \quad (5.28c)$$

$$P_{k|k} = P_{k|k-1} - K_{fx, k}R_{e, k}K'_{fx, k} \quad (5.28d)$$

Compute the filtered state

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{fx, k}e_k \quad (5.29)$$

Compute  $u_k = \mu(\hat{x}_{k|k}, u_{k-1}, \mathcal{D}_k, \mathcal{R}_k, \mathcal{F}_k)$  (5.23).

**return**  $u_k, \hat{x}_{k|k}, P_{k|k}$

---

where the vectors and matrices are defined as

$$Z_{i, k} = \begin{bmatrix} z_{i, k|k} \\ z_{i, k+1|k} \\ \vdots \\ z_{i, k+N|k} \end{bmatrix} \quad U_{i, k} = \begin{bmatrix} u_{i, k|k} \\ u_{i, k+1|k} \\ \vdots \\ u_{i, k+N-1|k} \end{bmatrix} \quad (5.31a)$$

$$D_{i, k} = \begin{bmatrix} d_{i, k|k} \\ d_{i, k+1|k} \\ \vdots \\ d_{i, k+N-1|k} \end{bmatrix} \quad \Phi_i = \begin{bmatrix} C_{z, i} \\ C_{z, i}A_i \\ C_{z, i}A_i^2 \\ \vdots \\ C_{z, i}A_i^N \end{bmatrix} \quad (5.31b)$$

$$\Gamma_{u,i} = \begin{bmatrix} D_{z,i} & 0 & \cdots & 0 \\ H_{1,i}^u & 0 & \cdots & 0 \\ H_{2,i}^u & H_{1,i}^u & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ H_{N,i}^u & H_{N-1,i}^u & \cdots & H_{1,i}^u \end{bmatrix} \quad \Gamma_{,i} = \begin{bmatrix} F_{z,i} & 0 & \cdots & 0 \\ H_{1,i}^d & 0 & \cdots & 0 \\ H_{2,i}^d & H_{1,i}^d & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ H_{N,i}^d & H_{N-1,i}^d & \cdots & H_{1,i}^d \end{bmatrix} \quad (5.32a)$$

The impulse response coefficients, also known as Markov parameters, are

$$H_{j,i}^u = C_i A_i^{j-1} B_i \quad H_{j,i}^d = C_{z,i} A_i^{j-1} E_i \quad j = 1, 2, \dots, N \quad (5.33)$$

Eliminating the states using (5.30) transforms the Economic MPC problem (5.23) into

$$\min_{U,S} \sum_{i \in \mathcal{M}} c_i^T U_i + \rho^T S \quad (5.34a)$$

$$s.t. \quad R_{min} - S \leq \sum_{i \in \mathcal{M}} Z_i \leq R_{max} + S \quad (5.34b)$$

$$\Delta U_i^{min} \leq \Delta U_i \leq \Delta U_i^{max} \quad (5.34c)$$

$$U_i^{min} \leq U_i \leq U_i^{max} \quad (5.34d)$$

$$S \geq 0 \quad (5.34e)$$

$c_i^T$  defines the production costs for operating the energy unit  $i$ ,  $\rho^T$  represents the penalty to pay every time that the slack variable  $s_k$  is non-zero

$$c_i^T = [c_{i,j} \quad c_{i,j+1} \quad \cdots \quad c_{i,j+N}] \quad \rho^T = [\rho_j \quad \rho_{j+1} \quad \cdots \quad \rho_{j+N}] \quad (5.35)$$

The constraint (5.34b) links all energy units output  $Z_i$  into the overall portfolio energy production, which must satisfy customers' consumption. Usually, this type of constraint is named coupling or connecting constraints.

At the time  $j$ , if the variable  $s_{j+n}$ , with  $n = 0, \dots, N$ , is non-zero then the overall energy production does not satisfy customers' demand interval defined by  $R_{min}$  and  $R_{max}$

$$S = \begin{bmatrix} s_j \\ s_{j+1} \\ \vdots \\ s_{j+N} \end{bmatrix} \quad R_{min} = \begin{bmatrix} r_{min,j} \\ r_{min,j+1} \\ \vdots \\ r_{min,j+N} \end{bmatrix} \quad R_{max} = \begin{bmatrix} r_{max,j} \\ r_{max,j+1} \\ \vdots \\ r_{max,j+N} \end{bmatrix} \quad (5.36)$$

In our scenario,  $R_{min}$  and  $R_{max}$  represent a customers' demand forecast in advance by external forecasting systems defined in (5.20).

5. LINEAR ECONOMIC MPC FOR POWER PLANT PORTFOLIO VIA  
DANTZIG-WOLFE DECOMPOSITION

---

Constraints (5.34c)-(5.34d) are named decoupling constraints because they represent an independent set of constraints for each energy unit  $i$ , with  $i \in \mathcal{M} = \{1, 2, \dots, M\}$ , and they are defined as

$$\Delta u_{i,j} = u_{i,j} - u_{i,j-1} \quad \Delta U_i = \begin{bmatrix} \Delta u_{i,j} \\ \Delta u_{i,j+1} \\ \vdots \\ \Delta u_{i,j+N-1} \end{bmatrix} \quad (5.37a)$$

$$\Delta U_i^{min} = \begin{bmatrix} \Delta u_{i,j}^{min} \\ \Delta u_{i,j+1}^{min} \\ \vdots \\ \Delta u_{i,j+N-1}^{min} \end{bmatrix} \quad \Delta U_i^{max} = \begin{bmatrix} \Delta u_{i,j}^{max} \\ \Delta u_{i,j+1}^{max} \\ \vdots \\ \Delta u_{i,j+N-1}^{max} \end{bmatrix} \quad (5.37b)$$

$$U_i^{min} = \begin{bmatrix} u_{i,j}^{min} \\ u_{i,j+1}^{min} \\ \vdots \\ u_{i,j+N-1}^{min} \end{bmatrix} \quad U_i^{max} = \begin{bmatrix} u_{i,j}^{max} \\ u_{i,j+1}^{max} \\ \vdots \\ u_{i,j+N-1}^{max} \end{bmatrix} \quad (5.37c)$$

### 5.2.6 Block-angular Economic MPC

Let us define the set  $\bar{\mathcal{M}} = \{1, \dots, M, \bar{M}\}$  which is equivalent to  $\mathcal{M} + 1$ . The EMPC problem (5.23) corresponds to the LP (5.34) and it can be expressed as

$$\min_{q_i} \varrho_1^T q_1 + \dots + \varrho_{\bar{M}}^T q_{\bar{M}} \quad (5.38a)$$

$$s.t. \quad F_1 q_1 + \dots + F_{\bar{M}} q_{\bar{M}} \geq f \quad (5.38b)$$

$$G_i q_i \geq g_i \quad (5.38c)$$

The coefficients and the optimisation vector  $q_i \in \mathbb{R}^n$  are defined as

$$\varrho_i = c_i \quad i \in \mathcal{M}, \quad \varrho_i = \rho \quad i = \bar{M} \quad (5.39)$$

$$q_i = u_i \quad i \in \mathcal{M}, \quad q_i = s \quad i = \bar{M} \quad (5.40)$$

The coupling constraints (5.34c)-(5.34d) define the set of matrices  $F_i \in \mathbb{R}^{n_f \times n}$  and  $f \in \mathbb{R}^{n_f}$  expressed as

$$F_i = \begin{bmatrix} \Gamma_{u,i} \\ -\Gamma_{u,i} \end{bmatrix} \quad i \in \mathcal{M} \quad , \quad F_i = \begin{bmatrix} I \\ -I \end{bmatrix} \quad i = \bar{M} \quad (5.41)$$

$$f = \begin{bmatrix} R_{min} - \sum_{i \in \mathcal{M}} (\Phi_i x_{0,i} - \Gamma_{d,i} D_i) \\ -R_{max} + \sum_{i \in \mathcal{M}} (\Phi_i x_{0,i} + \Gamma_{d,i} D_i) \end{bmatrix} \quad (5.42)$$

The decoupling constraints matrices are  $G_i \in \mathbb{R}^{n_{g_i} \times n}$  and  $g_i \in \mathbb{R}^{n_{g_i}}$  defined as for  $i \in \mathcal{M}$

$$G_i = \begin{bmatrix} I \\ -I \\ \Lambda \\ -\Lambda \end{bmatrix} \quad \Lambda = \begin{bmatrix} I_0^T \\ \Lambda_d \end{bmatrix} \quad I_0 = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (5.43)$$

$$\Lambda_d = \text{diag}([I_d \quad I_d \quad \dots \quad I_d]) \quad I_d = [-I \quad I] \quad (5.44)$$

In addition

$$\Delta U = \Lambda U - I_0 u_{-1} \quad (5.45)$$

$$g_i = \begin{bmatrix} U_{min,i} \\ -U_{max,i} \\ \Delta U_{min,i} + I_0 u_{-1,i} \\ -\Delta U_{max,i} - I_0 u_{-1,i} \end{bmatrix} \quad i \in \mathcal{M} \quad (5.46)$$

Similarly, for  $i = \bar{M}$

$$G_{\bar{M}} = [I] \quad (5.47)$$

$$g_{\bar{M}} = [0] \quad (5.48)$$

The Economic MPC problem (5.23) designs a control problem formulated as a block-angular LP (5.38) and its solution provides the optimal input signal for each energy unit of the energy system. A block-angular LP is characterised by two sets of constraints: coupling (5.38b) and decoupling (5.38c) constraints. As a consequence, the LP has a block-angular constraints matrix. Decomposition techniques easily apply to LPs with such a specific structure. It is worth noticing that the solution of LPs via decomposition techniques is fast, especially when the number of constraints and variables is high. Accordingly, a fast solution of the Economic MPC problem speeds up the entire control framework shown in Figure 5.2 because the main computational load is the solution of the control problem (5.23) which corresponds to the block-angular LP (5.38).

### 5.3 Dantzig-Wolfe decomposition

In this work, we apply the Dantzig-Wolfe decomposition technique to efficiently solve the Economic MPC LP (5.38). Chapter 4 outlines this decomposition technique.

The Dantzig-Wolfe decomposition algorithm is based on the theorem of convex combination expressed in Theorem 4.3.1.

## 5. LINEAR ECONOMIC MPC FOR POWER PLANT PORTFOLIO VIA DANTZIG-WOLFE DECOMPOSITION

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The control problem (5.38) defines a set of feasible polyhedron  $\mathcal{Q}_i$ , with  $i \in \bar{M}$ , that are bounded, closed and non-empty. Thus, we only include the extreme points in the convex combination and omit the extreme rays [26]. This yields to expressing the optimisation variable,  $q_i$ , as a convex combination of the vertices  $v_i^j$  of the polyhedron  $\mathcal{Q}_i$ ,  $\forall i \in \bar{M}$

$$q_i = \sum_{j=1}^V \alpha_{ij} v_i^j \quad \sum_{j=1}^V \alpha_{ij} = 1 \quad \alpha_{ij} \geq 0 \quad \forall i \in \bar{M} \quad (5.49)$$

The Master Problem (MP), defined in (4.4), is obtained by substituting the notation (5.49) for the optimisation variable  $q_i$  into the block-angular LP (5.38), which represents the Economic MPC control problem. However, it is not practical to solve the entire MP. Because of this, the Dantzig-Wolfe decomposition introduces a Reduced Master Problem RMP (5.50) including  $L \leq V$  vertices of each polyhedron  $\mathcal{Q}_i$  with  $i \in \bar{M}$

$$\min_{\alpha_{ij}} \quad \phi = \sum_{i=1}^{\bar{M}} \sum_{j=1}^L p_i^j \alpha_{ij} \quad (5.50a)$$

$$s.t. \quad \sum_{i=1}^{\bar{M}} \sum_{j=1}^L h_i^j \alpha_{ij} \geq f \quad (5.50b)$$

$$\sum_{j=1}^L \alpha_{ij} = 1 \quad i = 1, \dots, \bar{M} \quad (5.50c)$$

$$\alpha_{ij} \geq 0 \quad i = 1, \dots, \bar{M}, j = 1, \dots, L \quad (5.50d)$$

$\alpha_{ij}$  is the optimisation variable,  $\phi$  is the objective function, while the cost coefficients and the inequality constraints coefficients are

$$p_i^j = \varrho_i' v_i^j \quad h_i^j = F_i v_i^j \quad i \in \bar{M}, j = 1, \dots, L \quad (5.51)$$

Let us assume to have the initial feasible vertex  $v_i^0$  available  $\forall i \in \bar{M}$ . Section 5.3.1 addresses how to efficiently initialise the Dantzig-Wolfe algorithm. The initial feasible vertices are necessary to compute the RMP coefficients (5.51) and solve the related LP (5.50).

Additionally, the RMP (5.50) provides the simplex multipliers  $\pi$  and  $\gamma_i$  from the inequality constraints (5.50b) and the convexity constraints (5.50c). These simplex multipliers are sent from the RMP (5.50) and used to update the cost coefficients of the subproblems (5.52)  $\forall i \in \bar{M}$

$$\min_{q_i} \quad \xi_i = \left[ \varrho_i - F_i' \pi \right]' q_i \quad (5.52a)$$

$$s.t. \quad G_i q_i \geq g_i \quad (5.52b)$$

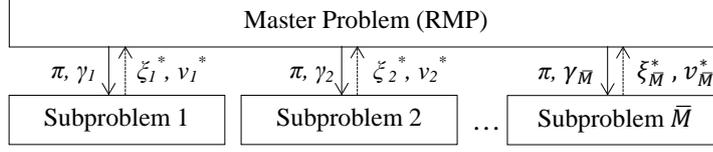


Figure 5.3: Dantzig-Wolfe decomposition framework.

$\xi_i$  denotes the objective function for the subproblem  $i$ . Let  $\xi_i^*$  denote the optimal value of the objective function  $\xi$  in (5.52) and  $q_i^* = v_i^*$  the optimal solution. If the optimality condition

$$\xi_i^* - \gamma_i \geq 0 \quad \forall i \in \bar{\mathcal{M}} \quad (5.53)$$

is satisfied, the solution  $q_i^* = v_i^*$  provides the optimal solution for the MP through the convex combination (5.49). The optimal solution for the block-angular LP (5.38) is given by

$$q_i^* = \sum_{j=1}^V \alpha_{ij} v_i^{j*} \quad (5.54)$$

If the optimality condition (5.53) is not satisfied, then the Dantzig-Wolfe decomposition adds columns into basis by augmenting the columns of the RMP

$$h_i^{j+1} = F_i v_i^{j*} \quad p_i^{j+1} = \varrho^T v_i^{j*} \quad \forall i \in \bar{\mathcal{M}} \quad (5.55)$$

Figure 5.3 represents the Dantzig-Wolfe framework. The RMP (5.50) sends its simplex multipliers  $\pi$  and  $\gamma_i$  to the subproblems (5.52). These LPs update their coefficients and send the optimal solution  $v_i^*$ , in order to update the coefficients (5.55) of the RMP for the next iteration. In addition, the subproblems (5.52) send the optimal value of the objective functions  $\xi_i^*$ , in order to check the optimality condition (5.53).

Algorithm 2 summarises the Dantzig-Wolfe algorithm that we utilise to efficiently solve the Economic MPC block-angular LP (5.38).

### 5.3.1 Warm-Start

In our work, the Dantzig-Wolfe algorithm is part of a MPC controller. As a consequence, the previous solution is always available and it can be used to compute the initial feasible vertex. In the Dantzig-Wolfe decomposition, the initial solution is necessary to compute the coefficients (5.51) in the RMP (5.50).

We derive a warm-start strategy from the constraints on the portfolio energy production (5.19) as Algorithm 3 illustrates. The solution of the EMPC is

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**Algorithm 2** Classic Dantzig-Wolfe decomposition

---

**Require:** Initial feasible vertex for the RMP (5.50).  
**if** No points are found **then**  
    Stop.  
**else**  
     $L=1$   
    **while**  $Converged == false$  **do**  
        Solve the  $L - th$  RMP (5.50).  
        Solve subproblem  $i$  (5.52).  
        **if** Optimality condition (5.53) is satisfied **then**  
             $Converged == true$ .  
            Optimal solution is given by (5.54).  
        **else**  
            Compute RMP coefficients (5.55).  
        **end if**  
         $L = L + 1$   
    **end while**  
**end if**

---

feasible and it satisfies the dynamic of the systems through the constraints (5.23b)-(5.23d). In the warm-start strategy, we use the previous optimal solution to estimate the power production for each power unit,  $\hat{z}_{i,k+j+1|k}$ , and the overall energy system production,  $\hat{z}_{k+j+1|k}$ . Moreover, we use these values to determine an initial vertex for the slack variable,  $\tilde{s}_{k+j+1|k}$ , through the soft constraints related to the output variables (5.23j)-(5.23l).

Accordingly, all the required initial feasible vertices are found without solving any LPs and this yields a reduction in computation time.

## 5.4 MATLAB Simulations

In Papers B, D, G, H, I and K we present our work where we propose the Dantzig-Wolfe decomposition algorithm to efficiently solve the Economic MPC control problem (5.38) to control large and distributed energy systems. In general, we apply the EMPC strategy to large and distributed energy systems that consist of multiple and independent energy units modelled in Chapter 2, (2.1).

Our intention is to show the benefits of implementing the Dantzig-Wolfe decomposition to the linear Economic MPC problem. The proposed controller satisfies customers' demand and reduces computation times even in large scale scenarios. For this purpose, we divide numerical results into two parts: power production and computational time. In the former part, we focus on the energy system power production and customers' consumption. In the latter

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**Algorithm 3** Initialisation technique for a Dantzig-Wolfe algorithm in a linear Economic MPC

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**Require:**  $\hat{u}_{i,k+j|k}$ .

**if**  $j = 0$  **then**

$$\hat{u}_{i,k|k} = u_i^{\min}$$

**end if**

Estimate the system outputs,  $\hat{z}_{i,k+j+1|k}$  (5.26c) and  $\hat{\tilde{z}}_{k+j+1|k}$  for the whole energy system.

Compute the initial feasible vertices for the next iteration from the global constraints (5.19)

$$\tilde{s}_{k+j+1|k,\min} = \max \left\{ \hat{r}_{k+j+1|k,\min} - \hat{\tilde{z}}_{k+j+1|k}, 0 \right\} \quad (5.56a)$$

$$\tilde{s}_{k+j+1|k,\max} = \max \left\{ \hat{\tilde{z}}_{k+j+1|k} - \hat{r}_{k+j+1|k,\max}, 0 \right\} \quad (5.56b)$$

$$\tilde{s}_{k+j+1|k} = \max(\tilde{s}_{k+j+1|k,\min}, \tilde{s}_{k+j+1|k,\max}) \quad (5.56c)$$

**return**  $\tilde{s}_{k+j+1|k}$ .

---

part of the results we present the computational performances of the proposed control algorithm.

It is worth noticing that we implement the warm-start strategy in Algorithm 3 in order to initialise the Dantzig-Wolfe decomposition.

### 5.4.1 Power production

The work in Paper H includes a scenario consisting of two power units. One of these has slow dynamic,  $\tau_1 = 15$ . The other power unit has fast dynamic,  $\tau_2 = 5$ . The manipulated variable of each power plant is subject to the following hard constraints

$$\begin{aligned} 0 \leq u_1 \leq 12 & & -1 \leq \Delta u_1 \leq 1 \\ 0 \leq u_2 \leq 12 & & -3 \leq \Delta u_2 \leq 3 \end{aligned}$$

Furthermore, the sampling time  $T_s$  is 1 s and the time horizon is  $N = 100$ . The noise  $w_k$  and  $d_k$  are normally distributed pseudorandom generated with the variance equal to 0.1.

Figure 5.4 shows the performances of the EMPC strategy on such an energy system in closed-loop simulations. The overall power produced by the energy system satisfies the energy consumption reference. The Dantzig-Wolfe decomposition algorithm computes, for each power plant, the reference signal, which is the magenta line visible in Figure 5.4. Figure 5.4 also displays the system output and its predicted value.

## 5. LINEAR ECONOMIC MPC FOR POWER PLANT PORTFOLIO VIA DANTZIG-WOLFE DECOMPOSITION

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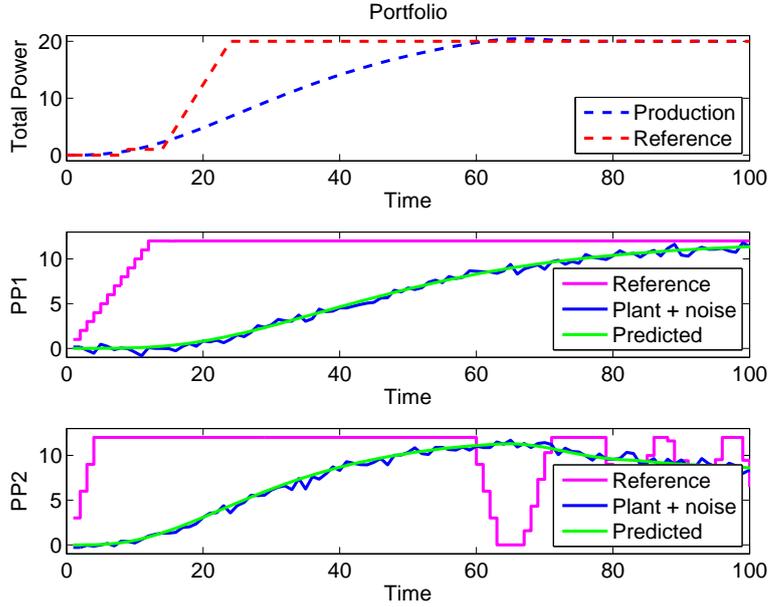


Figure 5.4: Energy production and consumption in closed-loop simulation with a scenario consisting of two power plants. PP1 denotes the performances of the slowest power plant, while PP2 is related to the fastest one.

In Paper [K](#) we analyse the performances of an energy system that consists of five power units modelled as (2.1), where the time constants are randomly generated over the interval  $[5, 30]$ , the sampling time is 1 s and time horizon is 50. The input bounds are set as

$$0 \leq u_i \leq 15 \quad -5 \leq \Delta u_i \leq 5 \quad i = 1, \dots, 5$$

The noise  $w_k$  and  $d_k$  variance is 0.1 and both variables are normally distributed pseudorandom generated. Figure 5.5 shows closed-loop simulations. The power produced by the energy system is within the customers' demand interval.

Deterministic simulations are in Paper [D](#) and [G](#). The scenario includes three power plants and the system is noise free. Figure 5.6 contains simulation results with sampling time  $T_s = 5$  s and time horizon is equal to 50. The time constants are  $\tau_1 = 40$ ,  $\tau_2 = 90$  and  $\tau_3 = 1000$ , while inputs bounds are defined as

$$\begin{aligned} 0 \leq u_1 \leq 50 & & -30 \leq \Delta u_1 \leq 30 \\ 0 \leq u_2 \leq 100 & & -20 \leq \Delta u_2 \leq 20 \\ 0 \leq u_3 \leq 200 & & -5 \leq \Delta u_3 \leq 5 \end{aligned}$$

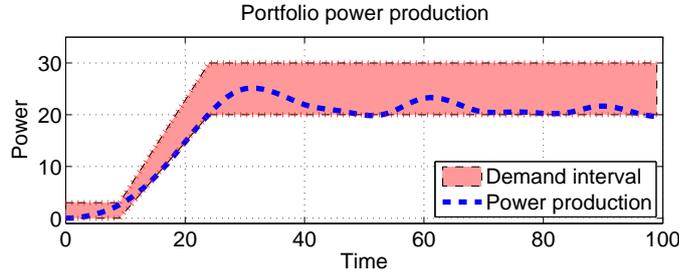


Figure 5.5: Power demand interval and overall power production of an energy system consisting of five power units in closed-loop simulation.

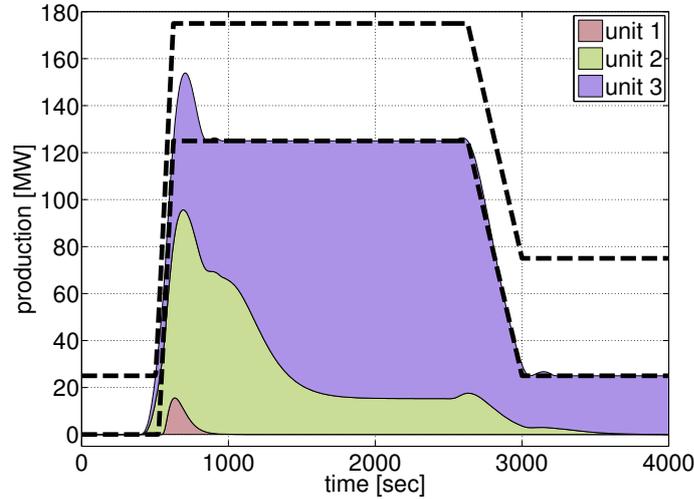


Figure 5.6: Closed-loop simulations noise free. Power demand interval and overall power production for an energy system consisting of three power units.

#### 5.4.2 Computation time

In Paper H, we utilise the MATLAB solver `linprog` to solve the block-angular LP control problem (5.38). In this work, the time constants  $\tau$  are randomly generated within the interval  $[2, 20]$ . Figure 5.7 shows that the considered number of power units in the energy system limits the solution of the control problem in a centralised way. A high number of units involves a high number of variables and constraints. This does not affect the performances of the proposed EMPC controller that implements the Dantzig-Wolfe decomposition algorithm. Figure 5.7 displays that the proposed controller computes the optimal input signals for all the case studies considered. With regard to computation times, the Dantzig-Wolfe decomposition outperforms the centralised solution of the control problem. In addition, solving in parallel the subproblems (5.52) yields to a substantial reduction in computation time.

5. LINEAR ECONOMIC MPC FOR POWER PLANT PORTFOLIO VIA DANTZIG-WOLFE DECOMPOSITION

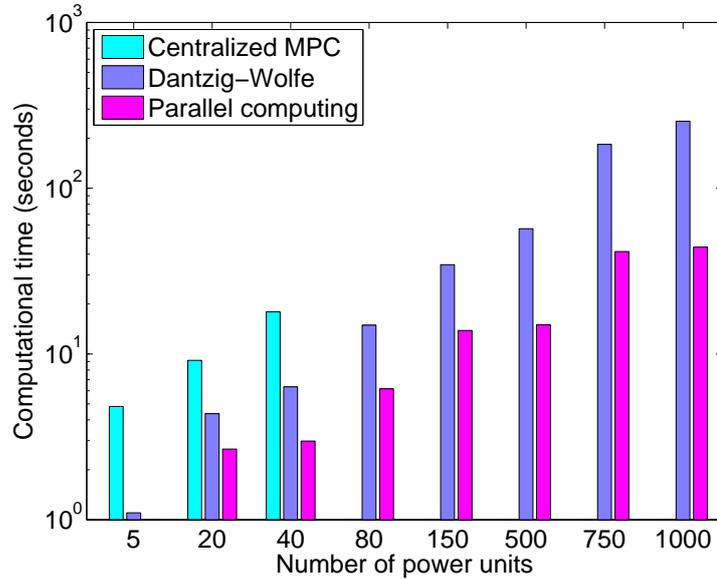


Figure 5.7: Centralized MPC and Dantzig-Wolfe computational times.

Papers [B](#), [D](#), [G](#) and [K](#) compare the performances of the proposed controller with the state-of-the-art LP solvers. Figure 5.8 reports that the solution of the control problem (5.38) in a distributed way via the Dantzig-Wolfe decomposition overcomes the state-of-the-art LP solvers. Interior Point (IP) and Active Search (AS) strategies applied to specific solvers, e.g. MOSEK, Gurobi and CPLEX solve the control problem (5.38) in a centralised way.

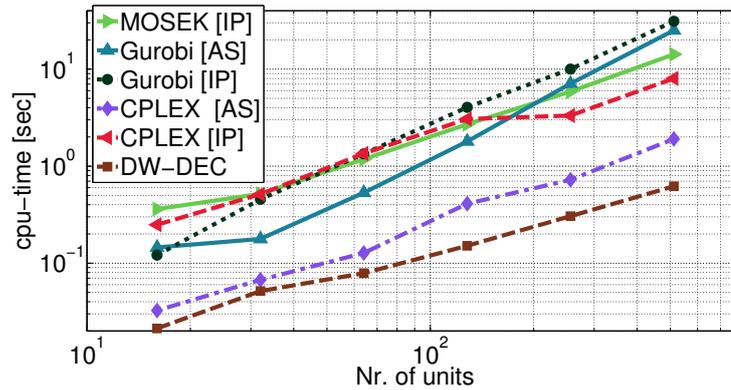


Figure 5.8: Dantzig-Wolfe vs. State-of-the-art LP solvers.

The work in Paper [B](#) investigates the performances of the Dantzig-Wolfe decomposition compared to the ADMM strategy. Figure 5.9 reveals that the Dantzig-Wolfe decomposition outperforms both centralised solution of the linear EMPC problem (5.38) and the distributed solution via ADMM.

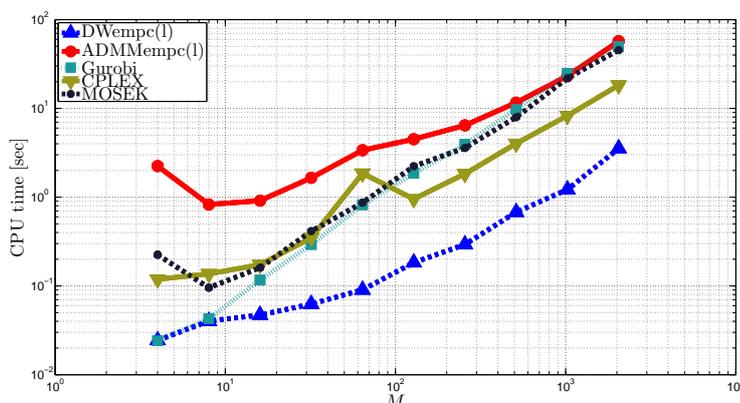


Figure 5.9: Dantzig-Wolfe vs. ADMM and State-of-the-art LP solvers.

## 5.5 Summary

In this chapter, we apply the Economic Model Predictive Control (EMPC) strategy to power systems operation. Moreover, we describe such a control problem as a LP with a block-angular constraint matrix. In order to make the EMPC problem computationally feasible we implement the Dantzig-Wolfe decomposition. We test the proposed algorithm in simulation and we use validated mathematical models of power plants in our scenario.

On the whole, the proposed algorithm provides the optimal control trajectories to each power unit in the energy system. As a result, the overall portfolio power production satisfies the consumption. Additionally, it is apparent in all the cases that the proposed control algorithm outperforms the other LP solvers and the decomposition techniques considered. The number of power units considered in the case study does not affect the performances of the Dantzig-Wolfe decomposition algorithm.



# Early Termination in Dantzig-Wolfe Decomposition for EMPC

In this chapter, we outline the early termination strategy applied to the Dantzig-Wolfe decomposition technique for the Economic Model Predictive Control (EMPC) problem. We introduce the feasibility of the solution given by such a strategy and we define the consequent stability. Then, we investigate the early termination performances in simulations. We apply the EMPC strategy to operate large and distributed energy systems. Simulations reveal that the early termination strategy causes a decrease in computation times and it affects optimality.

## 6.1 Early termination in Dantzig-Wolfe decomposition

The Economic Model Predictive Control (EMPC) strategy applies to large and distributed energy systems consisting of multiple and dynamically decoupled power units. These power systems are defined by linear models, process variables are subject to linear constraints, and the objective function is linear. Accordingly, the EMPC formulation is linear and it has a block-angular constraints matrix. Because of this, we implement the Dantzig-Wolfe decomposition in order to efficiently compute the optimal control trajectory for each power unit.

We extensively describe the Dantzig-Wolfe algorithm in Chapter 4 and in Chapter 5. In this work, we consider bounded, non-empty and closed polyhedra. Therefore, the optimisation variable,  $q_i$ , of the block-angular LP can be

## 6. EARLY TERMINATION IN DANTZIG-WOLFE DECOMPOSITION FOR EMPC

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expressed as the convex combination of the extreme points,  $v_i^j$ , of the related polyhedra

$$q_i = \sum_{j=1}^V \alpha_{ij} v_i^j \quad \sum_{j=1}^V \alpha_{ij} = 1 \quad \alpha_{ij} \geq 0 \quad \forall i \in \bar{M} \quad (6.1)$$

The early termination strategy in the Dantzig-Wolfe decomposition limits the number of iterations.

At each iteration, the Dantzig-Wolfe decomposition adds columns into basis while the optimality condition is not satisfied. The columns of the Reduced Master Problem (RMP) correspond to the basis and the optimisation variable of the RMP is  $\alpha_{ij}$  (6.1). As a result, the number of variables of the RMP increases by adding columns into basis. However, the extreme points of the polyhedra and the variables of the RMP constitute the convex combinations (6.1). Hence, the increase in the number of variables of the RMP yields to an increase in the number of extreme points of the polyhedra.

Accordingly, the limit on the number of iterations implicitly limits the number of extreme points considered.

The early termination strategy does not guarantee optimality because the limit on the number of iterations might stop the algorithm prematurely and before the optimality condition is satisfied.

### 6.1.1 Feasibility

The early termination strategy in the Dantzig-Wolfe decomposition provides a solution that is not optimal but it is feasible. In Chapter 4 we introduce Theorem 4.7.1 about the feasibility and optimality of the solution in the Dantzig-Wolfe decomposition technique.

We recall this theorem considering only the feasibility. We omit the extreme rays and we consider only the extreme points because we work with bounded, non-empty and closed polyhedra.

**Theorem 6.1.1** (Feasible solution with closed, bounded and non-empty polyhedra). *Any solution of the MP determines a feasible solution to the block-angular LP by convex combination.*

*Proof.* See [76]. □

Furthermore, the feasibility of the solution ensures stability of the algorithm developed.

The early termination strategy in the Dantzig-Wolfe decomposition technique is explained in Figure 6.1 where  $\mathcal{Y}$  denotes the maximum number of iterations.

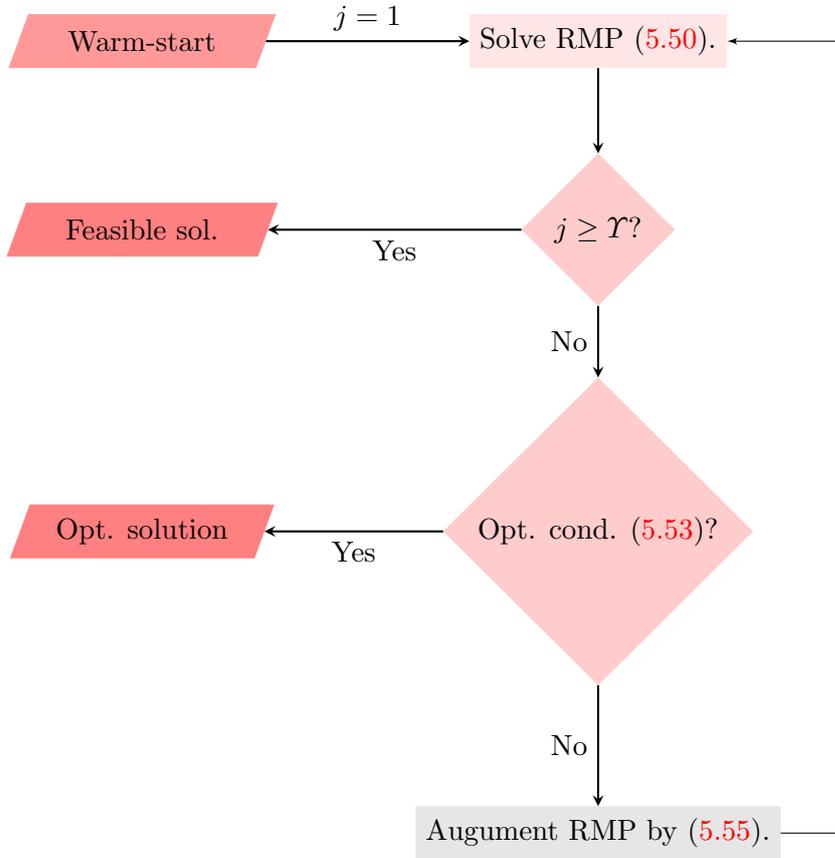


Figure 6.1: Early termination of Dantzig-Wolfe decomposition.

## 6.2 MATLAB simulations

Papers **C**, **E** and **F** investigate the performances of the early termination strategy in the Dantzig-Wolfe decomposition to operate power systems via the Economic Model Predictive Control (EMPC). The goal of our investigation is to show how the early termination strategy affects both the computational time and the optimality.

The scenario in Paper **C** includes large and distributed energy systems consisting of four multiple power units (2.1) with  $\tau_1 = 20$ ,  $\tau_2 = 10$ ,  $\tau_3 = 18$  and  $\tau_4 = 9$ . The input bounds are subject to the hard constraints

$$\begin{aligned} 0 &\leq u_{1,\dots,4} \leq 10 \\ -1 &\leq \Delta u_{1,3} \leq 1, \quad -3 \leq \Delta u_{2,4} \leq 3 \end{aligned}$$

The time horizon is  $N = 70$  and sampling time  $T_s = 1$  s. Additionally, we implement the warm-start strategy explained in Section 5.3.1 in the view of initialising the Dantzig-Wolfe algorithm. Moreover, simulations are closed-loop

## 6. EARLY TERMINATION IN DANTZIG-WOLFE DECOMPOSITION FOR EMPC

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and the noise  $w_k$  and  $d_k$  are normally distributed pseudorandom generated with the variance equal to 0.1. .

Figure 6.2 provides the performances of the early termination in the Dantzig-Wolfe algorithm. The results are divided into two parts as follows: CPU times, in Figure 6.2a, and costs, in Figure 6.2b. Figure 6.2a reveals that, in general, bounds on the number of iterations reduce computation times. Low bounds on the number of iterations yield to a left shift of the CPU time distributions, so computation times are decreasing. On the contrary, a right shift of the cost distributions can be identified in Figure 6.2b. Our investigations show that the limits on the number of iteration might stop the Dantzig-Wolfe algorithm prematurely and yield to extra costs. This is due to the number of extreme points of the feasible polyhedra included.

The early termination stops the Dantzig-Wolfe decomposition algorithm without considering the optimality condition. If the algorithm stops before the optimality condition is satisfied, then the solution is feasible and non-optimal. As a result, the non-optimality causes deterioration in the optimal values of the objective functions and this corresponds to extra costs.

In Papers E and F, the scenario is slightly different. We consider an energy system that consists of five power plants modelled as in Section 2.3.1 with  $\tau_1 = 8$ ,  $\tau_2 = 20$ ,  $\tau_3 = 10$ ,  $\tau_4 = 8$  and  $\tau_5 = 4$ . The input bounds are

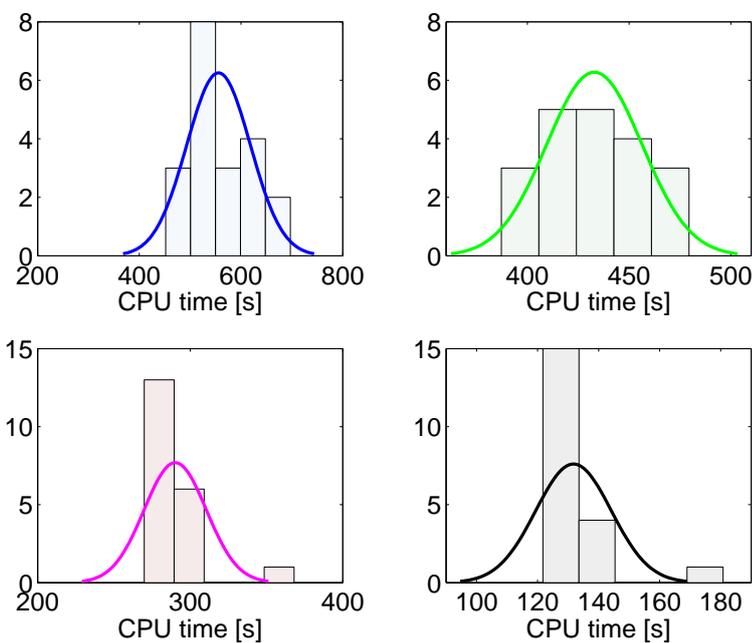
$$0 \leq u_{1,\dots,5} \leq 10 \quad -2 \leq \Delta u_{1,\dots,5} \leq 2$$

In addition, the time horizon is  $N = 50$  and sampling time is 1s. Figure 6.3 reports the decrease in computation time, blue bar, given by the early termination strategy. However, this strategy causes extra costs that the magenta bars represent. Consequently, a low number of extreme points of the feasible polyhedra leads to a decrease in CPU time and, at the same time, extra costs to pay. Stopping the Dantzig-Wolfe algorithm after 16 iterations causes a decrease of 50% in the CPU time and 20% of extra costs to pay. Similarly, a limit of four iterations reduces computation times up to 90% and leads to an increase of 50% in costs.

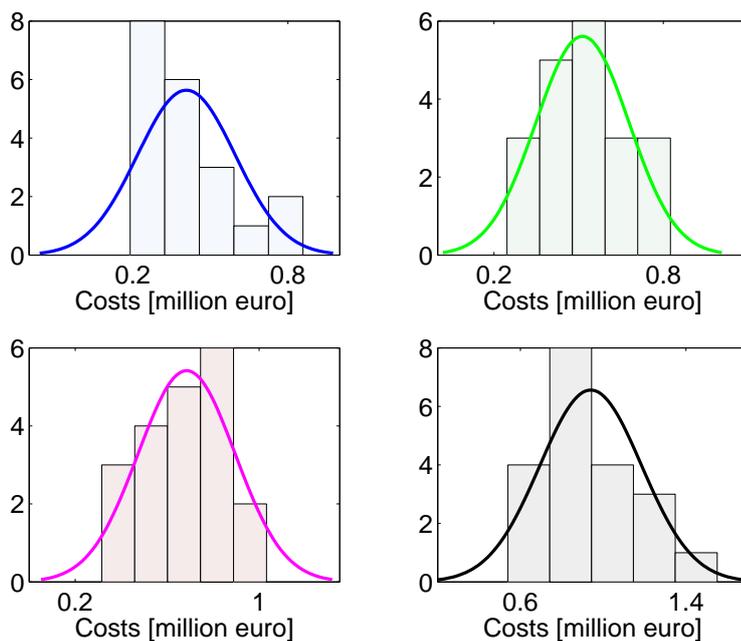
### 6.3 Summary

In this Chapter, we introduce the early termination strategy in the Dantzig-Wolfe decomposition to operate power systems via the Economic Model Predictive Control (EMPC). We prove that this approach ensures feasibility of the solution and this suffices for stability.

Moreover, we investigate the affects of the early termination strategy on computation times and optimality. In the main, this strategy causes reduction in the computation times. However, stopping the Dantzig-Wolfe algorithm prematurely affects the optimality and deterioration in the optimal values of the objective function occurs. Due to the implementation of the Dantzig-Wolfe



(a) CPU time



(b) Costs

Figure 6.2: CPU time and costs distributions based on 20 stochastic simulations. Blue distribution is for Economic MPC solved via exact Dantzig-Wolfe algorithm. Green distribution is for an early termination strategy with 15 vertices. Magenta distribution is for an early termination strategy with 10 vertices. Black distribution is for an early termination strategy with five vertices.

## 6. EARLY TERMINATION IN DANTZIG-WOLFE DECOMPOSITION FOR EMPC

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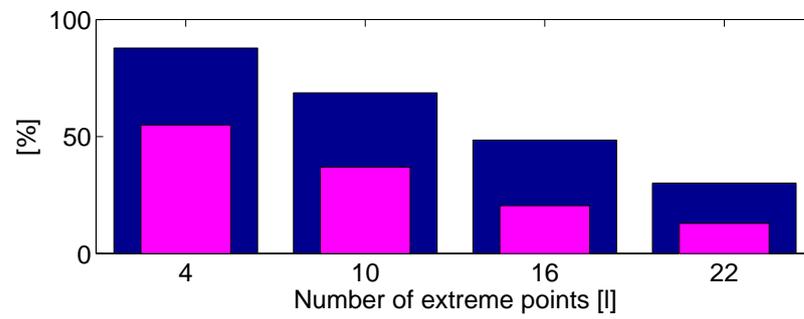


Figure 6.3: CPU time, blue bars vs. Extra Costs, magenta bars in the early termination strategy.

decomposition in the EMPC framework, deterioration in the optimal values corresponds to extra costs.

# Reduced Dantzig-Wolfe Decomposition and Partial Cycling Strategies for Linear EMPC

In this chapter, we outline the reduced version of the Dantzig-Wolfe decomposition aimed to fasten the algorithm. Moreover, we examine that this reduced decomposition technique computes a feasible and non-optimal solution. We describe that feasibility suffices for stability. Then, we implement the reduced Dantzig-Wolfe decomposition to the Economic Model Predictive Control (EMPC) problem for operating large and distributed energy systems. We provide numerical results from closed-loop simulations. In addition, we investigate the performances of the partial cycling strategies for the Dantzig-Wolfe decomposition in simulation.

## 7.1 Reduced Dantzig-Wolfe decomposition

In Paper [A](#), we investigate a reduced version of the Dantzig-Wolfe decomposition technique. Similarly to the classic Dantzig-Wolfe algorithm, the reduced version applies to the LPs that have a block-angular constraints matrix. Chapter [5](#) outlines that such a LP represents the Economic Model Predictive Control (EMPC) problem to operate large and distributed energy systems. The reduced Dantzig-Wolfe decomposition changes the optimality condition of the classic decomposition technique.

Let  $\xi_i^*$  denote the optimal value of the objective function of a subproblem  $i$  ([5.52](#)),  $\gamma_i$  the simplex multipliers given by the RMP ([5.50](#)),  $i \in \bar{\mathcal{M}}$ . If a

## 7. REDUCED DANTZIG-WOLFE DECOMPOSITION AND PARTIAL CYCLING STRATEGIES FOR LINEAR EMPC

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subproblem  $i$  satisfies the condition

$$\xi_i^* - \gamma_i \geq 0 \tag{7.1}$$

then the related reduced cost is non-negative. The classic Dantzig-Wolfe decomposition adds columns into basis until the condition (7.1) is not satisfied  $\forall i \in \bar{\mathcal{M}}$ .

The reduced Dantzig-Wolfe algorithm introduces a set  $\mathcal{S} \in \bar{\mathcal{M}}$  and this set  $\mathcal{S}$  includes those subproblems  $i$  that satisfy the condition (7.1).

The Dantzig-Wolfe decomposition requires the column with the most negative reduced cost to add into basis. When a subproblem  $i$  satisfies the condition (7.1) then the related reduced cost is non-negative and the Dantzig-Wolfe algorithm will not select the subproblem  $i$  to come into basis at the next iteration. Because of this, the reduced Dantzig-Wolfe decomposition includes the subproblem  $i$  into a set  $\mathcal{S} \subset \bar{\mathcal{M}}$ . The reduced Dantzig-Wolfe decomposition does not add columns to the RMP for those subproblem in the set  $\mathcal{S}$  because their reduced costs are non-negative. Therefore, solving these subproblems at the next iteration is not necessary.

The reduced Dantzig-Wolfe decomposition provides the solution for the block-angular LP if

$$\xi_i^* - \gamma_i \geq 0 \quad i \in \mathcal{S} \equiv \bar{\mathcal{M}} \tag{7.2}$$

and the solution is given by

$$q_i^* = \sum_{j=1}^L \alpha_{ij}^* v_i^j \quad i \in \bar{\mathcal{M}} \equiv \mathcal{S} \tag{7.3}$$

where  $v_i^j = q_i^*$  is the optimal solution of the subproblem (5.52) and  $\alpha_{ij}$  is the optimisation variable of the RMP.

We recall the optimality condition in the traditional Dantzig-Wolfe decomposition

$$\xi_i^* - \gamma_i \geq 0 \quad \forall i \in \bar{\mathcal{M}} \tag{7.4}$$

It is evident that the traditional Dantzig-Wolfe decomposition requires that all subproblems  $i$ ,  $\forall i \in \bar{\mathcal{M}}$ , satisfy the condition (7.4) at the same time. In comparison, the reduced Dantzig-Wolfe decomposition modifies the previous optimality condition (7.4) into (7.2) where all subproblems  $i$ ,  $\forall i \in \bar{\mathcal{M}}$ , must satisfy the condition but not at the same time.

The reduced Dantzig-Wolfe decomposition is described in Algorithm 4 and in Figure 7.1.

---

**Algorithm 4** Reduced Dantzig-Wolfe decomposition

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**Require:** Initial feasible vertex  $v_i^0$  for the RMP (5.50).  
**if** No points are found **then**  
    Stop.  
**else**  
     $\mathcal{S} = \{\emptyset\}$   
     $L=1$   
    **while** *Converged* == *false* **do**  
        Solve the  $L$  – *th* RMP (5.50).  
        Solve subproblem  $i$  (5.52), for  $i \in \bar{\mathcal{M}} \setminus \mathcal{S}$ .  
        **if**  $\mathcal{S} \equiv \bar{\mathcal{M}}$  **then**  
            *Converged* == *true*  
        **else**  
            **if** a subproblem  $i$ ,  $i \in \bar{\mathcal{M}} \setminus \mathcal{S}$ , satisfies the optimality condition (7.1)  
            **then**  
                 $\mathcal{S} = \{i\}$ ,  $\mathcal{S} \subset \bar{\mathcal{M}}$ .  
            **end if**  
            Compute RMP coefficients (5.55)  $\forall i \in \bar{\mathcal{M}} \setminus \mathcal{S}$ .  
        **end if**  
         $L = L + 1$   
    **end while**  
**end if**

---

**7.1.1 Suboptimality and stability in linear MPC via reduced Dantzig-Wolfe decomposition**

The reduced Dantzig-Wolfe decomposition changes the optimality condition (7.4) of the traditional decomposition technique into a condition that does not ensure optimality. As a result, the reduced Dantzig-Wolfe decomposition computes a suboptimal and feasible solution of the block-angular LP (5.38). In order to explain the suboptimality of the solution, we refer to Theorem 4.7.2. Refer to Corollary 1 in case of degeneracy in the block-angular LP.

The reduced Dantzig-Wolfe decomposition does not require all subproblems to satisfy the optimality condition at the same time and, because of this, it does not satisfy Theorem 4.7.2. Accordingly, the reduced Dantzig-Wolfe decomposition does not provide an optimal solution.

With reference to the feasibility of the solution, we recall Theorem 4.7.1. In this work, its formulation is different because we consider bounded, closed and non-empty polyhedra, hence, we consider exclusively extreme vertices and we omit extreme rays.

**Theorem 7.1.1** (Feasible and optimal solution with bounded, closed and non-empty polyhedra). *Any  $\alpha_{ij}$  solutions of the MP (5.50), determine an  $x$  as feasible solution to the block-angular problem (5.38) by convex combination*

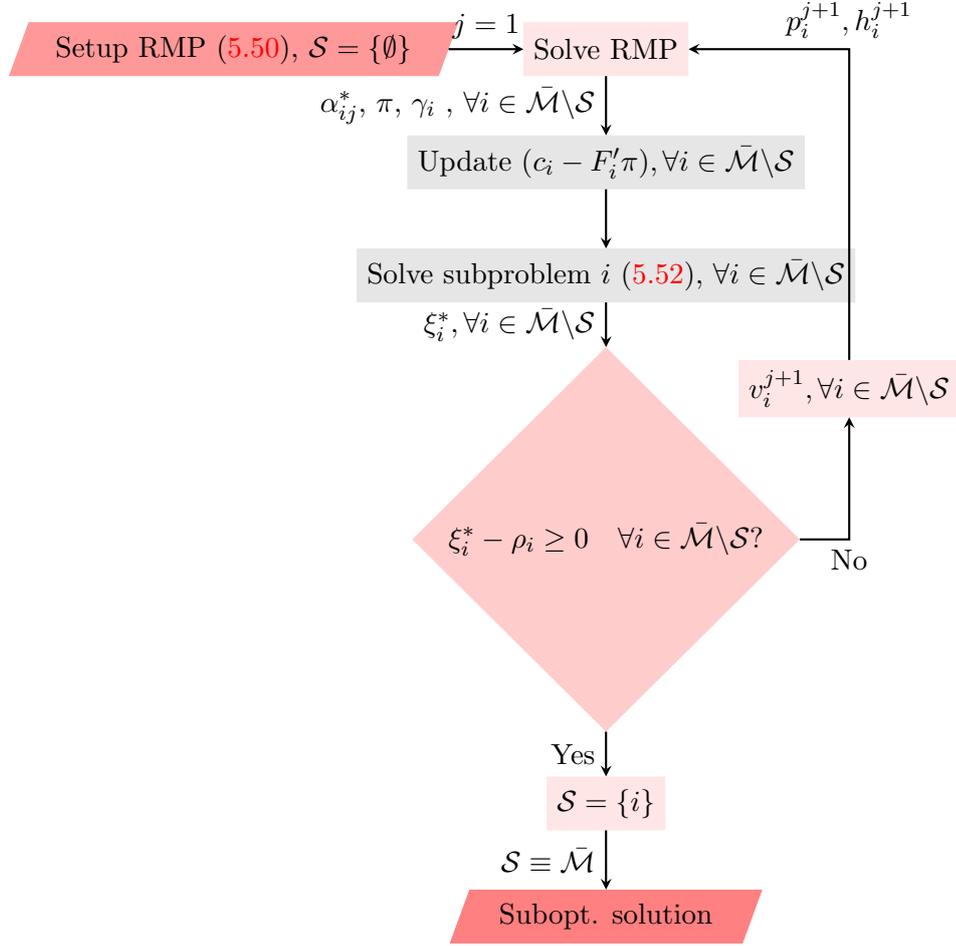


Figure 7.1: Reduced Dantzig-Wolfe decomposition.

(5.49). Likewise, if  $\phi^*$  is the minimum of MP (5.50) for  $\alpha_{ij}^*$ , then (5.49) provides an optimal feasible solution  $x^*$  to the original LP (5.38).

*Proof.* See [76]. □

Consequently, the reduced Dantzig-Wolfe decomposition computes a suboptimal solution for the block-angular LP, however the solution is feasible. Numerous studies have established that feasibility suffices for stability when achieving optimality is not required [22, 88–91].

### 7.1.2 MATLAB simulations: traditional DW vs. reduced DW

In Paper A, we investigate the performances of the reduced Dantzig-Wolfe decomposition on six case studies consisting of, respectively, 25, 50, 75, 100, 125 and 150 power units modelled in Chapter 2, (2.1). Time constants are

randomly selected inside the range  $[8, 25]$ , the time horizon is  $N = 70$  and the sampling time is  $T_s = 1$  s. Simulations are closed-loop and the noise  $w_k$  and  $d_k$  are normally distributed pseudorandomly generated with variance equal to 0.1. Moreover, the control variables are subject to the following hard constraints  $0 \leq u_k \leq 10$  and  $-2 \leq \Delta u_k \leq 2$ . The customers' demand is taken from the Nordic Electricity Market NordPool [92].

We investigate the performances of both traditional and reduced Dantzig-Wolfe decomposition applied to the Economic Model Predictive Control (EMPC) problem for operating a large and distributed energy system.

Figure 7.2 displays the performances in closed-loop simulations of both decomposition techniques applied to the Economic Model Control (EMPC) problem for operating an energy system consisting of 75 power units. It is evident in Figure 7.2a that the overall power production satisfies the customers' demand interval for both traditional and reduced Dantzig-Wolfe decomposition. Furthermore, 7.2b shows that the suboptimality of the solution causes the deterioration in the objective function optimal value. Figure 7.2b contains the total production costs and it reveals that the reduced Dantzig-Wolfe decomposition involves higher costs.

We consider all six case studies in order to illustrate the performances of the reduced Dantzig-Wolfe decomposition on large-scale energy systems. This decomposition technique, as expected, reduces computational times for all six case studies, according to Figure 7.3a. In addition, Figure 7.3b illustrates that objective function optimal values given by the reduced decomposition technique are higher than the traditional Dantzig-Wolfe. This is due to the non-optimality of the solutions.

Additionally, Figure 7.4 shows the percent decreases in the computational time and the percent changes in the optimal values of the objective function. The computation times decrease up to 80%, while the deteriorations in the objective function optimal values exceed 20% (upper dashed line) for only one case study. Moreover, the percent deterioration in the optimal value of the objective function is often below 10% (lower dashed line), even when the number of power units in the case study increases.

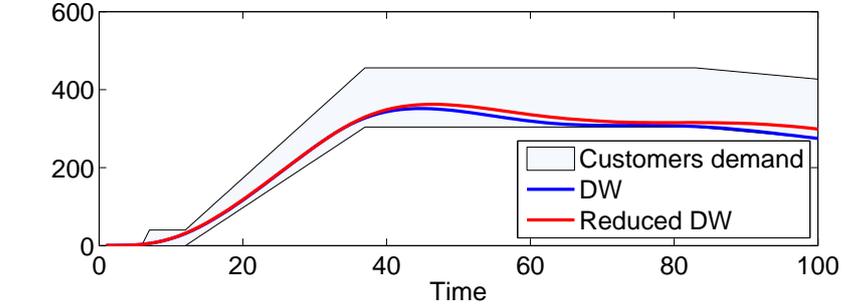
The overall response is that the reduced Dantzig-Wolfe decomposition causes reduction in the computation times. Closed-loop simulations show its stability. Moreover, we investigate the extra costs related to the deteriorations in the optimal values of the objective function due to the non-optimality in the reduced Dantzig-Wolfe algorithm.

## 7.2 Partial cycling strategies in Dantzig-Wolfe decomposition

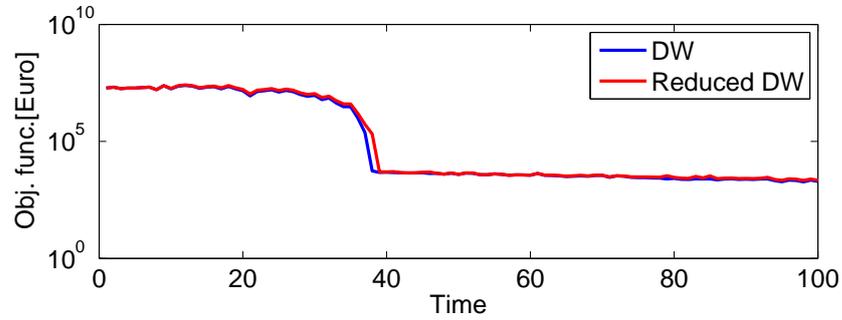
In Paper J, we investigate two partial cycling strategies applied to the Dantzig-Wolfe decomposition. The aim of these strategies is to fasten the solution

## 7. REDUCED DANTZIG-WOLFE DECOMPOSITION AND PARTIAL CYCLING STRATEGIES FOR LINEAR EMPC

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(a) Overall power production.



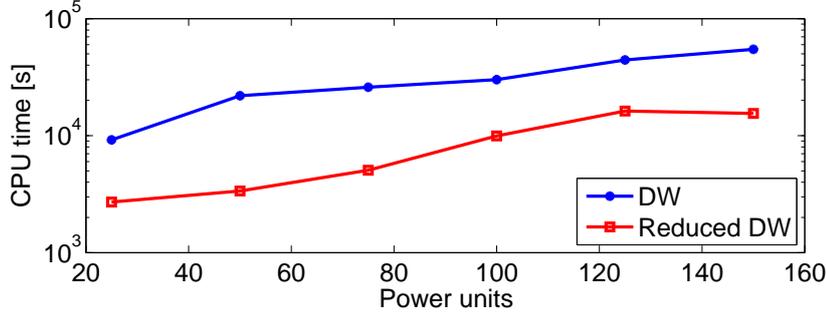
(b) Objective function (total production costs).

Figure 7.2: Closed-loop simulations run both the classic Dantzig-Wolfe, blue graph, and the novel reduced decomposition, red plot, on an energy system consisting of 75 power units.

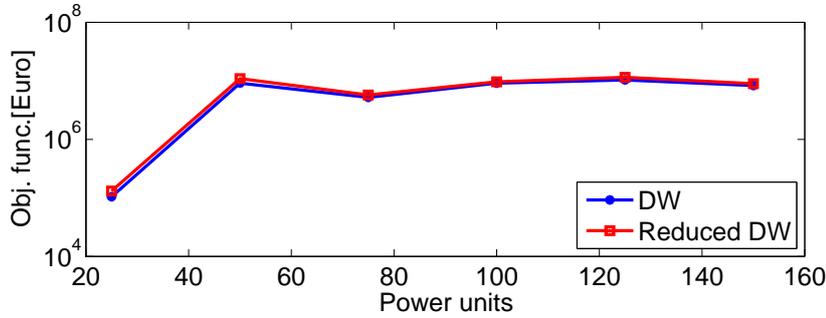
of the Economic Model Predictive Control (EMPC) problem. The two partial cycling strategies modify the optimality condition (7.4) of the traditional Dantzig-Wolfe decomposition and introduce two different cycling schemes. Accordingly, these strategies compute suboptimal solutions, but they ensure feasibility.

The traditional Dantzig-Wolfe decomposition adds columns to the Reduced Master Problem (RMP) (5.50) until all subproblems  $i \in \bar{\mathcal{M}}$  satisfy the optimality condition (7.1).

**Partial Strategy A** If a subproblem (5.52)  $i \in \bar{\mathcal{M}}$  satisfies the optimality condition (7.1) of the Dantzig-Wolfe algorithm, then the partial cycling strategy A does not add columns to the RMP related to subproblem  $i$ . The subproblem  $i$  is then included in a set  $\mathcal{A} \subset \bar{\mathcal{M}}$ . If all subproblems have satisfied the optimality condition, then,  $\mathcal{A} \equiv \bar{\mathcal{M}}$  and the partial cycling strategy has the solution to the block-angular LP. The partial cycling strategy A is described in Figure 7.5 and Algorithm 5.



(a) Computation time.



(b) Objective function optimal values.

Figure 7.3: Computation time and objective function optimal values for the classic Dantzig-Wolfe, blue line, and for the reduced version, red plot, vs. The number of power units in the energy system.

**Partial Strategy B** If a subproblem (5.52)  $i \in \bar{\mathcal{M}}$  has a decreasing objective function, then this problem will not be solved at the next iteration. The partial cycling strategy B is described in Figure 7.6 and in Algorithm 6. In this thesis, the number of extreme points  $\tilde{L}$  is heuristically defined.

Theorem 4.7.1 ensures the feasibility and the non-optimality of the solution given by both partial cycling strategies A and B.

It is worth noticing that when the subproblem  $i$  satisfies the optimality condition (7.1), then the related polytope  $\mathcal{Q}_i$  may attain the same local optimum given by the traditional implementation of the Dantzig-Wolfe algorithm. However, the partial strategies omit the contributions of the subproblem solutions to the coupling constraints and objective costs coefficients (5.51). Because of this, the solution cannot be optimal. Moreover, the lack of persistence of LPs influences the non-optimality of the solution. In particular, it is important to point out that, in general, the fact that a subproblem did not lead to the generation of a column

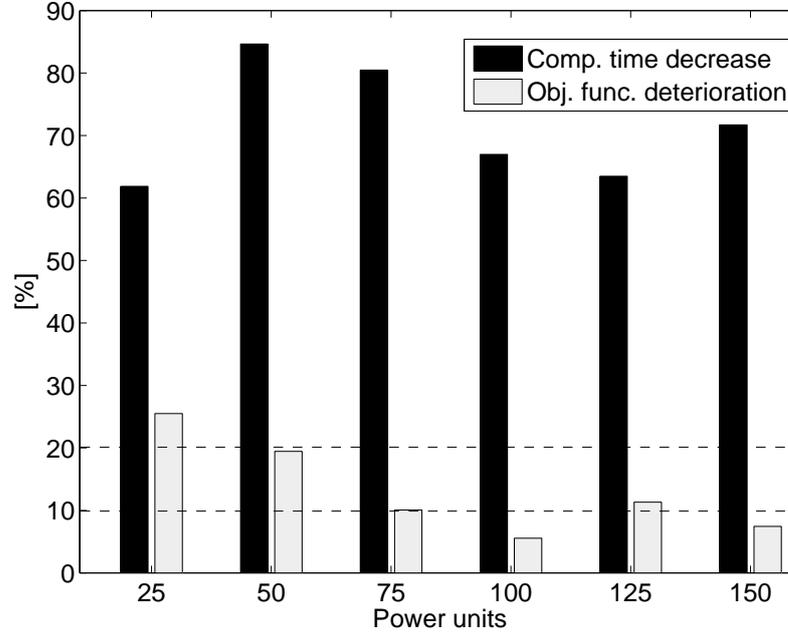


Figure 7.4: Percent changes, vs. Number of power units in the energy system. Black bars denote percent decrease in computation time, while grey bars denote percent change in objective function optimal values. These reductions in computational time and optimal values are the reductions compared with the classic Dantzig-Wolfe algorithm.

in one iteration does not imply that there will be no column generated in the next iteration.

### 7.2.1 MATLAB simulations

We compare the two partial cycling strategies to the classic and the reduced Dantzig-Wolfe decomposition. Simulations are open-loop, the time horizon is  $N = 100$  and the sampling time is  $T_s = 1$  s. We consider five case studies that represent large and distributed energy systems and they consist of, respectively, 10, 50, 100, 150 and 200 power units, modelled in Chapter 2, (2.1). Furthermore, external forecasting systems provide customers' demand from the Nordic Electricity Market NordPool [92].

The partial cycling strategy B requires the number of iterations  $\tilde{L}$  *a priori*. In these simulations,  $\tilde{L}$  is heuristically chosen and set to 15. This value set *a priori* limits the applicability of the strategy because a wrong value of  $\tilde{L}$

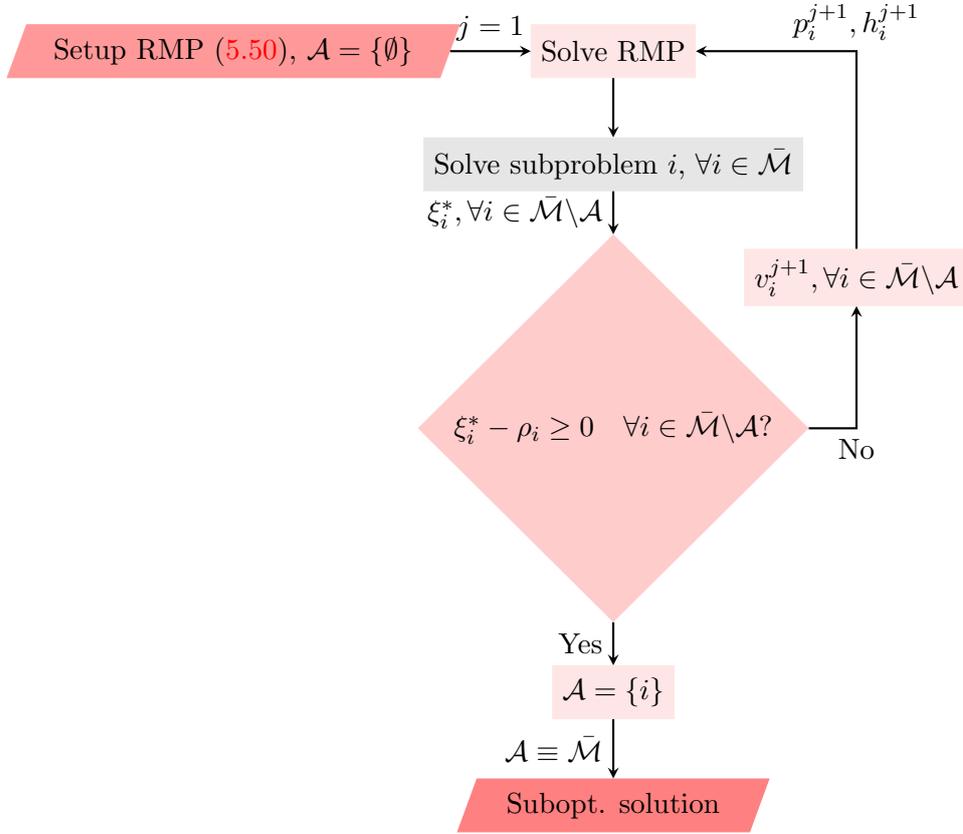


Figure 7.5: Partial cycling strategy A.

causes bad performances of the algorithm. However, in this work, we have not investigated methodologies on how to set efficiently this value  $\tilde{L}$ . Our choice is motivated by our knowledge of the case studies considered.

We investigate the performances of these partial strategies and we divide the numerical results into three parts: computation times, performances and data storage.

With reference to computation time, Figure 7.7a indicates that the partial cycling strategies reduce computation times for all five case studies. Furthermore, we measure the performances of the partial cycling strategies by considering the optimal value of the objective function. Figure 7.7b shows the deterioration of the partial cycling strategy B in the optimal value of the objective function is up to 12% in open-loop simulations.

In addition, we investigate the reduction in data storage that result from the two partial cycling strategies. The Dantzig-Wolfe algorithm includes columns into basis and this causes extra columns to the RMP. Therefore, most of the data storage is related to this part of the algorithm. Table 7.1 reports the number of columns of the RMP related to the number of power units included

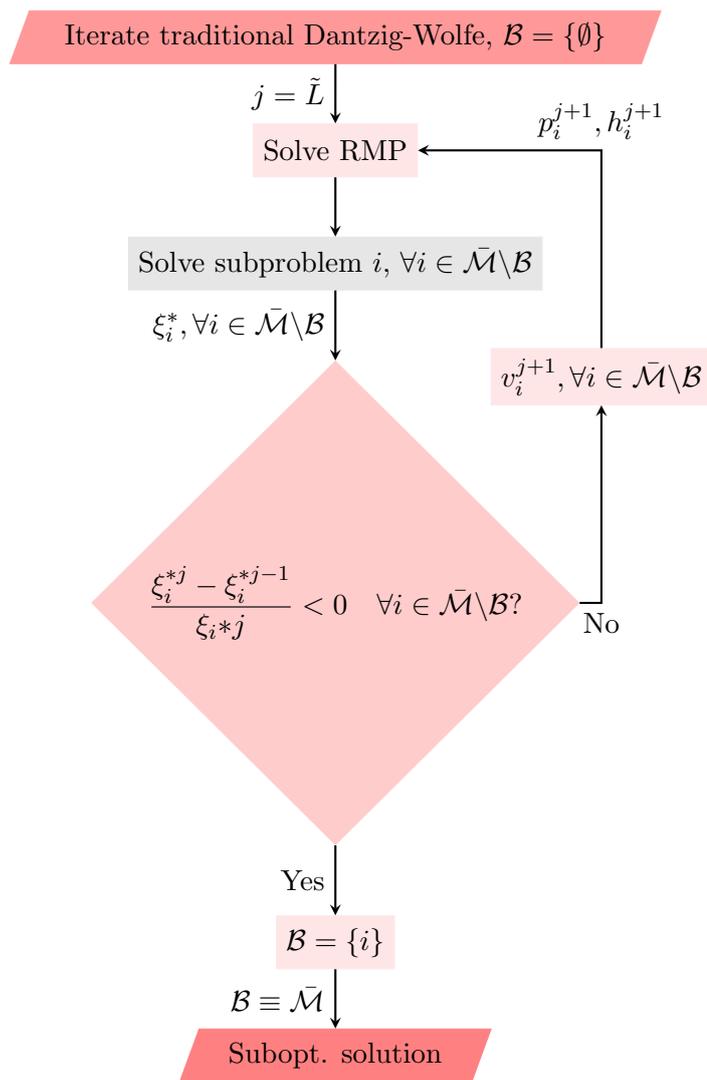


Figure 7.6: Partial cycling strategy B.

**Algorithm 5** Partial Cycling A

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Initialise the traditional Dantzig-Wolfe algorithm.
Solve the RMP (5.50).
 $\mathcal{A} = \{\emptyset\}$ .
while Not Converged do
  Solve subproblem  $i \in \bar{\mathcal{M}}$ .
  if a subproblem  $i \in \bar{\mathcal{M}}$  satisfies the optimality condition (7.1) then
    The subproblem  $i$  is included in the set  $\mathcal{A} \subset \bar{\mathcal{M}}$ .
  else
    Add more extreme points to the polytope  $\mathcal{Q}_i$  for all the subproblem
     $\forall i \in \bar{\mathcal{M}} \setminus \mathcal{A}$ .
  end if
  if  $\mathcal{A} \equiv \bar{\mathcal{M}}$  then
    Converged = true.
  end if
end while

```

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in the case study. We note that the classic Dantzig-Wolfe implementation requires the highest number of columns. As expected, the proposed partial cycling strategies cause reduction in the number of columns of the RMP.

Number of Units	DW	Strategy A	Reduced DW	Strategy B
10	1365	334	266	294
50	4750	916	1139	785
100	9500	3015	3285	2792
150	13500	3038	3315	2488
200	18400	6629	8117	6556

Table 7.1: Number of columns in RMP.

In all cases, the partial cycling strategies fasten the Dantzig-Wolfe algorithm. However, our investigations reveal that this causes deteriorations in the optimal value of the objective functions due to the non-optimality of the solutions.

## 7.3 Summary

In this Chapter, we explore techniques aimed to fasten the Dantzig-Wolfe algorithm. We apply these strategies to the Dantzig-Wolfe decomposition to operate the large and distributed energy systems via the Economic Model Predictive Control (EMPC) strategy.

First, we introduce a reduced Dantzig-Wolfe decomposition that ensures feasibility rather than optimality of the solution. We prove that the feasibility

7. REDUCED DANTZIG-WOLFE DECOMPOSITION AND PARTIAL CYCLING STRATEGIES FOR LINEAR EMPC

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**Algorithm 6** Partial Cycling B

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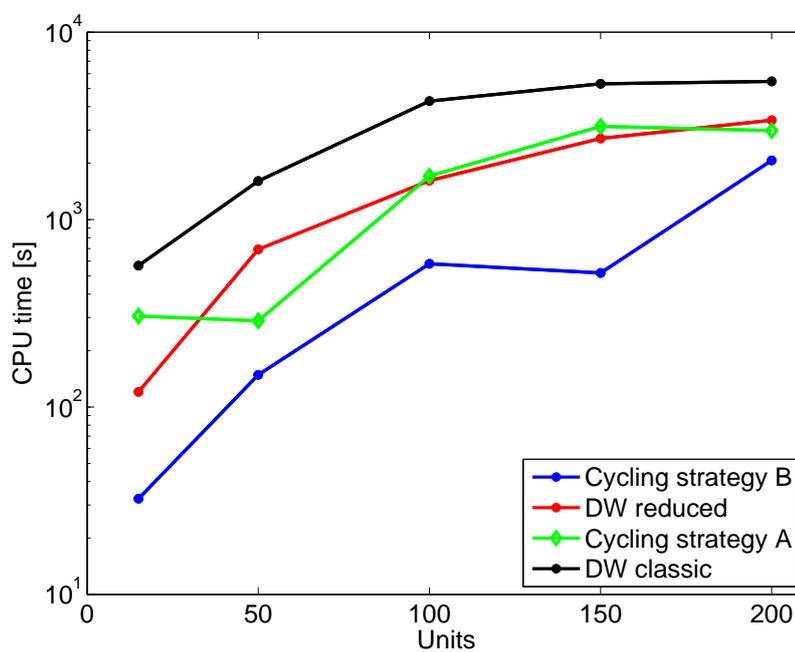
Iterate the Dantzig-Wolfe algorithm for  $\tilde{L}$  iterations.
 $j = \tilde{L}$ .
 $\mathcal{B} = \{\emptyset\}$ .
while Not Converged do
  Solve RMP (5.50).
  if  $i \notin \mathcal{B}$  then
    Solve subproblem  $i$ .
  end if
  if  $\frac{\xi_i^{*j} - \xi_i^{*j-1}}{\xi_i^{*j}} < 0 \quad \forall i \in \bar{\mathcal{M}} \setminus \mathcal{B}$  is decreasing then
    The subproblem  $i$  is included in the set  $\mathcal{B} \subset \bar{\mathcal{M}}$ .
  else
    Add more extreme points to the polytope  $\mathcal{Q}_i$  for all the subproblem
     $\forall i \in \bar{\mathcal{M}} \setminus \mathcal{B}$ .
  end if
  if  $\mathcal{B} \equiv \bar{\mathcal{M}}$  then
    Converged = true.
  end if
end while

```

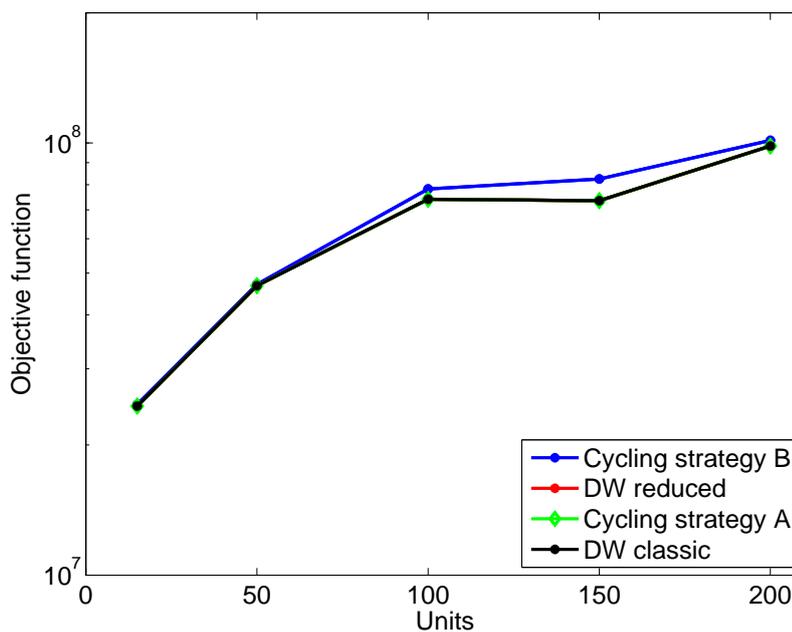
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is sufficient for stability in closed-loop simulations. Our investigations reveal that the reduced Dantzig-Wolfe decomposition reduces computation times. However, the suboptimality of its solution yields to deterioration of the optimal values of the objective functions.

In addition, we investigate partial cycling strategies for the Dantzig-Wolfe decomposition. These strategies compute feasible but non-optimal solutions. We explore the performances of the partial cycling strategies in open-loop simulations and we compare their performances to the classic and the reduced Dantzig-Wolfe algorithm. We prove that they fasten the solution of the block-angular LP. However, these strategies cause extra costs due to the non-optimality of the solutions.



(a) Computation time.



(b) Objective function optimal values.

Figure 7.7: CPU times and objective function optimal values vs. the number of power generator units in the energy system. The objective functions are provided by the implementation of: classic Dantzig-Wolfe (black), partial cycling strategy A (green), reduced Dantzig-Wolfe decomposition (red) and partial cycling strategy B (blue).



# Conclusions and Perspectives

In this thesis, we have provided insights into the design and the implementation of the Economic Model Predictive Control (EMPC) policy for operating the future power systems. We have outlined the current power systems and we have described their units, structures and control systems. In addition, we have offered an understanding of the future smart grids and their features. One of the key aspects discussed has been the Economic Model Predictive Control as a control strategy to coordinate and control large and distributed energy systems consisting of multiple and independent power units. We have revealed that the Dantzig-Wolfe decomposition technique efficiently solves the EMPC problem and outperforms the centralized solution of the control problem.

## 8.1 Large and distributed energy systems

In this thesis, we define smart grids as characterised by large and distributed energy systems. These embody various power units, e.g. thermal power plants, diesel generators, solar tanks, electric vehicles, wind farms, refrigeration systems and heat pumps for heating buildings. In Chapter 2, we provide realistic linear dynamical models of these flexible units. Moreover, we derive the related time space models and the linear constraints to fit into a predictive control framework. As a result, in Chapter 5, we include these linear models of the power units into the power plant portfolio. We consider these power units independent and with decoupled dynamics. However, in a power system operation, the power units cooperate to meet one common goal, which is to satisfy customers' demand. Additionally, we utilise the Finite Impulse Response (FIR) models of the system. Accordingly, we enable the Economic Model Predictive Control for power plant management. In particular, we choose the certainty equivalent MPC and we include the Kalman filter in the control framework. We also formulate the control problem as a LP due to the

linear models of power units, their linear constraints and the linear economic objective function.

## 8.2 Dantzig-Wolfe decomposition to solve linear EMPC

In Chapter 2, we reveal that the EMPC problem has two types of constraints: coupling and decoupling. The former type of constraints links all the power units of the energy system. The decoupling constraints refer to each power unit. As a result, the EMPC problem is expressed as a LP with a block-angular constraints matrix and this enables the implementation of a specific decomposition technique. Thus, we facilitate the solution of such an EMPC linear problem through the implementation of the Dantzig-Wolfe decomposition. This decomposition solves the control problem in a distributed way and computes the optimal input signals for each power unit in the energy system. In Chapter 2 and in Papers B, D, G, H, I and K, we ensure that the Economic MPC successfully coordinates the power units and the portfolio power production satisfies the customers' demand. Furthermore, we show that the Dantzig-Wolfe decomposition outperforms the state-of-the-art LP solvers. Also, we enhance the Dantzig-Wolfe decomposition performances by implementing parallel computing.

## 8.3 How to fasten the Dantzig-Wolfe decomposition

We provide insight into feasible and non-optimal solutions of the Dantzig-Wolfe algorithm in the EMPC problem to operate large and distributed energy systems. Moreover, we investigate their economic effects.

- \* In Chapter 6 and in Papers C, E and F, we explore the early termination strategy in the view of fastening the Dantzig-Wolfe algorithm. We prove that this strategy provides feasible solutions of the EMPC problem and it ensures stability. We efficiently solve the EMPC problem and we explore the performances of the early termination given an *a priori* limit on the number of iterations. The proposed strategy causes deterioration in the optimal values of the objective functions that corresponds to extra costs.
- \* In Chapter 7 and in Paper A, we investigate the reduced version of the Dantzig-Wolfe decomposition. This reduced technique does not require an *a priori* information. We prove that the reduced Dantzig-Wolfe decomposition ensures the feasibility that suffices for the stability. We investigate the performances of the proposed algorithm in closed-loop simulations. We succeed in reducing computation times up to 85% for

solving the EMPC problem. Our investigations reveal that this reduced version causes deterioration in the optimal values up to 20%.

- \* In Chapter 6 and in Paper J, we explore two partial cycling strategies for the Dantzig-Wolfe decomposition. These strategies do not compute the optimal input trajectories, but the solutions are suboptimal and feasible. We compare these strategies to the traditional Dantzig-Wolfe decomposition and to the reduced version. With reference to the computation times and the data storage, simulations reveal that the partial strategies outperform the traditional Dantzig-Wolfe decomposition. Instead, we investigate the effects of these strategies about deteriorations in the optimal values and the related extra costs.

## 8.4 Contributions

The scientific contributions in this thesis are:

- Mathematical models for large and distributed energy systems consisting of multiple power units that are dynamically decoupled.
- Economic MPC as a control strategy to coordinate and control all power units in large and distributed energy systems.
- Linear control problem tailored for the implementation of decomposition techniques aimed to fasten its solution.
- Efficient solution of the EMPC LP through Dantzig-Wolfe decomposition technique.
- Warm-start strategy for initialising the Dantzig-Wolfe algorithm applied to the EMPC problem.
- Investigations into suboptimal solutions of the EMPC problem through modified versions of the Dantzig-Wolfe algorithm.

## 8.5 Perspectives and future works

While governments, industries and academia are trying to enable the employment of the smart grid technologies, we are facing the fact that the gap between research and practice in such a technology is huge. For example, in this thesis we develop efficient control algorithms for large and distributed energy systems. In reality, the launch of large-scale demonstration projects is a challenge. Furthermore, the full deployment of smart grid technologies requires modern and adapted infrastructures. In my opinion, smart grids employed on

a small scale scenario represent the starting point of the energy revolution. A building, an apartment, a city, a district: all these systems comprise multiple energy units, such as heat pumps, ventilation systems, etc. Therefore, the control algorithm that we have developed in this work easily applies to these small scale implementation of the smart grid technologies as well.

In this thesis, we prove that the Economic Model Predictive Control (EMPC) policy is a potentially good control strategy for the future energy systems that implement smart grid technologies. Moreover, we succeed in efficiently solving the control problem via the Dantzig-Wolfe decomposition. This finding is promising and future studies should focus on massively parallel computing in the Dantzig-Wolfe decomposition for LPs. In addition, it would be beneficial to formulate the EMPC problem as a mixed-integer problem and solve it via the Dantzig-Wolfe decomposition. Another important question for future studies is to determine if the extra costs linked to the suboptimal solutions can be reduced.

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## Part III

# Published Papers and Technical Reports



# Appendices



APPENDIX

A

# Paper A - 19<sup>th</sup> IFAC World Congress, 2014

**A Reduced Dantzig-Wolfe Decomposition for a Suboptimal  
Linear MPC**

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# A Reduced Dantzig-Wolfe Decomposition for a Suboptimal Linear MPC<sup>\*</sup>

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**Abstract:** Linear Model Predictive Control (MPC) is an efficient control technique that repeatedly solves online constrained linear programs. In this work we propose an economic linear MPC strategy for operation of energy systems consisting of multiple and independent power units. These systems cooperate to meet the supply of power demand by minimizing production costs. The control problem can be formulated as a linear program with block-angular structure. To speed-up the solution of the optimization control problem, we propose a reduced Dantzig-Wolfe decomposition. This decomposition algorithm computes a suboptimal solution to the economic linear MPC control problem and guarantees feasibility and stability. Finally, six scenarios are performed to show the decrease in computation time in comparison with the classic Dantzig-Wolfe algorithm.

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## 1. INTRODUCTION

Recently, energy systems have evolved into highly integrated systems that deliver energy services to our homes and businesses. Electric power networks, also known as smart-grids, connect renewable energy sources (RES) to traditional power plants, cooling networks, as well as to other infrastructures. Increased reliability and performance, cost reduction, and minimized environmental impacts are the main benefits of the new energy systems. However, a major issue is the design of the controllers that coordinate and control the units of these energy systems to ensure that total energy production satisfies customer demand. Uncontrollable availability of renewable energy sources (RES), as well as fluctuations in consumer demand, yield power companies to utilize dynamic control of energy systems in the view of handling such variabilities.

This paper focuses on the design of a distributed algorithm to compute optimal control sequences for a centralized controller. We propose a Linear Economic Model Predictive Control (MPC) strategy to coordinate and control the independent and controllable units of energy systems in the most economic way. Linear Economic MPC requires repeated online solution of constrained linear optimization problems. Therefore, the computational speed limits the application of such a controller. Energy systems have independent units, so the control problem has a block-angular structure and the Dantzig-Wolfe distributed optimization efficiently solves this class of linear programs. With regard to speeding up the controller, we outline a reduced Dantzig-Wolfe decomposition that reduces computation times and guarantees feasibility and stability. This reduced Dantzig-Wolfe decomposition can be applied to the Linear

Economic MPC controller and calculates suboptimal local solutions.

MPC is a well-known control strategy that has been extensively used in several applications. Distributed model predictive control structures have attracted much attention, as shown in Scattolini (2009). Powerful tools to compute robust and efficient optimal control sequences were introduced by Conejo et al. (2006), who described how decomposition techniques can be applied to the control problems by exploiting their structures and efficiently solving the optimization problem. Sokoler et al. (2013) compared the Dantzig-Wolfe decentralized linear MPC with a centralized controller for large-scale systems and Standardi et al. (2013) introduced an early termination strategy to speed up the online computations; however, this approach involved unavoidable extra costs. With the aim of speeding up the control algorithms, suboptimal approaches were developed, guaranteeing feasibility and stability as reported in Scokaert et al. (1999); Zeilinger et al. (2008); Pannocchia et al. (2011). Rawlings et al. (2012) introduced the fundamentals of Economic MPC, the closed-loop properties that can be achieved, such as stability and convergence. However, few studies have addressed computational aspects of the Dantzig-Wolfe decomposition, and most of these works are about mixed integer and binary problems, see Kavinesh et al. (2009); Klein and Young (1999); Rios and Ross (2014). Little work has been done on speeding up Dantzig-Wolfe decomposition for LPs. Burger et al. (2012) developed a distributed simplex algorithm for degenerate LPs, while Frangioni and Gendron (2013) introduced a stabilized Dantzig-Wolfe decomposition subject to several assumptions.

The outline of this paper is as follows. Section 2 introduces Linear Economic MPC. Dantzig-Wolfe decomposition and its novel reduced version are formulated in Section 3. Suboptimality and stability of the proposed algorithm are

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illustrated in Section 4. We show the performances of our approach using numerical examples in Section 5, while conclusions are in Section 6.

## 2. LINEAR ECONOMIC MPC COORDINATION OF ENERGY SYSTEMS

The new energy systems are built by connecting individual and controllable power units that need a controller to satisfy the customer demand. The control problem must compute for each power unit the most economic and optimal production plan. We introduce the Economic MPC strategy that balances power supply and demand for such energy systems.

The following stochastic discrete state-space model describes a power unit in energy systems

$$x_{k+1} = Ax_k + Bu_k + Gw_k + Ed_k \quad (1a)$$

$$y_k = Cx_k + v_k \quad (1b)$$

$$z_k = C_z x_k. \quad (1c)$$

$x_k$  denotes the state variable and  $y_k$  is the measurement. Moreover, Standardi et al. (2012) includes process and measurement noises, respectively  $w_k$  and  $v_k$ , being distributed as  $\sim N_{iid}(0, R_{ww})$  and  $\sim N_{iid}(0, R_{vv})$ . Due to the large shares of renewable energy sources (RES), the model needs to consider weather forecasts  $d_k \sim N(\bar{d}_k, R_{dd,k})$  predicted by external prognosis systems. The manipulated variable,  $u_k$ , denotes the input signal and it is subject to hard constraints

$$u_{min} \leq u_k \leq u_{max} \quad (2a)$$

$$\Delta u_{min} \leq \Delta u_k \leq \Delta u_{max} \quad (2b)$$

$z_k$  indicates the system output and it must be within the interval  $[r_{min,k}, r_{max,k}]$ ; this interval may represent forecast consumer demand, or it can define indoor temperature in a building, or temperatures in a refrigeration system, or state-of-charge of a battery

$$r_{min,k} \leq z_k \leq r_{max,k} \quad (3)$$

A decoupled Kalman filter estimates the state and the output variables, while the certainty equivalence principle substitutes all the variables with their mean values as described in Standardi et al. (2012). It is worth noting that this observer works locally for each unit and does not involve the entire energy system.

The control strategy computes the control trajectory in the most economic way, thus minimizing the production costs. For each power unit, the cost of following the production plan  $u_k$  is

$$\phi_{i,k} = \sum_{j=0}^{N-1} \hat{c}'_{i,k+j|k} \hat{u}_{i,k+j|k} \quad (4)$$

where  $\hat{c}_{i,k+j|k}$  denotes the production costs and is forecast by external systems.

Altogether, the control problem is a linear problem because it applies to linear systems (1) subject to linear constraints (2)-(3) and it minimizes a linear cost function (4). Due to this economic objective, the controller optimizes directly online the economic performances of the energy systems computing the control sequences for each power unit. Therefore, the Economic MPC policy applied

to an energy system consisting of  $P$  power units (1) can be expressed as

$$\min_{\hat{u}_{i,k+j|k}, \hat{s}_{k+j+1|k}} \phi_k = \sum_{i=1}^P \phi_{i,k} + \sum_{j=0}^{N-1} \hat{\rho}'_{k+j+1|k} \hat{s}_{k+j+1|k} \quad (5)$$

subject to the local constraints  $\forall i \in P$  and  $\forall j \in \mathcal{N}$

$$\hat{x}_{i,k+j+1|k} = A_i \hat{x}_{i,k+j|k} + B_i \hat{u}_{i,k+j|k} + E_i \hat{d}_{i,k+j|k} \quad (6a)$$

$$\hat{z}_{i,k+j+1|k} = C_{z,i} \hat{x}_{i,k+j+1|k} \quad (6b)$$

$$u_{min,i} \leq \hat{u}_{i,k+j|k} \leq u_{max,i} \quad (6c)$$

$$\Delta u_{min,i} \leq \Delta \hat{u}_{i,k+j|k} \leq \Delta u_{max,i} \quad (6d)$$

$$\hat{r}_{min,i,k+j+1|k} \leq \hat{z}_{i,k+j+1|k} \leq \hat{r}_{max,i,k+j+1|k} \quad (6e)$$

and subject to the following connecting constraints  $\forall j \in \mathcal{N}$  and  $\forall i \in P$

$$\hat{z}_{k+j+1|k} = \sum_{i=1}^P \tilde{C}_{z,i} \hat{x}_{i,k+j+1|k} \quad (7a)$$

$$\hat{z}_{k+j+1|k} + \hat{s}_{k+j+1|k} \geq \hat{r}_{min,k+j+1|k} \quad (7b)$$

$$\hat{z}_{k+j+1|k} - \hat{s}_{k+j+1|k} \leq \hat{r}_{max,k+j+1|k} \quad (7c)$$

$$\hat{s}_{k+j+1|k} \geq 0 \quad (7d)$$

where  $\hat{z}_{k+j+1|k}$  denotes the overall power production, and  $\hat{r}_{min,k}$  and  $\hat{r}_{max,k}$  define customer demand forecasts. The connecting constraints include slack variables  $\hat{s}_{k+j+1|k}$ ; non-zero slack variables involve penalties  $\hat{\rho}_{k+j+1|k}$  to pay, as expressed in the objective function (5).

For large-scale energy systems consisting of multiple power units, the control problem (5)-(7) includes several variables and constraints; for this reason, decomposition techniques are investigated to efficiently compute the optimal control trajectories. Furthermore, the optimization control problem (5)-(7) consists of two sets of constraints: local constraints (6) for each power unit, and connecting constraints (7) for the overall energy system. This linear programming problem has a block-angular structure tailored for the implementation of the Dantzig-Wolfe decomposition to solve the control linear program. Section 3 introduces the Dantzig-Wolfe decomposition technique.

## 3. THE REDUCED DANTZIG-WOLFE DECOMPOSITION

The Dantzig-Wolfe decomposition is a specialized version of the Simplex Method to solve linear programming problems that have a block-matrix structure, see Dantzig and Thapa (2003). Among these systems, the block-angular systems have independent blocks defining local constraints and one set of coupling constraints. The linear programming problem (5)-(7) has a block-angular structure that defines local constraints (6) and a set of global constraints (7).

We consider the linear program (8) with the block-angular structure for  $i \in \mathcal{M}$ , where  $\mathcal{M} = \{1, \dots, M\}$

$$\min_{q_i} c'_1 q_1 + \dots + c'_M q_M \quad (8a)$$

$$s.t. \quad F_1 q_1 + \dots + F_M q_M \geq f \quad (8b)$$

$$G_i q_i \geq g_i \quad (8c)$$

This LP has  $i \in \mathcal{M}$ , with  $\mathcal{M} = \{1, \dots, M\}$ , blocks and each block defines a set of local constraints (8c) coupled through the connecting constraints (8b). Moreover,  $q_i \in \mathbb{R}^n$  defines the vector of variables to be determined and  $c_i \in \mathbb{R}^n$  is the vector of objective function coefficients. The block-angular constraints matrix consists of  $F_i \in \mathbb{R}^{n_f \times n}$ , representing the coupling constraints, and  $G_i \in \mathbb{R}^{n_{g_i} \times n}$ , denoting the local constraints; moreover,  $f \in \mathbb{R}^{n_f}$  and  $g_i \in \mathbb{R}^{n_{g_i}}$  are involved in the connecting and local constraints, respectively.

We briefly outline the classic Dantzig-Wolfe in Section 3.1 and the novel reduced form is introduced in Section 3.2.

### 3.1 Dantzig-Wolfe decomposition

The Dantzig-Wolfe algorithm is applied to the block-angular linear program (8), in which each column of coefficients can be freely chosen as any point from a convex set  $\mathcal{Q}$  as stated in the Theorem 1 of convex combination.

*Theorem 1.* Let  $\mathcal{Q}_i = \{q_i | G_i q_i \geq g_i\}$ , with  $i \in \mathcal{M}$  and  $\mathcal{M} = \{1, \dots, M\}$ , be a polyhedral set in  $\mathbb{R}^n$ . Every point  $q_i$  in the polyhedral set  $\mathcal{Q}_i$  can be expressed as a convex combination of the finite set  $\mathcal{V} = \{1, \dots, V\}$  of its extreme points  $v_i^j$  and a non-negative linear combination of the finite set  $\mathcal{K} = \{1, \dots, K\}$  of extreme rays  $r_i^k$

$$q_i = \sum_{j=1}^V \alpha_{ij} v_i^j + \sum_{k=1}^K \beta_{ik} r_i^k, \quad \sum_{j=1}^V \alpha_{ij} = 1 \quad (9)$$

with  $\alpha_{ij}, \beta_{ik} \geq 0$ .

**Proof.** See Dantzig and Thapa, 2003.

For the block-angular LP (8), each set of feasible polyhedra  $\mathcal{Q}_i$  is bounded, closed and non-empty, thus we only include the extreme points in the problem formulation as in Cheng et al. (2008). However, Dantzig and Thapa (2003) included the extreme rays in the problem formulation.

Substituting the convex combination (9) into the block-angular LP (8) formulates the master problem (MP) or extremal problem. It is worth noting that the MP has fewer rows in the coefficients matrix than the original block-angular program (8). However, the number of columns, and therefore also the number of variables, in the MP is larger, corresponding to all  $V$  extreme points of all  $M$  polyhedra.

The Dantzig-Wolfe does not solve the impractical full MP and generates at each iteration of the Simplex algorithm only the column of the MP that has been selected to come into basis. As a result, the algorithm formulates the reduced master problem (RMP) (10) for  $L$  vertices of the polyhedra, where  $L \leq V$

$$\min_{\alpha_{ij}} \quad \gamma = \sum_{i=1}^M \sum_{j=1}^L p_i^j \alpha_{ij} \quad (10a)$$

$$s.t. \quad \sum_{i=1}^M h_i^j \alpha_{ij} \geq f \quad (10b)$$

$$\sum_{j=1}^L \alpha_{ij} = 1 \quad i = 1, \dots, M \quad (10c)$$

$$\alpha_{ij} \geq 0 \quad i = 1, \dots, M, j = 1, \dots, L \quad (10d)$$

where  $\alpha_{ij}$  is the optimization variable,  $\gamma$  is the objective function, and  $p_i^j = c_i v_i^j$  and  $h_i^j = F_i v_i^j$  denote the cost and the inequality constraints coefficients. However, in order to select which column has to come into basis, the RMP needs an initial basic feasible solution  $v_i^0$ . Dantzig and Thapa (2003) proposed an algorithm to obtain such a starting basic solution via Simplex Phase I. Similarly, Standardi et al. (2012) introduced a warm-start strategy specialized to the MPC strategy that provides initial basic feasible solutions without solving any linear problems.

Let us assume that the initial extreme points  $v_i^0$  are available for each polyhedron  $i \in \mathcal{M}$ . Thus, the RMP provides the dual variables  $\pi$  and  $\mu$ , respectively, for linking (10b) and convexity constraints (10c). The algorithm utilizes these dual variables to generate only the column having the most negative reduced cost without having to generate all the remaining columns of the MP. This pricing problem is expressed in the following subproblems

$$\min_{q_i} \quad \xi_i = [c_i - F_i' \pi]' q_i \quad (11a)$$

$$s.t. \quad G_i q_i \geq g_i \quad (11b)$$

where  $\xi_i$  denotes objective function for the subproblem  $i$ . It is evident that each subproblem  $i \in \mathcal{M}$  (11) is independent and decoupled; hence, parallel computing techniques can efficiently compute these  $i$  optimal solutions. The optimal solution of the subproblem (11) identifies which column has the smallest reduced cost for the MP. Thus, if the optimal objective function value  $\xi_i^*$  satisfies the following condition

$$\xi_i^* - \mu_i \geq 0 \quad \forall i \in \mathcal{M} \quad (12)$$

then all the reduced costs for the MP will be non-negative. Hence, the Dantzig-Wolfe algorithm has an optimal solution to the MP and, consequently, to the original block-angular problem (8) through convex combination (9).

In contrast, if  $\xi_i^* - \mu_i < 0$ , then we augment the columns of the RMP by

$$p_i^{j+1} = c_i v_i^j \quad h_i^{j+1} = F_i v_i^j \quad (13)$$

where  $v_i^j = q_i^j$  is the optimal basic feasible solution of (11).

The classic Dantzig-Wolfe decomposition is illustrated in Algorithm 1.

### 3.2 Reduced Dantzig-Wolfe decomposition

In this work, we propose a reduced version of the Dantzig-Wolfe decomposition.

At each iteration of the Simplex algorithm, the Dantzig-Wolfe decomposition computes only the column of the RMP (10), which has to come into basis. This column has the most negative reduced cost. Moreover, let us assume that at iteration  $t$ , only a set of subproblems  $\mathcal{S} \subset \mathcal{M}$  satisfies the optimality condition (12)

$$\xi_s^* - \mu_s \geq 0 \quad s \in \mathcal{S} \subset \mathcal{M} \quad (14)$$

In such a scenario, the classic Dantzig-Wolfe brings variables into basis by adding columns to the RMP (10) for every subproblem  $i \in \mathcal{M}$ , hence even for the set  $\mathcal{S}$  of subproblems.

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**Algorithm 1** Classic Dantzig-Wolfe

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**Require:** Initial feasible vertex for the RMP (10), see Section 3.1.  
**if** Any points are found **then**  
    Stop.  
**else**  
    L=1  
    **while** *Converged* == *false* **do**  
        Solve the *L* – *th* RMP (10).  
        Solve subproblem *i* (11),  $\forall i \in \mathcal{M}$ .  
        **if** optimality condition (12) is satisfied  $\forall i \in \mathcal{M}$   
            **then**  
                *Converged* == *true*  
            **else**  
                Compute RMP coefficients  $\forall i \in \mathcal{M}$  (13).  
            **end if**  
        L = L + 1  
    **end while**  
**end if**

---

In contrast, if condition (14) holds, then the reduced Dantzig-Wolfe does not add columns to the RMP (10) for the set  $\mathcal{S}$  of subproblems; this yields to update the coefficients of the RMP as

$$p_i^{t+1} = c_i v_i^t \quad h_i^{t+1} = F_i v_i^t, \quad i \in \mathcal{M} \setminus \mathcal{S} \quad (15)$$

At iteration  $t + 1$ , the reduced Dantzig-Wolfe solves the following subproblems

$$\min_{q_i} \quad \xi_i = [c_i - F_i' \pi]' q_i \quad i \in \mathcal{M} \setminus \mathcal{S} \quad (16a)$$

$$s.t \quad G_i q_i \geq g_i \quad (16b)$$

Consequently, the Dantzig-Wolfe applies the pricing problem on a reduced set of subproblems  $\mathcal{S}$ . As a result, by applying this reduced Dantzig-Wolfe decomposition, the number of iterations decreases. Algorithm 2 illustrates the reduced Dantzig-Wolfe decomposition.

#### 4. SUBOPTIMALITY AND STABILITY IN LINEAR MPC VIA REDUCED DANTZIG-WOLFE

The reduced Dantzig-Wolfe decomposition computes a solution to the block-angular problem (8) that is not optimal but it is feasible. In this Section we illustrate suboptimality of the reduced Dantzig-Wolfe decomposition; moreover, we demonstrate that this decomposition technique does not affect the convergence, thereby it guarantees feasibility and stability.

*Suboptimality* The reduced Dantzig-Wolfe computes a suboptimal solution to the block-angular problem (8). In order to explain this suboptimality, we introduce Theorem 2 that provides the optimal solution for the MP in the Dantzig-Wolfe decomposition.

*Theorem 2.* An optimal basic feasible solution of the RMP (10) is also optimal for the MP if

$$\xi_i^* = \mu_i \quad \forall i \in \mathcal{M} \quad (17)$$

Then the algorithm computes the optimum in a finite number of iterations.

**Proof.** See Dantzig and Thapa, 2003.

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**Algorithm 2** Reduced Dantzig-Wolfe

---

**Require:** Initial feasible vertex for the RMP (10), see Section 3.1.  
**if** Any points are found **then**  
    Stop.  
**else**  
     $\mathcal{S} = \{\emptyset\}$   
    L=1  
    **while** *Converged* == *false* **do**  
        Solve the *L* – *th* RMP (10).  
        Solve subproblem *i* (16), for  $i \in \mathcal{M} \setminus \mathcal{S}$ .  
        **if** optimality condition (12) is satisfied  $\forall i \in \mathcal{M}$   
            **then**  
                *Converged* == *true*  
            **else**  
                **if** a subproblem *s*,  $s \in \mathcal{M}$ , satisfies the optimality condition (12) **then**  
                     $\mathcal{S} = \{s\}, \mathcal{S} \subset \mathcal{M}$ .  
                    Compute RMP coefficients  $\forall i \in \mathcal{M} \setminus \mathcal{S}$ .  
                **else**  
                    Compute RMP coefficients (13)  $\forall i \in \mathcal{M} \setminus \mathcal{S}$ .  
                **end if**  
            **end if**  
        L = L + 1  
    **end while**  
**end if**

---

Theorem 2 states that the optimal basic feasible solution of the MP is given when every subproblem  $i$ ,  $i \in \mathcal{M}$ , satisfies the condition (17). The reduced Dantzig-Wolfe decomposition does not compute the optimality condition (17) for every subproblem at the same time. Instead, this reduced version stops computing the optimal solution for a subproblem when this satisfies the optimality condition (12), even if the other subproblems do not provide an optimal solution. Accordingly, the reduced Dantzig-Wolfe decomposition does not provide an optimal solution, as Theorem 2 is not satisfied; thus, the solution is suboptimal, however feasible.

*Feasibility and stability* Scolaert et al. (1999); Muske and Rawlings (1993); Mayne et al. (2000); Chisci et al. (1996) demonstrated how feasibility implies stability for a linear MPC strategy. Because of this, the following theorem illustrates the feasibility of the reduced Dantzig-Wolfe decomposition.

*Theorem 3.* Any  $\alpha_{ij}$  that solves the RMP (10) determines a feasible solution  $q_i$  for the block-angular program (8) by the convex combination (9). Moreover, if  $\gamma$  has the minimum of the RMP (10) for  $\alpha_i^*$ , then the convex combination (9) generates an optimal feasible solution  $q_i^*$  to the original problem (8).

**Proof.** See Dantzig and Thapa, 2003.

Therefore, the reduced Dantzig-Wolfe decomposition guarantees feasibility that suffices for stability.

#### 5. COMPUTATIONAL RESULTS

As mentioned previously, our intention is to show that the novel reduced Dantzig-Wolfe decomposition speeds up the algorithm, guaranteeing feasibility and stability. In this section, we compare the performances of both classic

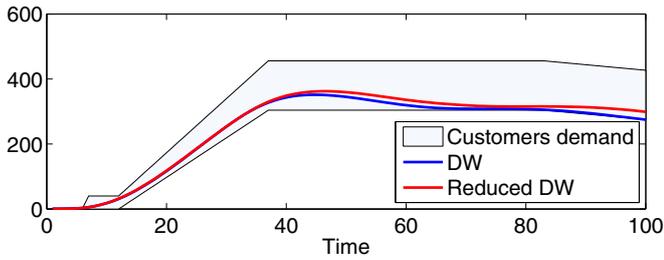


Fig. 1. Overall power production of energy system consisting of 75 power units in closed-loop simulations. The grey area defines the customer demand interval. The control problem is solved applying both classic Dantzig-Wolfe, blue graph, and novel reduced decomposition, red plot.

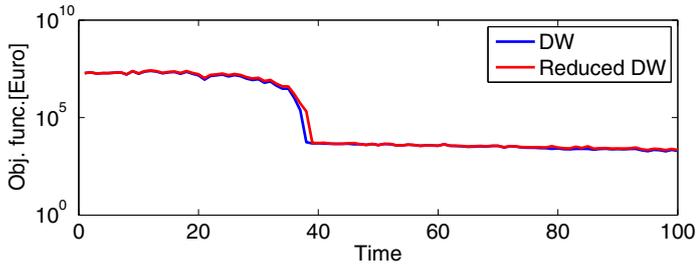


Fig. 2. Objective function (total production costs) of energy system consisting of 75 power units in closed-loop simulations. Simulations run both classic Dantzig-Wolfe, blue graph, and novel reduced decomposition, red plot.

and reduced Dantzig-Wolfe decomposition, as described in the previous section. These are implemented in MATLAB in closed-loop simulations. Section 3.1 introduces the need of initial basic feasible solutions for the RMP. We apply the warm-start technique described in Standardi et al. (2012). Moreover, as mentioned in Section 2, we assume to have the forecasts for weather  $d_k$ , costs  $\hat{c}_{i,k}$  and penalties  $\hat{\rho}$ . The output bounds  $\hat{r}_{\min,k}$  and  $\hat{r}_{\max,k}$  represent customer demand interval; these power consumption forecasts are taken from the Nord Pool Spot Power Market and the bounds are derived according to real data from Nord Pool Spot (2012).

Our case studies are energy systems consisting of multiple power units. In particular, these controllable units might represent thermal power plants, gas turbines and diesel generators. We model these units as described in Edlund et al. (2010)

$$Z_i(s) = \frac{1}{(\tau_i s + 1)^3} (U_i(s) + D_i(s)) \quad (18)$$

where  $U_i(s)$  denotes the control signal,  $D_i(s)$  is the process noise, and  $Z_i(s)$  is the power produced. We consider six energy systems consisting of: 25, 50, 75, 100, 125 and 150 power units. Furthermore, the time horizon is  $N = 70$ , sampling time is 1 second and time steps are 100. The reduced Dantzig-Wolfe decomposition computes the control trajectories for each power units of the energy system considered. We observe from Figure 1 that the overall power production given by the implementation of the classic Dantzig-Wolfe decomposition satisfies customer demand as well. As expected, the suboptimal control sequence

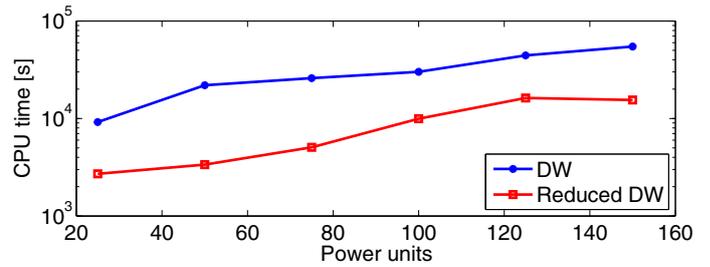


Fig. 3. Computation time for the classic Dantzig-Wolfe, blue graph, and for the reduced version, red plot, Vs. Number of power units in the energy system.

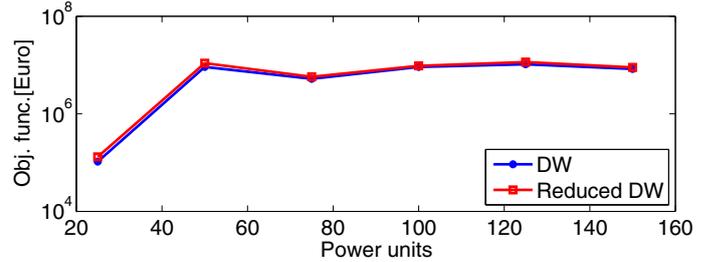


Fig. 4. Objective function optimal values for the classic Dantzig-Wolfe, blue plot, and for the reduced version, red plot, Vs. Number of power units in the energy system.

given by the reduced Dantzig-Wolfe decomposition makes the overall power production meet the customer demand. For the sake of completeness, results show the effect of sub optimality in the deterioration of the objective function. Figure 2 shows the objective function values of this case study including 75 power units. The reduced Dantzig-Wolfe decomposition is more expensive as it has higher costs (red graph in the plot) than the classic Dantzig-Wolfe decomposition (blue graph).

Let us consider all six case studies. Figure 3 shows that the reduced Dantzig-Wolfe decomposition quickens the controller, reducing computation times for all the study cases. Moreover, Figure 4 illustrates the objective function optimal values given by the reduced Dantzig-Wolfe decomposition and the classic algorithm. In order to examine the algorithm performances, Figure 5 shows the percent decrease in the computational time and the percent change in the optimal values of the objective function. The computation times decrease up to 80%, while the deterioration in the objective function optimal value exceeds 20% (upper dashed line) for only 1 case study. Moreover, the percent deterioration of the objective function is often below 10% (lower dashed line), even when the number of power unit in the case study increases.

## 6. CONCLUSIONS

In this paper we have introduced a reduced Dantzig-Wolfe decomposition for linear Economic MPC controllers. The problem formulation has been formulated as a linear economic MPC strategy to control energy systems consisting of multiple independent units. We have briefly described the classic Dantzig-Wolfe optimization and then derived the reduced version. We have demonstrated how the novel reduced Dantzig-Wolfe decomposition supports

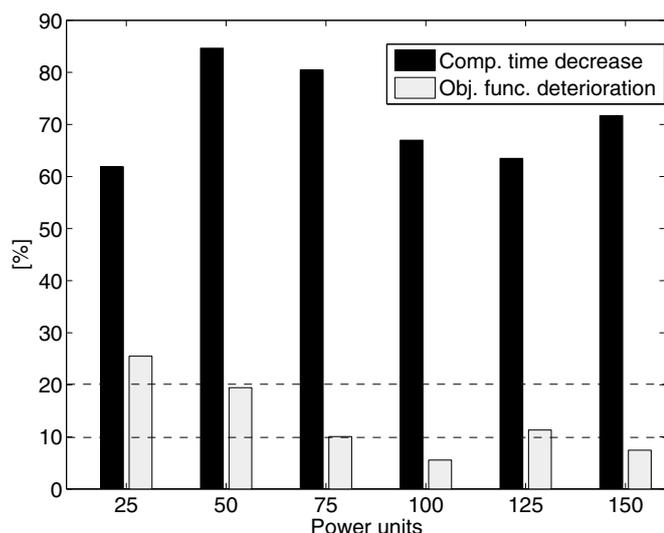


Fig. 5. Percent changes, Vs. Number of power units in the energy system. Black bars denote percent decrease in computation time, while grey bars denote percent change in objective function optimal values. These reductions in computational time and optimal values are the reductions compared with the classic Dantzig-Wolfe algorithm.

suboptimal and feasible solution for LPs; moreover, we have illustrated the stability of the proposed algorithm. We have collected the reduced Dantzig-Wolfe decomposition computation results for six case studies in closed-loop simulations. Results have demonstrated that the proposed reduced Dantzig-Wolfe decomposition speeds up the algorithm. Our study represents a new approach to the solution of linear MPC and improves its applicability. The proposed algorithm guarantees feasibility and stability computing a suboptimal solution. The reduced Dantzig-Wolfe decomposition can be applied to a wide range of systems and it has potential in areas such as independent units building up a larger system.

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APPENDIX

B

## Paper B - Journal of Process Control, 2014

**A Dantzig–Wolfe Decomposition Algorithm for Linear  
Economic Model Predictive Control of Dynamically  
Decoupled Subsystems**

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# A Dantzig–Wolfe decomposition algorithm for linear economic model predictive control of dynamically decoupled subsystems



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## ABSTRACT

This paper presents a warm-started Dantzig–Wolfe decomposition algorithm tailored to economic model predictive control of dynamically decoupled subsystems. We formulate the constrained optimal control problem solved at each sampling instant as a linear program with state space constraints, input limits, input rate limits, and soft output limits. The objective function of the linear program is related directly to the cost of operating the subsystems, and the cost of violating the soft output constraints. Simulations for large-scale economic power dispatch problems show that the proposed algorithm is significantly faster than both state-of-the-art linear programming solvers, and a structure exploiting implementation of the alternating direction method of multipliers. It is also demonstrated that the control strategy presented in this paper can be tuned using a weighted  $\ell_1$ -regularization term. In the presence of process and measurement noise, such a regularization term is critical for achieving a well-behaved closed-loop performance.

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## 1. Introduction

Conventionally, the optimal control problem (OCP) solved in model predictive control (MPC) is formulated as a convex program that penalizes deviations between the controlled output and a setpoint [1–4]. While this approach ensures that the setpoint is reached in a reasonable amount of time, it does not guarantee that the transition between setpoints is performed in an economically efficient way. To overcome this problem, MPC has been extended to solve OCPs with more general cost functions, providing a systematic method for optimizing economic performance [5–11]. Stability and other properties of such economic MPC (EMPC) schemes have been addressed in [5–9,12–14].

The main contribution of this paper is a Dantzig–Wolfe decomposition algorithm for EMPC of dynamically decoupled subsystems that solves the OCP in an efficient and reliable way. As the control law is computed in real-time, such an algorithm allows EMPC to be employed even for applications with thousands of subsystems. In particular, we consider an  $\ell_1$ -regularized linear type of OCP with input constraints, input rate constraints and soft

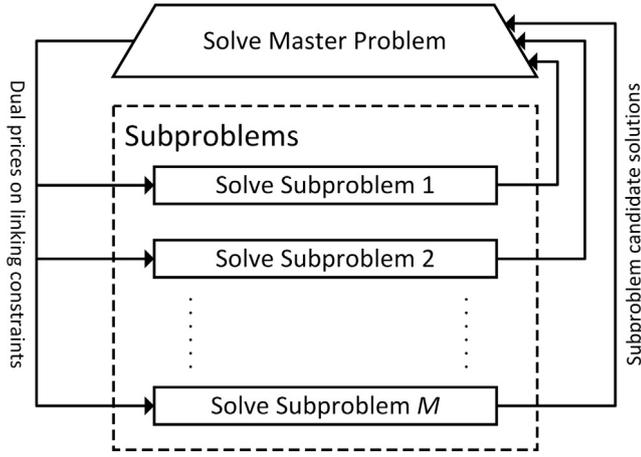
output constraints. Each subsystem is governed by a discrete state space model. The coupling of the subsystems occurs through a set of aggregated variables.

The Dantzig–Wolfe decomposition algorithm, presented in this paper, exploits that dynamically decoupled subsystems give rise to a block-angular structure in the OCP constraint matrix. This allows the OCP to be decomposed into a master problem and a number of subproblems [15–17]. The master problem includes a set of linking constraints which couples the subsystems, whereas the subproblems are concerned only with the individual subsystems. Using an iterative approach illustrated in Fig. 1, the decomposed problem can be solved via a delayed column generation procedure. Such techniques have previously been applied to conventional norm-based MPC in [18–20].

The block-angular constraint matrix structure appears for dynamically decoupled subsystems with linking constraints [21]. Dynamic multi-plant models as well as dynamic multi-product models are examples of such models [22]. Dynamic multi-plant models occur e.g. in the production planning for multiple refineries [23]. For process systems, dynamically decoupled systems with linking constraints occur when independent units are connected to shared process equipment such as pipes. A boiler-turbine system producing high pressure (HP), middle pressure (MP) and low pressure (LP) steam as well as electricity is a common example of a system that can be modeled as dynamically decoupled subsystems (the boilers) that have linking constraints (the demand for

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**Fig. 1.** Flowchart of the delayed column generation procedure used in the Dantzig–Wolfe decomposition algorithm for EMPC of dynamically decoupled subsystems with linking constraints. In each iteration, dual prices associated with the linking constraints are obtained by solving the master problem. These prices are used by the subsystems to compute an updated solution that improves the overall objective function.

various steam qualities and electrical power) [18,24]. In upstream offshore oil production, the compressors and pumps of a number of production wells share the pipeline, separators and compressors to bring the oil onshore [25,26]. This is also an example of a system that can be modeled as dynamically decoupled subsystems with linking constraints. Smart Grid systems in which a number of independent energy producers and consumers are controlled to balance power production and consumption represent yet another instance of dynamically decoupled systems with linking constraints [20,27]. The temperature regulation of multi-room buildings can also be formulated as dynamically decoupled subsystems with linking constraints [28]. As is evident by this list of examples, dynamically decoupled subsystems with linking constraints are common in process systems.

To test a MATLAB implementation of the Dantzig–Wolfe decomposition algorithm, denoted  $DW_{EMPC}$ , a simple energy systems management case study is presented. We show that as more units are added to a network of controllable generators,  $DW_{EMPC}$  becomes increasingly favorable over state-of-the-art sparse linear programming solvers provided by Gurobi, CPLEX, and MOSEK. It is further demonstrated that a nearly optimal solution can be acquired, even if  $DW_{EMPC}$  is terminated early. This is an attractive property in real-time applications such as EMPC, since only a limited amount of time is available for solving the OCP.

In addition to the general purpose solvers,  $DW_{EMPC}$  is compared to a structure exploiting implementation of the alternating direction method of multipliers (ADMM) [29–32], denoted  $ADMM_{EMPC}$ , with similar parallelization capabilities to  $DW_{EMPC}$ . Simulations illustrate that unless a highly suboptimal control performance is tolerated,  $DW_{EMPC}$  outperforms  $ADMM_{EMPC}$  with a significant margin. Results also show that for both algorithms, a simple warm-start strategy yields a substantial improvement over cold start, and that the performance of this strategy increases with the weights on the  $\ell_1$ -regularization term. Inclusion of the regularization term is critical for the controller performance in the face of stochastic process and measurement noise as well as model-plant mismatch.

### 1.1. Paper organization

We have organized the paper as follows. In Section 2, the OCP solved in this paper is introduced. We decompose the problem using Dantzig–Wolfe decomposition in Section 3, and a column

generation procedure for solving the decomposed problem is presented. Section 4 describes a distributed implementation of ADMM for solving the OCP. Section 5 reports performance indicators for the proposed algorithms. These performance indicators are computed using a conceptual energy systems management case study. Concluding remarks are given in Section 6.

## 2. Problem definition

We consider  $M$  dynamically decoupled discrete state space models in the form

$$x_{j,k+1} = A_j x_{j,k} + B_j u_{j,k}, \quad j \in \mathcal{M}, \quad (1a)$$

$$y_{j,k} = C_j x_{j,k}, \quad j \in \mathcal{M}, \quad (1b)$$

where  $\mathcal{M} = \{1, 2, \dots, M\}$ . The state space matrices are denoted by  $(A_j, B_j, C_j)$ , the states by  $x_{j,k} \in \mathbb{R}^{n_x(j)}$ , the inputs by  $u_{j,k} \in \mathbb{R}^{n_u(j)}$ , and the outputs by  $y_{j,k} \in \mathbb{R}^{n_y(j)}$ . Moreover, we define the aggregated variables

$$y_{T,k} = \sum_{j \in \mathcal{M}} \gamma_j y_{j,k} = \sum_{j \in \mathcal{M}} \gamma_j C_j x_{j,k}, \quad (2)$$

in which  $\gamma_j \in \mathbb{R}^{n_{yT} \times n_{y(j)}}$  are subsystem multipliers.

The OCP defining the EMPC control law for the subsystems (1), is in this paper defined as

$$\min_{u,x,y,y_T,\rho,\gamma} \psi = \psi_{eco} + \psi_{reg}, \quad (3a)$$

with

$$\psi_{eco} = \sum_{k \in \mathcal{N}_0} \left( q_{k+1}^T \rho_{k+1} + \sum_{j \in \mathcal{M}} p_{j,k}^T u_{j,k} + r_{j,k+1}^T \gamma_{j,k+1} \right), \quad (3b)$$

$$\psi_{reg} = \sum_{k \in \mathcal{N}_0} \sum_{j \in \mathcal{M}} w_{j,k} \|\Delta u_{j,k}\|_1, \quad (3c)$$

and subject to the constraints

$$x_{j,k+1} = A_j x_{j,k} + B_j u_{j,k}, \quad k \in \mathcal{N}_0, \quad j \in \mathcal{M}, \quad (3d)$$

$$y_{j,k} = C_j x_{j,k}, \quad k \in \mathcal{N}_1, \quad j \in \mathcal{M}, \quad (3e)$$

$$y_{T,k} = \sum_{j \in \mathcal{M}} \gamma_j C_j x_{j,k}, \quad k \in \mathcal{N}_1, \quad (3f)$$

$$\underline{u}_{j,k} \leq u_{j,k} \leq \bar{u}_{j,k}, \quad k \in \mathcal{N}_0, \quad j \in \mathcal{M}, \quad (3g)$$

$$\Delta u_{j,k} \leq \Delta u_{j,k} \leq \Delta \bar{u}_{j,k}, \quad k \in \mathcal{N}_0, \quad j \in \mathcal{M}, \quad (3h)$$

$$\underline{y}_{j,k} - \gamma_{j,k} \leq y_{j,k} \leq \bar{y}_{j,k} + \gamma_{j,k}, \quad k \in \mathcal{N}_1, \quad j \in \mathcal{M}, \quad (3i)$$

$$0 \leq \gamma_{j,k} \leq \bar{\gamma}_{j,k}, \quad k \in \mathcal{N}_1, \quad j \in \mathcal{M}, \quad (3j)$$

$$\underline{y}_{T,k} - \rho_k \leq y_{T,k} \leq \bar{y}_{T,k} + \rho_k, \quad k \in \mathcal{N}_1, \quad (3k)$$

$$0 \leq \rho_k \leq \bar{\rho}_k, \quad k \in \mathcal{N}_1, \quad (3l)$$

where  $\leq$  and  $\geq$  denote element-wise inequalities. The input rate is defined as  $\Delta u_k = u_k - u_{k-1}$  and  $\mathcal{N}_i = \{0 + i, 1 + i, \dots, N - 1 + i\}$ , with  $N$  being the length of the control and prediction horizon.

The input data to (3) are the input limits,  $(\underline{u}_{j,k}, \bar{u}_{j,k})$ , the input rate limits,  $(\Delta \underline{u}_{j,k}, \Delta \bar{u}_{j,k})$ , the subsystem output limits,  $(\underline{y}_{j,k}, \bar{y}_{j,k})$ , the aggregated variable limits,  $(\underline{y}_{T,k}, \bar{y}_{T,k})$ , the input prices,  $p_{j,k}$ , the price for violating the subsystem output limits,  $r_{j,k}$ , and the price for violating the aggregated variable limits,  $q_k$ . The slack variables,  $\gamma_{j,k}$  and  $\rho_k$ , account for the violation of the soft output constraints. We impose upper limits,  $(\bar{\gamma}_{j,k}, \bar{\rho}_k)$ , on these variables, as this simplifies later computations considerably.

The objective function (3a) consists of an economic term (3b) and a regularization term (3c). The economic term (3b) represents the cost of operating the subsystems and the cost of violating the soft output constraints. The regularization term (3c) is included to obtain a well behaved solution. In our paper, the regularization term is formulated as a weighted  $\ell_1$ -penalty on the input rate. Using an  $\ell_1$ -penalty ensures that the resulting OCP is a linear program that can be solved using Dantzig–Wolfe decomposition.

**Remark 1.** An alternative way of expressing the OCP objective function (3a) is as a trade-off between the economic term and the regularization term, such that

$$\psi = \alpha\psi_{\text{eco}} + (1 - \alpha)\psi_{\text{reg}}, \quad \alpha \in [0, 1], \quad (4)$$

where  $\alpha$  is a user-defined parameter. Amrit et al. [12] discuss the trade-off between the economic term and an  $\ell_2$ -regularization term.

The regularization term (3c) is a special case of

$$\psi_{\text{reg}} = \sum_{k \in \mathcal{N}_0} \left( \sum_{j \in \mathcal{M}} w_{j,k+1}^x \|x_{j,k+1} - \bar{x}_{j,k+1}\|_1 + \sum_{j \in \mathcal{M}} w_{j,k}^u \|u_{j,k} - \bar{u}_{j,k}\|_1 + w_{j,k}^{\Delta u} \|\Delta u_{j,k}\|_1 \right), \quad (5)$$

in which  $\{\bar{x}_{j,k+1}, \bar{u}_{j,k}\}_{k \in \mathcal{N}_0, j \in \mathcal{M}}$  are target values that may be computed by a target calculator or a real-time optimization layer. An objective function consisting only of (5) corresponds to conventional  $\ell_1$  norm-based MPC. Edlund et al. [20] solves such problems using Dantzig–Wolfe decomposition.

**Remark 2.** The objective function (4) is similar to the mean-variance-based economic objective function introduced in [33] for production optimization in an oil field. For a random cost variable,  $\psi_{\text{eco}}$ , the mean-variance optimization criterion is

$$\psi_{\text{MV}} = \alpha E[\psi_{\text{eco}}] + (1 - \alpha)V[\psi_{\text{eco}}].$$

$E[\psi_{\text{eco}}]$  is the cost expectation and  $V[\psi_{\text{eco}}]$  is the cost variance. In (4),  $\psi_{\text{eco}}$  can be interpreted as a certainty-equivalent approximation of the mean of the random cost variable,  $\psi_{\text{eco}}$ , while the regularization term,  $\psi_{\text{reg}}$ , is included to make the controller less sensitive to noise. The key advantage in using the deterministic formulation (4) is that the computational load is significantly reduced compared to a mean-variance approach based on Monte Carlo simulations. Other measures of risk than the mean-variance formulation that can be used to regularize the solution are Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) [34].

**Remark 3.** The Dantzig–Wolfe decomposition algorithm is an algorithm for solving linear programs. Consequently, the approach described in this paper is limited to solve OCPs with a linear objective function, linear dynamics, and linear constraints. Rao and Rawlings [35] provide a number of penalty functions that can be expressed as linear programs. Penalty functions based on  $\ell_1$  norms, such as (3c) and (5), as well as  $\ell_\infty$  norms can be expressed as linear programs. Piecewise linear approximations accommodate the need for solving OCPs with more general convex economic cost functions [26,36,37]. The disadvantage of using piecewise linear approximations is that the size of the resulting linear program may increase considerably.

**Remark 4.** The expression (2) for the aggregated variables is tailored to dynamically decoupled subsystems that collaborate to

meet a common objective. The expression (2) is a special case of the more general expression

$$y_{T,k} = \sum_{j \in \mathcal{M}} \gamma_j^y y_{j,k} + \gamma_j^u u_{j,k}, \quad k \in \mathcal{N}_1, \quad (6)$$

for the aggregated variables. The general expression (6) may be used to describe couplings between subsystems (e.g. interactions between 1) process units in a process system; and 2) the transmission lines coupling producers and consumers in a power system) [18]. When the number of aggregated variables increases, the number of linking constraints increases. The Dantzig–Wolfe decomposition algorithm is most efficient when the number of linking constraints is small compared to the total number of constraints.

### 2.1. Compact formulation

By eliminating the states using Eq. (1a), we can write the output Eq. (1b), as

$$y_{j,k} = C_j A_j^k x_{j,0} + \sum_{i \in \mathcal{N}_0} H_{j,k-i} u_{j,i}, \quad j \in \mathcal{M}, \quad k \in \mathcal{N}_1,$$

where the impulse response coefficients are given by

$$H_{j,k} = C_j A_j^{k-1} B_j, \quad j \in \mathcal{M}, \quad k \in \mathcal{N}_1.$$

Consequently

$$y_{T,k} = \sum_{j \in \mathcal{M}} \left( \gamma_j C_j A_j^k x_{j,0} + \sum_{i \in \mathcal{N}_0} \gamma_j H_{j,k-i} u_{j,i} \right), \quad k \in \mathcal{N}_1.$$

Define the vectors

$$y_j = [y_{j,1}^T \quad y_{j,2}^T \quad \cdots \quad y_{j,N}^T]^T, \quad j \in \mathcal{M}, \quad (7a)$$

$$u_j = [u_{j,0}^T \quad u_{j,1}^T \quad \cdots \quad u_{j,N-1}^T]^T, \quad j \in \mathcal{M}, \quad (7b)$$

and the matrices

$$\Gamma_j = \begin{bmatrix} H_{j,1} & 0 & \cdots & 0 \\ H_{j,2} & H_{j,1} & & \\ \vdots & \vdots & \ddots & \\ H_{j,N} & H_{j,N-1} & \cdots & H_{j,1} \end{bmatrix}, \quad \Phi_j = \begin{bmatrix} C_j A_j \\ C_j A_j^2 \\ \vdots \\ C_j A_j^{N-1} \end{bmatrix},$$

for  $j \in \mathcal{M}$ . Then, for each of the subsystems (7a)

$$y_j = \Gamma_j u_j + \Phi_j x_{j,0}, \quad j \in \mathcal{M}. \quad (8)$$

By introducing  $\tilde{\Gamma}_j$  and  $\tilde{\Phi}_j$  accordingly, it follows that  $y_T = \sum_{j \in \mathcal{M}} \tilde{\Gamma}_j u_j + \tilde{\Phi}_j x_{j,0}$ .

The notation is simplified further with

$$u_j = [u_{j,0}^T \quad u_{j,1}^T \quad \cdots \quad u_{j,N-1}^T]^T, \quad j \in \mathcal{M},$$

$$\bar{u}_j = [\bar{u}_{j,0}^T \quad \bar{u}_{j,1}^T \quad \cdots \quad \bar{u}_{j,N-1}^T]^T, \quad j \in \mathcal{M},$$

and similarly we define  $\Delta u_j$ ,  $\Delta \bar{u}_j$ ,  $y_j$ ,  $\bar{y}_j$ ,  $y_T$ ,  $\bar{y}_T$ ,  $\gamma_j$ ,  $\bar{\gamma}_j$ ,  $\rho$ ,  $\bar{\rho}$ ,  $\eta_j$ ,  $q$ ,  $p_j$ ,  $r_j$ ,  $w_j$  and  $\gamma_j$ . Using these definitions, the OCP (3) may be written in the form

$$\min_{u, \rho, \gamma, \eta} \psi = q^T \rho + \sum_{j \in \mathcal{M}} p_j^T u_j + r_j^T \gamma_j + w_j^T \eta_j \quad (9a)$$

subject to a set of decoupled constraints

$$u_j \leq u_j \leq \bar{u}_j, \quad j \in \mathcal{M}, \quad (9b)$$

$$\Delta u_j - I_0 u_{j,-1} \leq \Lambda u_j \leq \Delta \bar{u}_j - I_0 u_{j,-1}, \quad j \in \mathcal{M}, \quad (9c)$$

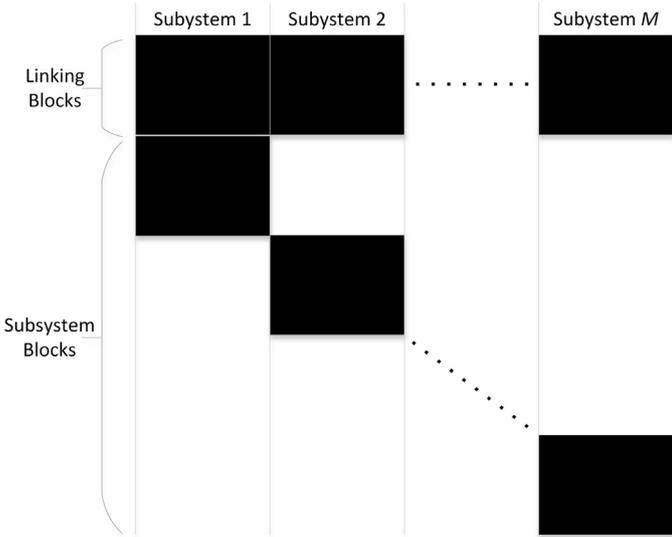


Fig. 2. The block-angular structure of the constraint matrix in (9). The efficiency of the Dantzig–Wolfe decomposition method depends on this structure.

$$y_j - \gamma_j \leq \Gamma_j u_j + \Phi_j x_{j,0} \leq \bar{y}_j + \gamma_j, \quad j \in \mathcal{M}, \tag{9d}$$

$$I_0 u_{j,-1} - \eta_j \leq \Lambda u_j \leq I_0 u_{j,-1} + \eta_j, \quad j \in \mathcal{M}, \tag{9e}$$

$$0 \leq \eta_j \leq \bar{\eta}, \quad j \in \mathcal{M}, \tag{9f}$$

$$0 \leq \gamma_j \leq \bar{\gamma}, \quad j \in \mathcal{M}, \tag{9g}$$

and a set of linking constraints

$$y_T - \rho \leq \sum_{j \in \mathcal{M}} \tilde{\Gamma}_j u_j + \tilde{\Phi}_j x_{j,0} \leq \bar{y}_T + \rho. \tag{9h}$$

$$0 \leq \rho \leq \bar{\rho}, \tag{9i}$$

where  $\Lambda$  and  $I_0$  are defined as

$$\Lambda_j = \begin{bmatrix} I & & & & & \\ -I & I & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & -I & I \end{bmatrix}, \quad I_0 = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We remark that (9e) and (9f) imply that  $\eta_{j,k} \geq |\Delta u_{j,k}|$ . Note also, that the structure of the constraint matrix in (9), can be stated in the block-angular form illustrated in Fig. 2.

In particular, (9) is written as

$$\min_z \psi = \sum_{j \in \bar{\mathcal{M}}} c_j^T z_j, \tag{10a}$$

$$\text{s.t. } G_j z_j \geq g_j, \quad j \in \bar{\mathcal{M}}, \tag{10b}$$

$$\sum_{j \in \bar{\mathcal{M}}} H_j z_j \geq h, \tag{10c}$$

with  $\bar{\mathcal{M}} = 1, 2, \dots, M + 1$ , and

$$z_j = [u_j^T \quad \gamma_j^T \quad \eta_j^T]^T, \quad c_j = [p_j^T \quad r_j^T \quad w_j^T]^T, \quad j \in \mathcal{M},$$

$$z_{M+1} = \rho^T, \quad c_{M+1} = q^T.$$

(10b) represents the decoupled constraints (9b)–(9g), and (10c) represents the linking constraints (9h) and (9i).

The data structures in (10) are defined as

$$G_j = \begin{bmatrix} \bar{G}_j \\ -\bar{G}_j \end{bmatrix}, \quad g_j = \begin{bmatrix} \underline{g}_j \\ -\bar{g}_j \end{bmatrix}, \quad H_j = \begin{bmatrix} \bar{H}_j \\ -\bar{H}_j \end{bmatrix}, \quad h = \begin{bmatrix} \underline{h} \\ -\bar{h} \end{bmatrix},$$

in which

$$[\bar{G}_j \quad \underline{g}_j \quad \bar{g}_j] = \begin{bmatrix} I & 0 & 0 & u_j & \bar{u}_j \\ \Lambda & 0 & 0 & \Delta \underline{u}_j & \Delta \bar{u}_j \\ \Gamma_j & I & 0 & y_j & \infty \\ \Gamma_j & -I & 0 & -\infty & \tilde{y}_j \\ 0 & I & 0 & 0 & \bar{\gamma}_j \\ -\Lambda & 0 & I & -I_0 u_{j,-1} & \infty \\ \Lambda & 0 & I & I_0 u_{j,-1} & \infty \\ 0 & 0 & I & 0 & \bar{\eta}_j \end{bmatrix},$$

$$[\bar{H}_j \quad \underline{h} \quad \bar{h}] = \begin{bmatrix} \tilde{\Gamma}_j & 0 & y_T & \infty \\ \tilde{\Gamma}_j & 0 & -\infty & \tilde{y}_T \end{bmatrix},$$

for  $j \in \mathcal{M}$ , with

$$\underline{y}_T = y_T - \sum_{j \in \mathcal{M}} \tilde{\Phi}_j x_{j,0}, \quad \tilde{y}_T = \bar{y}_T - \sum_{j \in \mathcal{M}} \tilde{\Phi}_j x_{j,0},$$

$$\underline{y}_j = y_j - \Phi_j x_{j,0}, \quad \tilde{y}_j = \bar{y}_j - \Phi_j x_{j,0}, \quad j \in \mathcal{M},$$

$$\Delta u_j = \Delta \underline{u}_j + I_0 u_{j,-1}, \quad \Delta \bar{u}_j = \Delta \bar{u}_j + I_0 u_{j,-1}, \quad j \in \mathcal{M},$$

In the special case  $j = M + 1$

$$[\bar{G}_{M+1} \quad \underline{g}_{M+1} \quad \bar{g}_{M+1}] = [I \quad 0 \quad \bar{\rho}],$$

$$\bar{H}_{M+1} = [I \quad -I]^T.$$

**Remark 5.** We only use (10) to have a convenient notation. In the actual solution of all the linear and quadratic programs reported in this paper, the bound constraints are exploited.

### 3. Dantzig–Wolfe decomposition

Dantzig–Wolfe decomposition utilizes the fact that a convex set can be characterized by its extreme points and its extreme rays [15–17]. In particular, for each  $j \in \mathcal{M}$ , the set of points satisfying the decoupled constraints (10b) may be written as

$$\mathcal{G}_j = \{z_j | G_j z_j \geq g_j\} = \left\{ z_j | z_j = \sum_{i \in \mathcal{P}} \lambda_i^j z_i^j, \sum_{i \in \mathcal{P}} \lambda_i^j = 1, \lambda_i^j \geq 0 \quad \forall i \in \mathcal{P} \right\}, \tag{11}$$

where  $z_i^j$  are the extreme points of  $\mathcal{G}_j$ , and  $\lambda_i^j$  are convex combination multipliers. Note that since each of the sets  $\mathcal{G}_j$  are bounded, extreme rays are not needed in their representation.  $\mathcal{P}$  is a set defined such that all extreme points of the set defined by (10b) can be represented as  $z^i = [z_j^i]_{j \in \bar{\mathcal{M}}} = [z_1^i; z_2^i; \dots; z_{|\bar{\mathcal{M}}}|^i]$  for  $i \in \mathcal{P}$ . Note that with this definition, the same extreme point,  $z_j^i$ , may appear several times in (11). This mathematical representation, with the possibility that the same subproblem extreme point,  $z_j^i$ , is represented several times, facilitates a computationally efficient implementation of the Dantzig–Wolfe decomposition algorithm.

By replacing the decision variables in (10) by convex combination multipliers, we obtain the master problem formulation

$$\min_{\lambda} \psi = \sum_{j \in \overline{\mathcal{M}}} \sum_{i \in \mathcal{P}} c_j^i \lambda_j^i, \quad (12a)$$

$$\text{s.t.} \quad \sum_{j \in \overline{\mathcal{M}}} \sum_{i \in \mathcal{P}} H_j^i \lambda_j^i \geq h, \quad (12b)$$

$$\sum_{i \in \mathcal{P}} \lambda_j^i = 1, \quad j \in \overline{\mathcal{M}}, \quad (12c)$$

$$\lambda_j^i \geq 0, \quad j \in \overline{\mathcal{M}}, i \in \mathcal{P}, \quad (12d)$$

where we have defined

$$H_j^i = H_j z_j^i, \quad j \in \overline{\mathcal{M}}, i \in \mathcal{P}, \quad (13a)$$

$$c_j^i = c_j^T z_j^i, \quad j \in \overline{\mathcal{M}}, i \in \mathcal{P}. \quad (13b)$$

Given a solution,  $\lambda^*$ , to the master problem (12), a solution to the original problem (10) can be obtained as

$$z_j^* = \sum_{i \in \mathcal{P}} (\lambda^*)_j^i z_j^i, \quad j \in \overline{\mathcal{M}}.$$

The number of extreme points,  $|\mathcal{P}|$ , can increase exponentially with the size of the original problem. In such cases, it is computationally inefficient to solve the master problem directly. In the following section, we overcome this issue by employing a column generation procedure that replaces  $\mathcal{P}$  by a subset  $\tilde{\mathcal{P}}$ .

### 3.1. Column generation procedure

The dual linear program of (12) may be stated as

$$\max_{\alpha, \beta} \phi = \alpha^T h + \sum_{j \in \overline{\mathcal{M}}} \beta_j, \quad (14a)$$

$$\text{s.t.} \quad (H_j^i)^T \alpha + \beta_j \leq c_j^i, \quad j \in \overline{\mathcal{M}}, i \in \mathcal{P}, \quad (14b)$$

$$\alpha \geq 0. \quad (14c)$$

$\alpha \in \mathbb{R}^{4N}$  and  $\beta \in \mathbb{R}^{M+1}$  are the Lagrange multipliers associated with the linking constraints (12b) and the convexity constraints (12c), respectively.

The necessary and sufficient optimality conditions for (12) and (14) are

$$\sum_{j \in \overline{\mathcal{M}}} \sum_{i \in \mathcal{P}} H_j^i \lambda_j^i \geq h, \quad (15a)$$

$$\sum_{i \in \mathcal{P}} \lambda_j^i = 1, \quad j \in \overline{\mathcal{M}}, \quad (15b)$$

$$\lambda_j^i \geq 0, \quad j \in \overline{\mathcal{M}}, i \in \mathcal{P}, \quad (15c)$$

$$c_j^i - (H_j^i)^T \alpha - \beta_j \geq 0, \quad j \in \overline{\mathcal{M}}, i \in \mathcal{P}, \quad (15d)$$

$$\alpha \geq 0, \quad (15e)$$

$$\lambda_j^i (c_j^i - (H_j^i)^T \alpha - \beta_j) = 0, \quad j \in \overline{\mathcal{M}}, i \in \mathcal{P}, \quad (15f)$$

Proposition 1 shows that a solution to the master problem (12) can be obtained by solving a restricted master problem in which  $\mathcal{P}$  in (12) is replaced by  $\tilde{\mathcal{P}} \subseteq \mathcal{P}$ . This implies that a solution to (12) can be obtained by solving a linear program that is often much smaller than (12).

**Proposition 1.** Let  $\tilde{\mathcal{P}} \subseteq \mathcal{P}$  and define  $(\tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$  as a primal-dual solution to (12) and (14) with  $\mathcal{P}$  replaced by  $\tilde{\mathcal{P}}$ . Define  $(\lambda^*, \alpha^*, \beta^*)$  as

$$\begin{aligned} \alpha^* &= \alpha, \\ \beta_j^* &= \beta_j, & j \in \overline{\mathcal{M}}, \\ (\lambda^*)_j^i &= \begin{cases} \tilde{\lambda}_j^i & \text{if } i \in \tilde{\mathcal{P}} \\ 0 & \text{if } i \in \mathcal{P} \setminus \tilde{\mathcal{P}} \end{cases}, & j \in \overline{\mathcal{M}}, i \in \mathcal{P}. \end{aligned}$$

If the optimal objective value of the subproblem

$$\min_{\tilde{z}_j} \varphi_j = (c_j - H_j^T \alpha^*)^T \tilde{z}_j - \beta_j^* \quad (16a)$$

$$\text{s.t.} \quad G_j \tilde{z}_j \geq g_j, \quad (16b)$$

is non-negative for each  $j \in \overline{\mathcal{M}}$ , i.e.  $\varphi_j^* \geq 0 \forall j \in \overline{\mathcal{M}}$ , then  $(\lambda^*, \alpha^*, \beta^*)$  satisfies the necessary and sufficient optimality conditions (15),  $\lambda^*$  is a minimizer of (12), and  $(\alpha^*, \beta^*)$  is a maximizer of (14).

**Proof.** The solution  $(\lambda^*, \alpha^*, \beta^*)$  satisfies (15a) since

$$\sum_{j \in \overline{\mathcal{M}}} \sum_{i \in \mathcal{P}} H_j^i (\lambda^*)_j^i = \sum_{j \in \overline{\mathcal{M}}} \sum_{i \in \tilde{\mathcal{P}}} H_j^i \tilde{\lambda}_j^i \geq h,$$

which follows from the definition of  $(\tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$ . Similarly, it can be verified that the conditions (15b), (15c), (15e) and (15f) are fulfilled.

Provided that  $(\lambda^*, \alpha^*, \beta^*)$  is optimal, (15d) yields

$$c_j^i - (H_j^i)^T \alpha^* - \beta_j^* = (c_j - H_j^T \alpha^*)^T z_j^i - \beta_j^* \geq 0, \quad (17)$$

for all  $j \in \overline{\mathcal{M}}$  and  $i \in \mathcal{P}$ . By construction of  $(\lambda^*, \alpha^*, \beta^*)$ , (17) is satisfied for all  $i \in \tilde{\mathcal{P}}$ . To check that the condition holds for all  $i \in \mathcal{P} \setminus \tilde{\mathcal{P}}$ , we consider the optimization problem (16). Since this linear program minimizes the left hand side of (17) over all possible extreme points,  $\tilde{z}_j$ , of  $G_j$ ,  $(\lambda^*, \alpha^*, \beta^*)$  also satisfies the remaining optimality condition (17) if  $\varphi_j$  is non-negative for all  $j \in \overline{\mathcal{M}}$ .  $\square$

Algorithm 1 summarizes a column generation procedure based on Proposition 1.

**Remark 6.** The problem (16) is an OCP with linear constraints and a linear objective function. Refs. [10,11,38,39] provide an efficient

#### Algorithm 1. Column generation procedure for solution of (12).

```

Require:  $(i_{\max}, \varepsilon), \{z_j^0\}_{j \in \overline{\mathcal{M}}}$ 
 $i = 0, \text{ converged} = \text{false}$ 
while not converged and  $i < i_{\max}$  do
   $\tilde{\mathcal{P}} = \{0, 1, \dots, i\}$ 
  COMPUTE PROBLEM DATA
  for  $j \in \overline{\mathcal{M}}$  do
     $H_j^i = H_j z_j^i$ 
     $c_j^i = c_j^T z_j^i$ 
  end for
  SOLVE RESTRICTED MASTER PROBLEM
   $(\phi^*, \lambda^*, \alpha^*, \beta^*) \leftarrow \text{solve (12) with } \mathcal{P} = \tilde{\mathcal{P}}$ 
  SOLVE SUBPROBLEMS
  for  $j \in \overline{\mathcal{M}}$  do
     $(\varphi_j^*, \tilde{z}_j^*) \leftarrow \text{solve (16)}$ 
  end for
  CHECK IF CONVERGED
  if  $|\varphi_j| \geq \varepsilon \quad \forall j \in \overline{\mathcal{M}}$  then
    converged = true
  else
    UPDATE EXTREME POINTS
    for  $j \in \overline{\mathcal{M}}$  do
       $z_j^{i+1} = \tilde{z}_j^*$ 
    end for
  end if
   $i = i + 1$ 
end while

```

Riccati-based homogeneous and self-dual interior-point linear programming algorithm for such problems. Using the optimal interior point solution found by this algorithm, crossover methods can be applied to obtain an optimal extreme point for the column generation procedure [17].

### 3.2. Warm-start

A sequence of closely related OCPs are solved in a moving horizon implementation of EMPC. Therefore, in Algorithm 1 the feasible initial guess of the solution,  $\{z_j^0\}_{j \in \overline{\mathcal{M}}}$ , at the current sampling instant is constructed from the solution at the previous sampling instant.

Given the solution to (16)

$$\begin{aligned} u_j^* &= [u_{j,0}^{*T} \ \cdots \ u_{j,N-1}^{*T}]^T, \quad j \in \mathcal{M}, \\ \gamma_j^* &= [\gamma_{j,1}^{*T} \ \cdots \ \gamma_{j,N}^{*T}]^T, \quad j \in \mathcal{M}, \\ \eta_j^* &= [\eta_{j,0}^{*T} \ \cdots \ \eta_{j,N-1}^{*T}]^T, \quad j \in \mathcal{M}, \\ \rho^* &= [\rho_1^{*T} \ \cdots \ \rho_N^{*T}]^T, \end{aligned}$$

we construct an initial point for the following sampling instant as

$$z_j^0 = \left[ (u_j^0)^T \ (\gamma_j^0)^T \ (\eta_j^0)^T \right]^T, \quad j \in \mathcal{M},$$

where

$$\begin{aligned} u_j^0 &= [u_{j,1}^{*T} \ \cdots \ u_{j,N-1}^{*T} \ (u_{j,N}^0)^T]^T, \quad j \in \mathcal{M}, \\ \gamma_j^0 &= [\gamma_{j,2}^{*T} \ \cdots \ \gamma_{j,N}^{*T} \ (\gamma_{j,N+1}^0)^T]^T, \quad j \in \mathcal{M}, \\ \eta_j^0 &= [\eta_{j,1}^{*T} \ \cdots \ \eta_{j,N-1}^{*T} \ (\eta_{j,N}^0)^T]^T, \quad j \in \mathcal{M}. \end{aligned}$$

Finally

$$z_{M+1}^0 = \rho^0 = [\rho_2^{*T} \ \cdots \ \rho_N^{*T} \ (\rho_{N+1}^0)^T]^T.$$

The original solution values are thus shifted forward in time, and  $u_{j,N}^0, \gamma_{j,N+1}^0, \eta_{j,N}^0$  and  $\rho_{N+1}^0$  are appended to the resulting initial point.

In our implementation, we use

$$\begin{aligned} u_{j,N}^0 &= u_{j,N-1}^*, \quad j \in \mathcal{M}, \\ \eta_{j,N}^0 &= 0, \quad j \in \mathcal{M}. \end{aligned}$$

We use the state space equations (1) and (2) to compute  $y_{j,N+1}^0$  and  $y_{T,N+1}^0$  associated with the input sequence  $\{u_j^0\}_{j \in \mathcal{M}}$ . We construct the initial slack values as

$$\gamma_{j,N+1}^0 = \max(y_{j,N+1} - y_{j,N+1}^0, 0) + \max(y_{j,N+1}^0 - \bar{y}_{j,N+1}, 0),$$

for each  $j \in \mathcal{M}$ , and

$$\rho_{N+1}^0 = \max(y_{T,N+1} - y_{T,N+1}^0, 0) + \max(y_{T,N+1}^0 - \bar{y}_{T,N+1}, 0).$$

As the solution to the OCP often only differs slightly between successive sampling instants, the initial point generated as above provides a warm-start for Algorithm 1.

### 3.3. Cold-start

In the case that no previous solution is available for generating a warm start, a feasible initial guess of the solution,  $\{z_j^0\}_{j \in \overline{\mathcal{M}}}$ , in Algorithm 1 can be constructed by adjusting the slack variables,  $\gamma_j^0$  and  $\rho^0$ . Let  $\{u_j^0\}_{j \in \mathcal{M}}$  be feasible with respect to the input constraints and the input-rate constraints. Such a point is easily obtained in

practice. As an example consider  $u_j^0 = \underline{u}_j$  for each  $j \in \mathcal{M}$ . Then, in a similar way as for the warm-start strategy, we compute

$$\gamma_{j,k}^0 = \max(y_{j,k} - y_{j,k}^0, 0) + \max(y_{j,k}^0 - \bar{y}_{j,k}, 0),$$

$$\rho_k^0 = \max(y_{T,k} - y_{T,k}^0, 0) + \max(y_{T,k}^0 - \bar{y}_{T,k}, 0),$$

where  $k \in \mathcal{N}_1, j \in \mathcal{M}$ . The values,  $y_{j,k}^0$  and  $y_{T,k}^0$  are the subsystem outputs and the aggregated variables associated with the inputs,  $\{u_j^0\}_{j \in \mathcal{M}}$ , computed via (1) and (2). Finally,  $\eta_j^0 = \Delta u_j^0$  for each  $j \in \mathcal{M}$ .

## 4. The alternating direction method of multipliers

ADMM has been demonstrated as a powerful algorithm for solving large-scale structured convex optimization problems [29]. The problems successfully solved by ADMM includes a range of OCPs arising in MPC applications [30–32]. In this section, we present a distributed ADMM scheme for solving the OCP (10) that exploits the block-angular structure of (10). We refer to [29] for details and proofs related to ADMM.

To solve (10) via ADMM, we first introduce the auxiliary variables  $v_j \in \mathbb{R}^{4Nn_{yT}}$  for  $j = 1, 2, \dots, M$  and  $v_{M+1} \in \mathbb{R}^{4N}$ , and write the OCP as

$$\begin{aligned} \min_{z,v} \quad & \psi = \sum_{j \in \overline{\mathcal{M}}} c_j^T z_j, \\ \text{s.t.} \quad & G_j z_j \geq g_j, \quad j \in \overline{\mathcal{M}}, \\ & H_j z_j = v_j, \quad j \in \overline{\mathcal{M}}, \\ & \sum_{j \in \overline{\mathcal{M}}} v_j \geq h, \end{aligned}$$

Using indicator functions, this problem can be stated in the standard ADMM form

$$\min_{z,v} \psi = \sum_{j \in \overline{\mathcal{M}}} (c_j^T z_j + I_{Z_j}(z_j)) + I_{\mathcal{V}}(v), \tag{18a}$$

$$\text{s.t.} \quad H_j z_j = v_j, \quad j \in \overline{\mathcal{M}} \tag{18b}$$

where  $Z_j = \{z_j | G_j z_j \geq g_j\}$ ,  $\mathcal{V} = \{v | \sum_{j \in \overline{\mathcal{M}}} v_j \geq h\}$ , and  $I_{\mathbb{A}}$  is the indicator function of a set  $\mathbb{A}$  defined as

$$I_{\mathbb{A}}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{A}, \\ \infty & \text{otherwise.} \end{cases}$$

For the problem (18), the ADMM recursions described in [29] becomes

$$z_j^{i+1} = \underset{z_j \in Z_j}{\text{argmin}} \ c_j^T z_j + \frac{\rho}{2} \|H_j z_j - v_j^i + u_j^i\|_2^2, \quad j \in \overline{\mathcal{M}}, \tag{19a}$$

$$v^{i+1} = \underset{v \in \mathcal{V}}{\text{argmin}} \ \frac{\rho}{2} \sum_{j \in \overline{\mathcal{M}}} \|H_j z_j^{i+1} - v_j + u_j^i\|_2^2, \tag{19b}$$

$$u_j^{i+1} = u_j^i + H_j z_j^{i+1} - v_j^{i+1}, \quad j \in \overline{\mathcal{M}}, \tag{19c}$$

where  $u^i$  is a scaled dual variable.

The update of  $z_j$ , (19a), thus consists of solving the constrained quadratic program

$$\min_{z_j} \ \frac{\rho}{2} z_j^T H_j^T H_j z_j + (c_j + \rho(-v_j^i + u_j^i))^T H_j z_j, \tag{20a}$$

$$\text{s.t.} \quad G_j z_j \geq g_j \tag{20b}$$

for each  $j \in \overline{\mathcal{M}}$ .

The update for  $v$ , (19b), yields the explicit solution

$$v_j^{i+1} = H_j z_j^{i+1} + u_j^i + \max(l/(M+1), 0), \quad j \in \overline{\mathcal{M}},$$

**Algorithm 2.** ADMM algorithm for the solution of (10)

**Require:**  $(\rho, \alpha, i_{\max}, \epsilon_P, \epsilon_D), (v^0, u^0), (\epsilon_P, \epsilon_D)$   
 $i = 0, \text{converged} = \text{false}$   
**while** converged = false **and**  $i < i_{\max}$  **do**  
  UPDATE VARIABLES  
  **for**  $j \in \mathcal{M}$  **do**  
     $z_j^{i+1} = \underset{z_j \in \mathcal{Z}_j}{\text{argmin}} c_j^T z_j + \frac{\rho}{2} \|H_j z_j - v_j^i + u_j^i\|_2^2$   
  **end for**  
   $l = h - \sum_{j \in \mathcal{M}} \alpha H_j z_j^{i+1} - (1 - \alpha)(-v_j^i) + u_j^i$   
  **for**  $j \in \mathcal{M}$  **do**  
     $v_j^{i+1} = \alpha H_j z_j^{i+1} - (1 - \alpha)(-v_j^i) + u_j^i + \max(l/(M+1), 0)$   
     $u_j^{i+1} = u_j^i + \alpha H_j z_j^{i+1} - (1 - \alpha)(-v_j^i) - v_j^{i+1}$   
  **end for**  
  COMPUTE RESIDUALS  
  **for**  $j \in \mathcal{M}$  **do**  
     $r_j^{i+1} = \rho H_j z_j^{i+1} - v_j^{i+1}$   
     $s_j^{i+1} = -\rho H_j^T (v_j^{i+1} - v_j^i)$   
  **end for**  
  CHECK IF CONVERGED  
  **if**  $\|r^{i+1}\|_2 \leq \epsilon_P$  **and**  $\|s^{i+1}\|_2 \leq \epsilon_D$  **then**  
    converged = true  
  **end if**  
   $i \leftarrow i + 1$   
**end while**

where  $l = h - \sum_{j \in \mathcal{M}} \alpha H_j z_j^{i+1} + u_j^i$ . Each subsystem can thus perform its own update of  $z_j$ . Having computed  $l$  with a contribution from all the subsystems,  $v_j$  and  $u_j$  can be determined individually as well.

Algorithm 2 provides an overview of the ADMM steps described above. Under mild assumptions, the ADMM algorithm converges with a linear convergence rate to the optimal solution of the OCP [29,40]. Note that we have replaced  $H_j z_j^{i+1}$  with  $\alpha H_j z_j^{i+1} - (1 - \alpha)(-v_j^i)$  in the recursions for  $v_j$  and  $u_j$ . As described in [29,41] such a relaxation often speeds up convergence. The relaxation parameter  $\alpha \in [0, 2]$  is tuned to the particular application.

To detect an optimal solution in Algorithm 2, we have adopted the stopping criteria proposed in [29]. For the specific problem formulation (18), these criteria can be written as

$$\|r^i\|_2 \leq \epsilon_P, \quad \|s^i\|_2 \leq \epsilon_D,$$

in which

$$s_j^{i+1} = -\rho H_j^T (v_j^{i+1} - v_j^i), \quad r_j^{i+1} = \rho H_j z_j^{i+1} - v_j^{i+1},$$

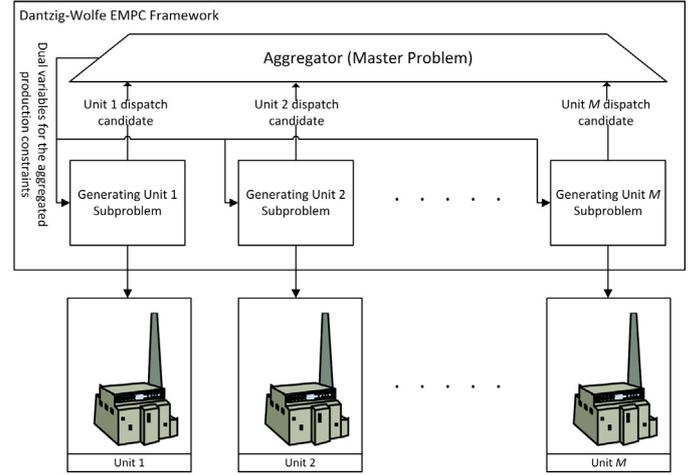
measure the primal and dual residual. These stopping criteria may be extended to include a relative measure as well [29].

As for the Dantzig–Wolfe decomposition algorithm, a warm start for Algorithm 2 can be constructed by shifting the closed-loop solution values,  $v^*$  and  $u^*$ , forward in time. If such a solution is not available, the standard cold-starting point  $(v^0, u^0) = (\mathbf{0}, \mathbf{0})$  is used. We remark that in comparison to the Dantzig–Wolfe decomposition algorithm, the initial point does not need to be feasible. Moreover, the extensions of Algorithm 2 are not restricted only to linear programming [30–32]. One could consider more general regularization terms in (4), e.g.  $\ell_2$ -regularization terms.

**Remark 7.** The optimization problem (20) is an OCP with a quadratic cost function and linear constraints. Efficient algorithms for such structured QPs include active-set methods [42–44], interior-point methods [10,45–50] and first-order methods [49,51].

## 5. Smart energy systems case study

To handle the volatile and unpredictable power generation associated with technologies such as wind, solar and wave power, energy systems management has emerged as a promising application area for EMPC. In EMPC of energy systems, the power



**Fig. 3.** EMPC diagram for the Dantzig–Wolfe decomposition algorithm for a dynamic multi-plant system with linking constraints.

production planning is handled in real-time by computing an optimal production plan based on the most recent information available such as forecasts of energy prices, wind power production, and district heating consumption [52–57].

In this section we use a conceptual energy systems management case study to test a MATLAB implementation of Algorithm 1,  $DW_{EMPC}$ , and a MATLAB implementation of Algorithm 2,  $ADMM_{EMPC}$ . The energy system considered, consists of a collection of power generating units in the form

$$Y_j(s) = \frac{1}{(\tau_j s + 1)^3} (U_j(s) + D_j(s)) + E_j(s), \quad j \in \mathcal{M}, \quad (21)$$

where  $D_j(s)$  is the process noise,  $E_j(s)$  is the measurement noise,  $U_j(s)$  is the input (power production setpoint) to the  $j$ th power unit and  $Y_j(s)$  is its power production. The third order model, (21), has been validated against actual measurement data in [58]. This system is a dynamic multi-plant system. Fig. 3 illustrates the Dantzig–Wolfe decomposition algorithm for a dynamic multi-plant system.

To represent different types of power generating units, we vary the time constants,  $\tau_j$ ; values in the range 40–80 are associated with slow units such as centralized thermal power plants, while values in the range 20–40 represent units with faster dynamics such as diesel generators and gas turbines.

In the case study, the controller must compute the most cost-efficient feasible power setpoint for each power generating unit such that the total power production satisfies the time varying power demand.

The total power produced by the  $M$  generating units is

$$Y_T(s) = \sum_{j=1}^M \frac{1}{(\tau_j s + 1)^3} (U_j(s) + D_j(s)). \quad (22)$$

Using a discrete state space representation, (21) and (22) may be expressed as

$$x_{j,k+1} = A_j x_{j,k} + B_j u_{j,k} + E_j d_{j,k}, \quad j \in \mathcal{M}, \quad (23a)$$

$$y_{j,k} = C_j x_{j,k} + e_{j,k}, \quad j \in \mathcal{M}, \quad (23b)$$

$$y_{T,k} = \sum_{j \in \mathcal{M}} C_j x_{j,k}, \quad (23c)$$

In the resulting model structure,  $u_{j,k} \in \mathbb{R}$  is the unit input (power setpoint),  $y_{j,k} \in \mathbb{R}$  is the unit power production, and  $y_{T,k} \in \mathbb{R}$  is the total power production. We assume that  $\mathbf{x}_{j,0} \sim N(\hat{x}_{j,0}, P_{j,0})$ ,

$\mathbf{d}_{j,k} \sim N(0, R_{j,d})$ , and that  $\mathbf{e}_{j,k} \sim N(0, R_{j,e})$ . By employing the Kalman filter, the separation principle, and the certainty equivalence principle, the OCP in EMPC for (23) can be stated in the form (3) with  $\Upsilon_j = 1$  for all  $j \in \mathcal{M}$ , see e.g. [38].

5.1. Suboptimality measure

The Dantzig–Wolfe decomposition algorithm and the ADMM algorithm satisfy the subsystem constraints (10b) in every iteration. Therefore, a set of feasible but not necessarily optimal inputs,  $\{\hat{u}_j\}_{j=1}^M$ , is available for the power generating units at each iteration of the algorithms. Consequently, the algorithms may be terminated early and still provide a feasible suboptimal solution. Using (9), we can compute the cost associated with the suboptimal inputs as

$$\hat{\psi} = q^T \hat{\rho} + \sum_{j \in \mathcal{M}} p_j^T \hat{u}_j + r_j^T \hat{\gamma}_j + w_j^T \hat{\eta}_j,$$

where  $\hat{\rho}$ ,  $\hat{\gamma}_j$  and  $\hat{\eta}_j$  are completely determined by  $\hat{u}_j$ . Based on  $\hat{\psi}$  and the optimal value  $\psi^*$ , we define the level of suboptimality as

$$\omega = 100 \frac{\hat{\psi} - \psi^*}{\max(|\psi^*|, 1)}. \tag{24}$$

This definition of suboptimality provides a quality measure of the current available inputs.

**Remark 8.** In Dantzig–Wolfe decomposition, the solution to the restricted master problem, (12) with  $\mathcal{P}$  replaced by  $\hat{\mathcal{P}} \subset \mathcal{P}$ , provides an upper bound on the optimal objective value. Moreover, a lower bound can be determined without much extra work via the Lagrangian relaxation techniques described in [59]. Therefore, a bound on (24) can be computed in each iteration of Algorithm 1.

5.2. Simulation parameters

In the simulations presented below, the control and prediction horizon is  $N=60$  time steps, and a sampling time of  $T_s=5$  s is used. Each generating unit is represented by a system in the form (21) with a time constant,  $\tau_j$ , sampled from the uniform distribution over the interval [20, 80]. For simplicity, it is assumed that  $\mathbf{d}_{j,k} \sim N(0, (10\sigma)^2 I)$ ,  $\mathbf{e}_{j,k} \sim N(0, \sigma^2 I)$ , and that full initial state information is given such that  $x_{j,0} \sim (0, 0)$ .

The power generating unit input price is  $p_{j,k} = 1/\tau_j$ . This implies that fast units are more expensive to use than slow units. The conflict between response time and operating costs represents a common situation in the power industry: Large thermal power plants often produce a majority of the electricity, while the use of units with faster dynamics such as diesel generators and gas turbines are limited to critical peak periods.

We define the input limits and the input rate limits as

$$(\underline{u}_{j,k}, \bar{u}_{j,k}, \Delta \underline{u}_{j,k}, \Delta \bar{u}_{j,k}) = (0, 8/M, -M/4, M/4).$$

In this way, the possible contribution from each unit to the overall power production diminishes as the number of units is increased. Local output constraints in the form (3i) and (3j) are not present. The local output variables,  $y_{j,k}$ , and the local slack variables,  $\gamma_{j,k}$ , are thus excluded from the optimization problem.

The penalty for not satisfying the electricity demand (3k) is fixed to  $\rho_k = 10$ . For ADMMmpc, we use the algorithm parameters  $\rho = 1$  and  $\alpha = 1.8$ . These parameters have been carefully tuned to this particular application. The tolerance parameter for DWmpc is set to  $\varepsilon = 1e-4$ . ADMMmpc uses the following primal and dual tolerance specification:  $\epsilon_p = \epsilon_d = 1e-2$ . Both DWmpc and ADMMmpc use CPLEX for solving the subproblems. Although the subproblems are solved sequentially, we refer to their effective CPU time in this paper, assuming that the subproblems are solved in parallel. The

**Table 1**  
Case study simulation and controller parameters.

	$\tau_i$	$p_k$	$\underline{u}_k$	$\bar{u}_k$	$\Delta \underline{u}_k$	$\Delta \bar{u}_k$
Generating Unit 1	65	1/65	0	4	-1	1
Generating Unit 2	75	1/75	0	4	-1	1

reason for this is to report the full potential of the distributed optimization algorithms.

5.3. Closed-loop simulations

We first consider an example with  $M = 2$  power generating units. Table 1 lists the system and controller parameters.

Fig. 4 illustrates closed-loop simulations for different values of the noise parameter,  $\sigma$ , and the regularization weights,  $w = w_{j,k}$ . As indicated in Fig. 4(b), the closed-loop input variance increases significantly if no penalty is imposed on the input rate. This happens even for small values of the noise parameter. By assigning a penalty to the input-rate, the solution becomes more well-behaved and better suited for practical applications. Table 2 shows that the addition of regularization also reduces the computing time for DWmpc as well as for ADMMmpc. E.g. for scenario  $s_6$ , corresponding to  $\sigma = 0.01$  and  $w = 0.1$ , the average number of iterations performed by DWmpc is reduced by more than 40% compared to the case without regularization, i.e. the case with  $w = 0$ . Also observe that while warm-start only leads to a marginal improvement in the iteration count for DWmpc, a substantial reduction in the number of iterations is achieved for ADMMmpc.

Fig. 5 shows the level of suboptimality,  $\omega$ , computed via (24), for scenario  $s_5$  when the run time of DWmpc and ADMMmpc is limited to 0.01 s.

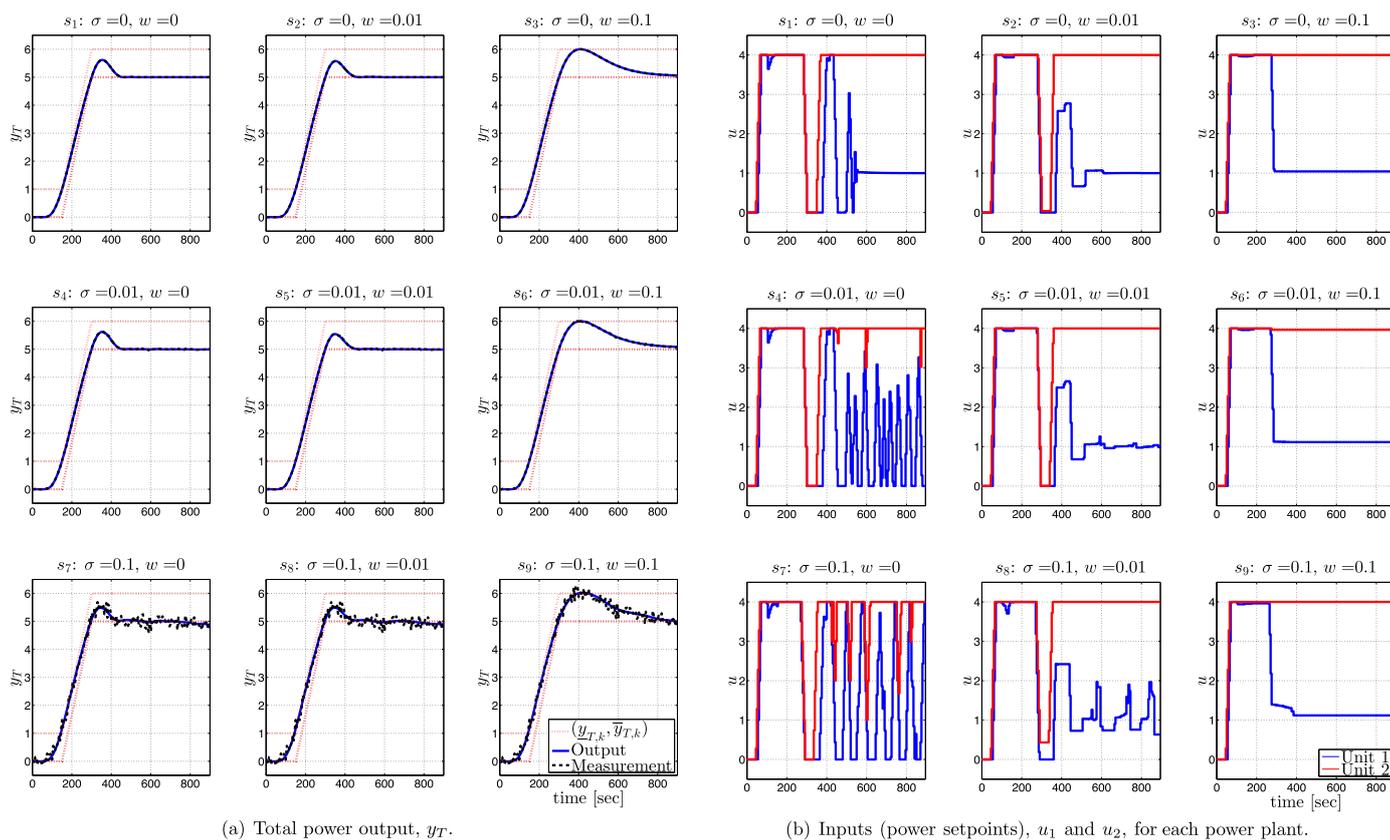
We observe that DWmpc is up to approximately 30% suboptimal when cold-started, and not more than 5% suboptimal when warm-started. Hence, although the number of iterations only decreases slightly when DWmpc is warm-started, the quality of the solution obtained after terminating early improves significantly. By the same token, warm-start reduces the level of suboptimality for ADMMmpc by several orders of magnitude.

Provided that the number of iterations is small, the effort per iteration is approximately equal for DWmpc and ADMMmpc. Table 2 reports that ADMMmpc requires many more iterations than DWmpc. Accordingly, we expect DWmpc to provide a more accurate solution than ADMMmpc within the same time frame. This is confirmed by Fig. 5. Note however, that the computing time per iteration is constant for ADMMmpc, while each iteration of DWmpc requires an increasing work-load since extreme points are added to the restricted master problem on the fly. Nonetheless, in all our simulations DWmpc outperforms ADMMmpc by a significant margin.

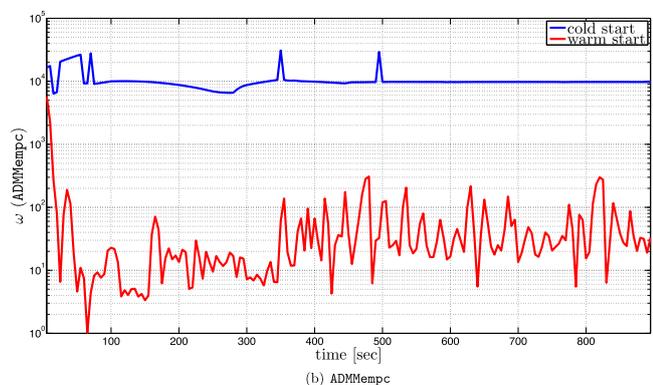
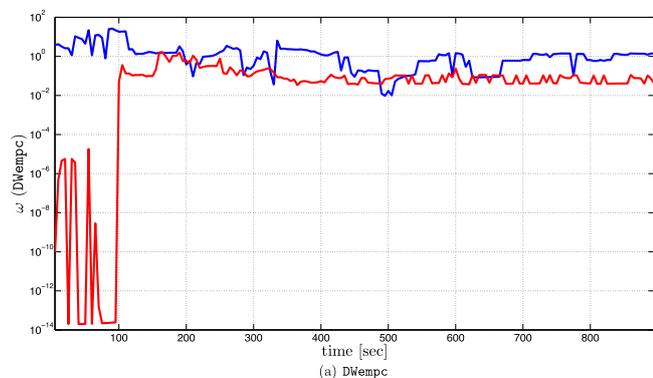
Fig. 6 depicts the level of suboptimality as a function of the CPU time for DWmpc and ADMMmpc. A single instance of the OCP with 128 generating units is solved.

**Table 2**  
Iteration information table for the closed-loop simulation scenarios depicted in Fig. 4. The minimum, maximum and average number of iterations is listed for both cold start and for warm start (in parentheses).

	$\sigma$	$w$	DWmpc	ADMMmpc
$s_1$	0	0	[6(2), 16(17), 12(11)]	[47(2), 485(410), 097(66)]
$s_2$	0	0.01	[6(2), 15(18), 10(09)]	[35(3), 469(410), 088(56)]
$s_3$	0	0.1	[5(2), 15(17), 07(07)]	[33(6), 359(280), 149(48)]
$s_4$	0.01	0	[7(2), 18(19), 13(11)]	[47(2), 485(410), 094(65)]
$s_5$	0.01	0.01	[6(2), 17(17), 10(09)]	[35(2), 469(410), 088(58)]
$s_6$	0.01	0.1	[5(2), 13(16), 07(06)]	[32(6), 380(290), 145(50)]
$s_7$	0.1	0	[7(2), 17(20), 12(11)]	[46(2), 485(410), 091(66)]
$s_8$	0.1	0.01	[6(2), 17(16), 09(09)]	[35(2), 469(410), 084(60)]
$s_9$	0.1	0.1	[5(2), 14(14), 07(06)]	[32(6), 359(279), 144(47)]



**Fig. 4.** Closed-loop simulations of the system (23) controlled by EMPC. The OCP (3) representing the EMPC is solved to a specified tolerance using CPLEX. The figures illustrate the total output and the inputs for different values of the noise parameter,  $\sigma$ , and the regularization weights,  $w$ . The effect of the regularization is most clearly observed in the inputs. At the expense of slightly less tight control on the total power output, the inputs become less volatile when the regularization weight is increased.

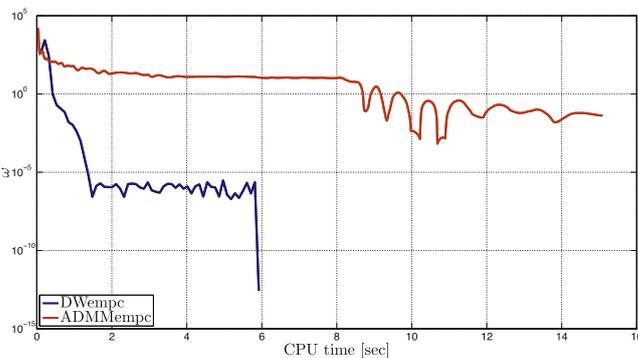


**Fig. 5.** Suboptimality level of the closed-loop solution obtained by  $DWempc$  and  $ADMMempc$  when terminated after 0.01 s.

Initially,  $ADMMempc$  finds the best solution. The quality of this solution is however far from optimal, making it economically very inefficient. For  $DWempc$ , fast convergence is observed after 0.2 s, and at 0.3 s a solution which is less than 1% suboptimal is found. Moreover, while  $DWempc$  keeps improving until a highly accurate solution is found,  $ADMMempc$  suffers from a much slower convergence rate. Only after 10 s is a solution with a suboptimality level of 1% found by this algorithm.

#### 5.4. Large-scale simulations

We compare the performance of the algorithms presented in this paper to the performance of Gurobi, CPLEX and MOSEK. These state-of-the-art linear programming solvers are invoked via a



**Fig. 6.** Level of suboptimality as a function of the CPU time, for a single instance of the OCP with 128 generating units.

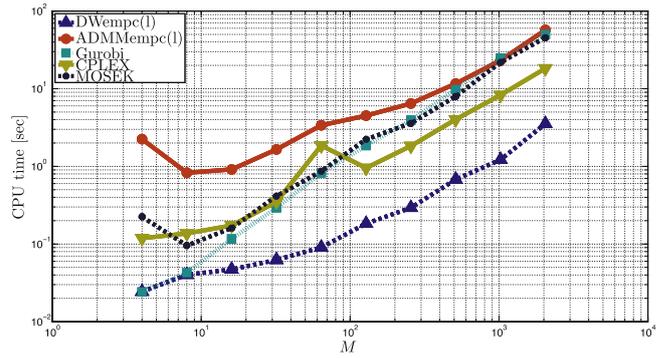
**Table 3**  
Tolerance specifications for DWempc and ADMMempc.

Accuracy	$\epsilon$	$\epsilon_p$	$\epsilon_D$
High (h)	1e-6	1e-4	1e-4
Medium (m)	1e-5	1e-3	1e-3
Low (l)	1e-4	1e-2	1e-2

MEX interface in MATLAB. We use their default tolerance settings. The algorithms are run on an Intel(R) Core(TM) i7-4770K CPU @ 3.50 GHz with 16 GB RAM running a 64-bit Windows 8.1 Pro operating system. For each solver, the computation time of solving the OCP (3) is reported as a function of the number of generating units. Table 3 lists the different accuracy settings used by DWempc and ADMMempc in our benchmarks.

Fig. 7 and Table 4 report the CPU time of solving the OCP for different number of generating units and optimization algorithms. For large problems, ADMMempc does not converge to high accuracy solutions within a reasonable amount of time. Therefore, Table 4 is incomplete.

For large problems, DWempc is faster than all other solvers tested in our case study. Observe also that Gurobi, CPLEX and MOSEK perform almost as well as DWempc in terms of CPU time. For high accuracy solutions, DWempc is 2 times faster than CPLEX and 5 times faster than Gurobi. DWempc and ADMMempc can easily accommodate very large problems in memory while Gurobi, CPLEX and MOSEK fail due to insufficient memory. The threshold when memory becomes an issue is around  $M = 3000$  generating units. Consequently, when considering both CPU time and memory requirements, DWempc is



**Fig. 7.** CPU time for the different solvers as a function of the number of power generating units,  $M$ .

an attractive optimization algorithm for large scale dynamically decoupled energy management problems.

Note from Fig. 6 that ADMMempc needs many more iterations to converge than DWempc for the high accuracy tolerance specification, (h). Table 5 further shows that the number of iterations increases with the problem size for ADMMempc. Therefore, ADMMempc is less attractive from a scalability point of view. Apparently, the number of iterations used by DWempc does not depend on the number of generating units,  $M$ .

Table 6 lists the suboptimality level of the solution determined by DWempc and ADMMempc for different values of  $M$ . As observed from Table 6, DWempc is not only faster than ADMMempc for the tolerance specifications listed in Table 3, but the solution accuracy is also significantly better.

**Table 4**  
CPU time for solving (3) with an increasing number of generating units,  $M$ .

solver/ $M$	16	32	64	128	256	512	1024	2048
Gurobi	1.16e-1	2.93e-1	8.22e-1	1.85	3.94	9.76	2.49e1	5.00e1
CPLEX	1.73e-1	3.49e-1	1.86	9.54e-1	1.83	4.02	8.28	1.83e1
MOSEK	1.60e-1	4.14e-1	8.66e-1	2.23	3.59	7.96	2.19e1	4.54e1
DWempc (l)	4.70e-2	6.25e-2	9.04e-2	1.83e-1	2.94e-1	6.78e-1	1.22	3.54
DWempc (m)	4.95e-2	7.20e-2	1.54e-1	2.61e-1	4.59e-1	1.28	2.78	5.20
DWempc (h)	6.48e-2	9.48e-2	2.03e-1	3.58e-1	6.74e-1	1.80	4.91	9.61
ADMMempc (l)	9.13e-1	1.65	3.38	4.50	6.44	1.16e1	2.32e1	5.70e1
ADMMempc (m)	1.76	3.06	4.28	1.03e1	2.45e1	7.76e1	2.58e2	-
ADMMempc (h)	8.49	7.88e1	5.59e2	-	-	-	-	-

**Table 5**  
The number of iterations performed by DWempc and ADMMempc in solving (3) for an increasing number of generating units,  $M$ .

solver/ $M$	16	32	64	128	256	512	1024	2048
DWempc (l)	9	8	7	7	6	6	5	5
DWempc (m)	10	9	10	9	8	9	8	6
DWempc (h)	12	12	11	12	11	10	11	11
ADMMempc (l)	93	157	303	352	395	436	530	615
ADMMempc (m)	178	286	370	705	1,284	2,520	4,994	-
ADMMempc (h)	865	7,468	50,000	-	-	-	-	-

**Table 6**  
Suboptimality level of DWempc and ADMMempc in the solution of (3) for an increasing number of generating units,  $M$ .

solver/ $M$	16	32	64	128	256	512	1024	2048
DWempc (l)	2.08e-2	1.67e-2	1.53e-1	1.86e-1	9.65e-1	9.44e-1	3.18	3.13
DWempc (m)	1.66e-5	3.87e-3	7.40e-3	1.94e-2	4.74e-2	5.29e-2	1.63e-1	9.37e-1
DWempc (h)	-2.46e-10	-1.13e-9	6.46e-6	1.98e-4	7.55e-4	1.06e-3	2.32e-3	4.82e-2
ADMMempc (l)	4.58e-1	6.65e-1	7.92e1	2.09e2	2.84e2	5.50e2	8.13e2	1.33e3
ADMMempc (m)	8.16e-2	8.36e-2	9.97	1.38e-1	1.53e1	6.62e-1	6.96	-
ADMMempc (h)	1.87e-1	1.86e-2	3.36e-5	-	-	-	-	-

## 6. Conclusions

In this paper, we developed and presented a warm-started possibly early terminated Dantzig–Wolfe decomposition algorithm for  $\ell_1$ -regularized linear EMPC of dynamically decoupled subsystems. Simulations show that a MATLAB implementation of the proposed algorithm, denoted  $DW_{empc}$ , is faster than CPLEX, Gurobi and MOSEK, as well as a special-purpose implementation of ADMM denoted  $ADMM_{empc}$ . Both  $DW_{empc}$  and  $ADMM_{empc}$  have similar parallelization capabilities. They are able to handle much larger problems than the general purpose solvers. The simulations also demonstrate that in combination with warm-start, early termination of  $DW_{empc}$  yields a highly accurate solution after only a few iterations. In contrast to  $ADMM_{empc}$ , the number of iterations required by  $DW_{empc}$  to achieve a certain tolerance level does not grow with the problem size.

For cases when the number of  $DW_{empc}$  iterations is large,  $DW_{empc}$  may be slower than  $ADMM_{empc}$ . The reason is that the computing time per iteration of  $DW_{empc}$  grows with the iteration number. Conversely, the time spent per iteration by  $ADMM_{empc}$  is constant. Although this is a potential drawback of the Dantzig–Wolfe decomposition algorithm that favors the ADMM algorithm, we have not observed this being the case in any of our simulations. In all our simulations,  $DW_{empc}$  outperforms  $ADMM_{empc}$ ; in some cases by several orders of magnitude.

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Computational Efficiency of Economic MPC for Power  
Systems Operation

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# Computational Efficiency of Economic MPC for Power Systems Operation

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**Abstract**—In this work, we propose an Economic Model Predictive Control (MPC) strategy to operate power systems that consist of independent power units. The controller balances the power supply and demand, minimizing production costs. The control problem is formulated as a linear program that is solved by a computationally efficient implementation of the Dantzig-Wolfe decomposition. To make the controller suitable for real-time applications, we investigate a suboptimal MPC scheme, introducing an early termination strategy to the Dantzig-Wolfe algorithm. Simulations demonstrate that the early termination technique substantially reduces the computation time.

## I. INTRODUCTION

For the last two centuries, mankind has depended on fossil fuel. The consequences of the extensive use of these fuels, i.e. global warming and rising costs of fossil fuels, affect our life. Furthermore, the greatest source of CO<sub>2</sub> emissions is the combustion of fossil fuels to generate electricity. A change in energy systems is clearly necessary if we are to become free of dependence on fossil fuels. The introduction of Smart Grids is crucial for future energy systems, as these grids will connect consumers and producers through, for example, renewable energy sources (RES). This innovative scenario requires control actions to ensure that the total energy production can satisfy customer demand.

In this paper we propose an optimization-based controller for dynamic load balancing of a power system consisting of multiple power units that are dynamically decoupled. The control strategy is an Economic MPC strategy that is used to balance power supply and demand in the most economical way. The control problem may be expressed as a linear program tailored for the Dantzig-Wolfe decomposition technique. However, real-time applications require fast computation of the optimal control trajectory; because of this, an early termination strategy is applied to the Dantzig-Wolfe decomposition algorithm. Such early termination significantly reduces the computational time for the MPC at the cost of obtaining only a suboptimal solution.

Economic MPC operates many energy systems, for example refrigeration systems, heat pumps for residential buildings, solar-heated water tanks and batteries in electric vehicles. However, with regard to large-scale systems, a distributed controller to overcome the communication limitation and computational complexity is a better choice [1]. Distributed

MPC successfully applies decomposition techniques, i.e. the Dantzig-Wolfe decomposition [2], [3]. Such a decomposition algorithm has been utilized in several applications, including power balancing based on an  $\ell_1$ -penalty function [4] and to operate large-scale power systems [5]. To strengthen the applicability of the controller to real-time applications, this work introduces a suboptimal MPC strategy [6]–[10].

The outline of this paper is as follows: Section II introduces power systems and formulates a linear Economic MPC for linear power systems. Section III describes the Dantzig-Wolfe decomposition algorithm. The early termination strategy is explained in Section IV. Section V-A proposes a model for the power generators included in the power system; Section V-B reports simulation results and, finally, the conclusion and suggestions for future work are presented in Section VI.

## II. POWER SYSTEMS AND ECONOMIC MPC

Future power systems will consist of independent power units connected to one common operation center that must control and coordinate such power units, balancing power production and consumption in an economical and reliable way. Operating such a power system means making real-time decisions such as planning the power production in response to customer demand and unpredicted production variations from renewable energy producers, e.g. wind turbines.

A power unit is assumed to be described as a linear stochastic discrete time state space model

$$x_{k+1} = Ax_k + Bu_k + Ed_k \quad (1a)$$

$$y_k = Cx_k + v_k \quad (1b)$$

$$z_k = C_z x_k. \quad (1c)$$

The measurement noise,  $v_k \sim N_{iid}(0, R_{vv})$ , and  $d_k \sim N(\bar{d}_k, R_{dd,k})$  are predicted by external prognosis systems [5]; in many power applications  $d_k$  might represent wind speed or sun radiation. While  $x_k$  denotes the states,  $u_k$  the manipulated variables (MVs),  $y_k$  denotes the measurement used for feedback, and  $z_k$  is the output variable. The manipulated variable,  $u_k$ , is subject to hard constraints

$$u_{min} \leq u_k \leq u_{max} \quad (2a)$$

$$\Delta u_{min} \leq \Delta u_k \leq \Delta u_{max} \quad (2b)$$

The system output  $z_k$  must be within an interval  $[r_{\min,k}, r_{\max,k}]$ ; such interval may represent electricity demand forecast in advance, or it can define indoor temperature in a building, or temperatures in a refrigeration system or state-of-charge of a battery. However, due to some disturbances or in a specific scenario, it might be impossible to obtain  $z_k$  within the defined interval; therefore, the constraints on the output variable include slack variables  $s_k$ . The slack variables,  $s_k$ , may represent selling or buying power from the short-term market, violation of temperature limits, or violation of state-of-charge limits. Every time  $s_k$  is non-zero, a penalty cost, e.g. the cost of buying or selling power on the short-term market must be paid.

$$r_{\min,k} - s_k \leq z_k \leq r_{\max,k} + s_k \quad (3a)$$

$$s_k \geq 0 \quad (3b)$$

In order to deal with the stochasticity of the power units, we apply the certainty equivalence principle: in this way, all variables are replaced by their conditional mean values [5]. Furthermore, we implement a Kalman filter to predict  $\hat{x}_{k+1+j|k} \forall j \in \mathcal{N}$  looking  $N$  periods ahead [5]. Consider a power system consisting of  $P$  power units (1), the cost of producing power for a power generator is  $\phi_{i,k}, \forall i \in P$ . This economic cost,  $\phi_{i,k}$ , consists of the cost of operating the power unit,  $\hat{c}_{i,k+j|k}$ , and the penalties,  $\hat{\rho}_{i,k+j+1|k}$ , related to the use of slack variables,  $\hat{s}_{i,k+j+1|k}$

$$\phi_{i,k} = \sum_{j=0}^{N-1} \hat{c}'_{i,k+j|k} \hat{u}_{i,k+j|k} + \hat{\rho}'_{i,k+j+1|k} \hat{s}_{i,k+j+1|k} \quad (4)$$

These unit prices are provided by external forecasting systems. The linear constraints and the linear objective functions lead to the formulation of the control problem as a linear program; accordingly, the Linear Economic MPC to operate a power system of  $P$  power units is formulated as

$$\min \phi_k = \sum_{i=1}^P \phi_{i,k} + \sum_{j=0}^{N-1} \hat{\rho}'_{k+j+1|k} \hat{s}_{k+j+1|k} \quad (5)$$

subject to the local constraints  $\forall i \in P$  and  $\forall j \in \mathcal{N}$

$$\hat{x}_{i,k+j+1|k} = A_i \hat{x}_{i,k+j|k} + B_i \hat{u}_{i,k+j|k} + E_i \hat{d}_{i,k+j|k} \quad (6a)$$

$$\hat{z}_{i,k+j+1|k} = C_{z,i} \hat{x}_{i,k+j+1|k} \quad (6b)$$

$$u_{\min,i} \leq \hat{u}_{i,k+j|k} \leq u_{\max,i} \quad (6c)$$

$$\Delta u_{\min,i} \leq \Delta \hat{u}_{i,k+j|k} \leq \Delta u_{\max,i} \quad (6d)$$

$$\hat{z}_{i,k+j+1|k} + \hat{s}_{i,k+j+1|k} \geq \hat{r}_{\min,i,k+j+1|k} \quad (6e)$$

$$\hat{z}_{i,k+j+1|k} - \hat{s}_{i,k+j+1|k} \leq \hat{r}_{\max,i,k+j+1|k} \quad (6f)$$

$$\hat{s}_{i,k+j+1|k} \geq 0 \quad (6g)$$

and subject to the following connecting constraints  $\forall j \in \mathcal{N}$  and  $\forall i \in P$ , where  $\hat{z}_{k+j+1|k}$  denotes the supply constraints connecting the produced and requested power,  $\hat{r}_{\min,k}$  and

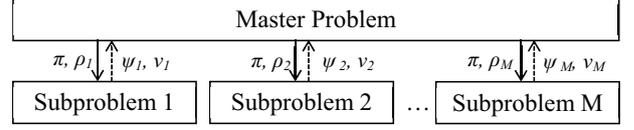


Fig. 1: Dantzig-Wolfe structure. Each subproblem communicates exclusively with the master problem (MP).

$\hat{r}_{\max,k}$  are provided by external forecasts

$$\hat{z}_{k+j+1|k} = \sum_{i=1}^P \tilde{C}_{z,i} \hat{x}_{i,k+j+1|k} + \tilde{D}_{z,i} \hat{u}_{i,k+j|k} \quad (7a)$$

$$\hat{z}_{k+j+1|k} + \hat{s}_{k+j+1|k} \geq \hat{r}_{\min,k+j+1|k} \quad (7b)$$

$$\hat{z}_{k+j+1|k} - \hat{s}_{k+j+1|k} \leq \hat{r}_{\max,k+j+1|k} \quad (7c)$$

$$\hat{s}_{k+j+1|k} \geq 0 \quad (7d)$$

The optimization control problem (5)-(7) has a block-angular structure tailored for the implementation of Dantzig-Wolfe decomposition to solve efficiently the control linear program.

### III. DANTZIG-WOLFE DECOMPOSITION TECHNIQUE

The Dantzig-Wolfe decomposition algorithm is a decomposition technique to solve efficiently linear programs that have a block-angular structure, such as (5)-(7), [2], [3]. The Economic MPC expressed, as a linear program in (5)-(7), can be formulated as

$$\min_{\{w_{i,k}\}_{i=1}^M} \varphi = \sum_{i=1}^M e'_i w_{i,k} \quad (8a)$$

$$s.t. \begin{bmatrix} F_1 & F_2 & \dots & F_M \\ G_1 & & & \\ & G_2 & & \\ & & \ddots & \\ & & & G_M \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_M \end{bmatrix} \geq \begin{bmatrix} g \\ h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix} \quad (8b)$$

where  $i \in \mathcal{M} = \{1, \dots, P, P+1\}$  as the slack variables  $\hat{s}_{k+j+1|k}$  in (5) and (7) are considered as an independent unit.

In this decomposition approach, the linear programming problem can be separated into independent subproblems, which are coordinated by a master problem (MP), as depicted in Figure 1. Within each iteration, the MP sends its Lagrange multipliers to all the subproblems to update their objective function. Then, the subproblems are solved and they send their solutions and objective function values to the MP. The solution to the original problem can be shown to be equivalent to solving the subproblems and the MP through a finite number of iterations [11].

When describing the Dantzig-Wolfe decomposition, it is necessary to introduce the theorem of convex combination, or Dantzig-Wolfe transformation [11], [12].

*Theorem 1 (Convex Combination):* Consider  $\mathcal{W} = \{w \mid Gw \geq h\}$  is a nonempty, bounded and closed set, i.e. a polytope.  $v^j$  denotes the extreme point of  $\mathcal{W}$  with

$j \in \{1, 2, \dots, V\}$ .

Then any point  $w$  in the polytope  $\mathcal{W}$  can be written as a convex combination of its extreme points

$$w = \sum_{j=1}^V \lambda_j v^j \quad (9a)$$

$$s.t. \lambda_j \geq 0, \quad j = 1, 2, \dots, V \quad (9b)$$

$$\sum_{j=1}^V \lambda_j = 1 \quad (9c)$$

*Proof:* See [11].  $\blacksquare$

Via the theorem of convex combination (9), the MPC can be formulated as follows, assuming that the feasible regions of subproblems are bounded

$$\min_{\lambda} \varphi = \sum_{i=1}^M \sum_{j=1}^{V_i} f_{ij} \lambda_{ij} \quad (10a)$$

$$s.t. \sum_{i=1}^M \sum_{j=1}^{V_i} p_{ij} \lambda_{ij} \geq g \quad (10b)$$

$$\sum_{j=1}^{V_i} \lambda_{ij} = 1, \quad i = 1, 2, \dots, M \quad (10c)$$

$$\lambda_{ij} \geq 0, \quad i = 1, 2, \dots, M; j = 1, 2, \dots, V_i \quad (10d)$$

where the coefficients are

$$f_{ij} = e_i' v_i^j, \quad p_{ij} = F_i' v_i^j \quad (11)$$

The linear program (10), known as master problem (MP), is equivalent to the block-angular linear problem (8). It is worth noting that (10) has fewer rows in the coefficient matrix than the original problem (8). However, in the MP the number of columns is larger due to an increase in the number of variables with the extreme points of all subproblems.

If the MP is solved via the Simplex method, then only the basic set is needed and it has the same number of basic variables as the number of rows. Hence, not all the extreme points are necessary to be known. This yields to the reduced master problem (RMP), dynamically constructed at a fixed size by incorporating column generation techniques

$$\min_{\lambda} \varphi = \sum_{i=1}^M \sum_{j=1}^l f_{ij} \lambda_{ij} \quad (12a)$$

$$s.t. \sum_{i=1}^M \sum_{j=1}^l p_{ij} \lambda_{ij} \geq g \quad (12b)$$

$$\sum_{j=1}^l \lambda_{ij} = 1, \quad i = 1, 2, \dots, M \quad (12c)$$

$$\lambda_{ij} \geq 0, \quad i = 1, 2, \dots, M; j = 1, 2, \dots, l \quad (12d)$$

where  $l \leq V_i$  for all  $i \in \{1, 2, \dots, M\}$ . Solving the RMP provides the Lagrange multipliers  $\pi$  associated with the inequality

constraint (12b), the Lagrange multipliers  $\rho$ , associated with equalities (12c), and the Lagrange multipliers  $\kappa$  for the positivity constraints (12d). As depicted in Figure 1, the MP sends the Lagrange multipliers to each subproblem.

The Lagrangian associated to the MP (10) yields to the following necessary and sufficient optimality conditions, for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, V_i$

$$\nabla_{\lambda_{ij}} \mathcal{L} = f_{ij} - p_{ij}' \pi - \rho_i - \kappa_{ij} = 0 \quad (13a)$$

$$\sum_{i=1}^M \sum_{j=1}^{V_i} p_{ij} \lambda_{ij} - g \geq 0 \quad \perp \quad \pi \geq 0 \quad (13b)$$

$$\sum_{j=1}^{V_i} \lambda_{ij} - 1 = 0 \quad (13c)$$

$$\lambda_{ij} \geq 0 \quad \perp \quad \kappa_{ij} \geq 0 \quad (13d)$$

The conditions (13a) and (13d) yield to

$$\kappa_{ij} = f_{ij} - p_{ij}' \pi - \rho_i = [e_i - F_i' \pi]' v_i^j - \rho_i \geq 0 \quad (14)$$

The Karush-Kuhn-Tucker conditions (KKT-conditions) for (10) are for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, V_i$

$$\sum_{i=1}^M \sum_{j=1}^{V_i} p_{ij} \lambda_{ij} - g \geq 0 \quad \perp \quad \pi \geq 0 \quad (15a)$$

$$\sum_{j=1}^{V_i} \lambda_{ij} - 1 = 0 \quad (15b)$$

$$\lambda_{ij} \geq 0 \quad \perp \quad \kappa_{ij} = [e_i - F_i' \pi]' v_i^j - \rho_i \geq 0 \quad (15c)$$

Initially, a feasible extreme point to the MP is necessary; we adopt an initialization technique that uses the previous optimal solution and the output constraints (3)-(7) to compute the initial vertex [5]. This initial point is then used to form the RMP (12) considering  $l = 1$ . Assuming  $\lambda_{ij}^{RMP}$  to be a solution of RMP, so that a feasible solution to MP (10) is

$$\lambda_{ij} = \lambda_{ij}^{RMP} \quad i = 1, 2, \dots, M; j = 1, 2, \dots, l \quad (16a)$$

$$\lambda_{ij} = 0 \quad i = 1, 2, \dots, M; j = l + 1, l + 2, \dots, V_i \quad (16b)$$

$\lambda_{ij}^{RMP}$  satisfies the KKT-conditions for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, l$ ; however, the optimal solution needs to satisfy these conditions for all  $i = 1, 2, \dots, M$  and  $j = l + 1, l + 2, \dots, V_i$ . We only know the extreme points,  $v_i^j$  for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, l$ . Because of this the KKT-conditions are satisfied for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, V_i$  if  $\min_i \psi_i - \rho_i \geq 0$  where

$$\psi_i = \min_{v_i^j} [e_i - F_i' \pi]' v_i^j \quad (17)$$

$v_i^j$  is an extreme point of the polytope  $\mathcal{W}_i$ . Then, we form the following linear program to solve (17)

$$\psi_i = \min_{w_i} \varphi = [e_i - F_i' \pi]' w_i \quad (18a)$$

$$s.t. \quad G_i w_i \geq h_i \quad (18b)$$

These linear programs are called subproblems and can be solved via either parallel or sequential computation. This possible parallel computation of the subproblems represents one of the advantages of the Dantzig-Wolfe decomposition algorithm. Let  $(\psi_i, w_i)$  be the optimal value-minimizer pair for the linear problem (18); an optimal solution is reached if the following condition is satisfied

$$\psi_i - \rho_i \geq 0 \quad i \in \{1, 2, \dots, M\} \quad (19)$$

Therefore the solution of the original control problem (8) is given by

$$w_i^* = \sum_{j=1}^l v_j^* \lambda_{ij} \quad i \in \{1, 2, \dots, M\} \quad (20)$$

When condition (19) is not satisfied, the number of extreme points considered,  $l$ , is not enough to satisfy the KKT-conditions and a new vertex  $v_i^{l+1}$  needs to be included.

#### IV. SUBOPTIMAL MPC STRATEGY

Real-time applications may restrict the applicability of the Economic MPC, especially because of the limits on the storage space and the computation time. Real-time MPC applications often involve warm-start, explicit MPC and early termination techniques to speed up the online computation [7], [8]; these approaches compute suboptimal solutions. The warm-start technique is used to initialize the Dantzig-Wolfe decomposition algorithm, as described in Section III and [5]. The novel step of this paper is the introduction of the early termination strategy to reduce computation time in the solution of the optimal control trajectory. Section III introduces the RMP (12): the algorithm adds a vertex of the polytope (9) to the RMP as long as the stopping criteria (19) is not satisfied. In a real-time scenario, the Dantzig-Wolfe algorithm tries to compute the optimal solution within a given sampling time. However, if too many vertices of the polytope (9) are necessary to get from the solution of the previous control problem to that of the current one, then the algorithm can be stopped. With regard to the early termination strategy, the CPU time can be limited [9]; for this purpose, in this work we use as heuristic a limit on the number of vertices of the polytope to include in the Dantzig-Wolfe algorithm. It is worth noting that, in the Dantzig-Wolfe algorithm, the subproblem (18) always has a feasible solution if the original linear program (8) does [13]; moreover, each polytope is assumed to be nonempty (9). As a result, the suboptimal solution obtained by early termination is feasible and the resulting MPC is both feasible and, therefore, stable [7], [8].

#### V. APPLICATION TO A POWER SYSTEM

In this section we apply the Economic MPC controller to a power system consisting of power plants, and the Dantzig-Wolfe decomposition computes the optimal control trajectory. In addition, we implement the early termination strategy in order to reduce computational times.

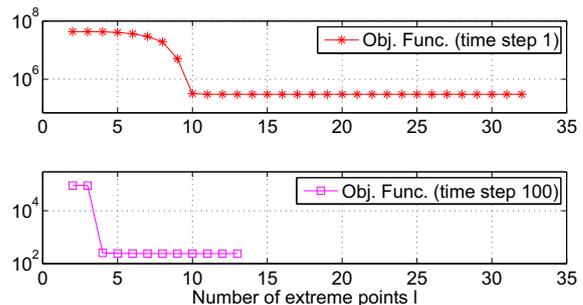


Fig. 2: Master problem (10) objective function and number of extreme points of the polytope (9) at each time step.

As a case study we consider a power system consisting of four power units, described below. Simulations are carried out for 120 time steps with time horizon  $T = 70$ .

##### A. Power Units

As a case study we consider a power system that consists of power generators. The individual power units are independent systems, and they can be modeled separately, as the actions in one of them do not directly affect the other units. They are coupled through the objective to follow the overall power system reference and activate secondary resources. Power units are modeled as in (21) [14]; in this way we address three kinds of power units: central thermal power plants, diesel generators and gas turbines. The first kind of power generator has a slow dynamic, while the remaining two show fast dynamics.

$$Z_i(s) = G_i(s)U_i(s) \quad G_i(s) = \frac{1}{(\tau_i s + 1)} \quad (21)$$

where  $z_i(t)$  is the produced power at unit  $i$ , while  $u_i(t)$  is the corresponding reference signal.

##### B. Simulations Results

The controller developed in this work implements an Economic MPC policy, where the Dantzig-Wolfe decomposition technique computes the optimal control sequence. Section IV introduces the early termination strategy that leads to a suboptimal MPC. Simulation reveals that ten vertices of the feasible polytope (9) provide the optimal value of the objective function, as shown in Figure 2: the algorithm reduces the objective function until it reaches the minimum. However, the Dantzig-Wolfe algorithm takes 32 extreme points to satisfy the stopping criteria (19). As a result, the controller can terminate iteration before the stopping criterion is satisfied and the solution obtained is, therefore, suboptimal. Furthermore, at each time step, the Dantzig-Wolfe algorithm implements the warm-start described in Section III; because of this, the algorithm has a better initial point and the optimal value of the objective function decreases at each time step, see Figure 2. In order to demonstrate the early termination effectiveness, we simulate four different scenarios: the first is the exact Dantzig-Wolfe algorithm, while the remaining three include limits on the number of vertices of the polytope, respectively 15, 10 and

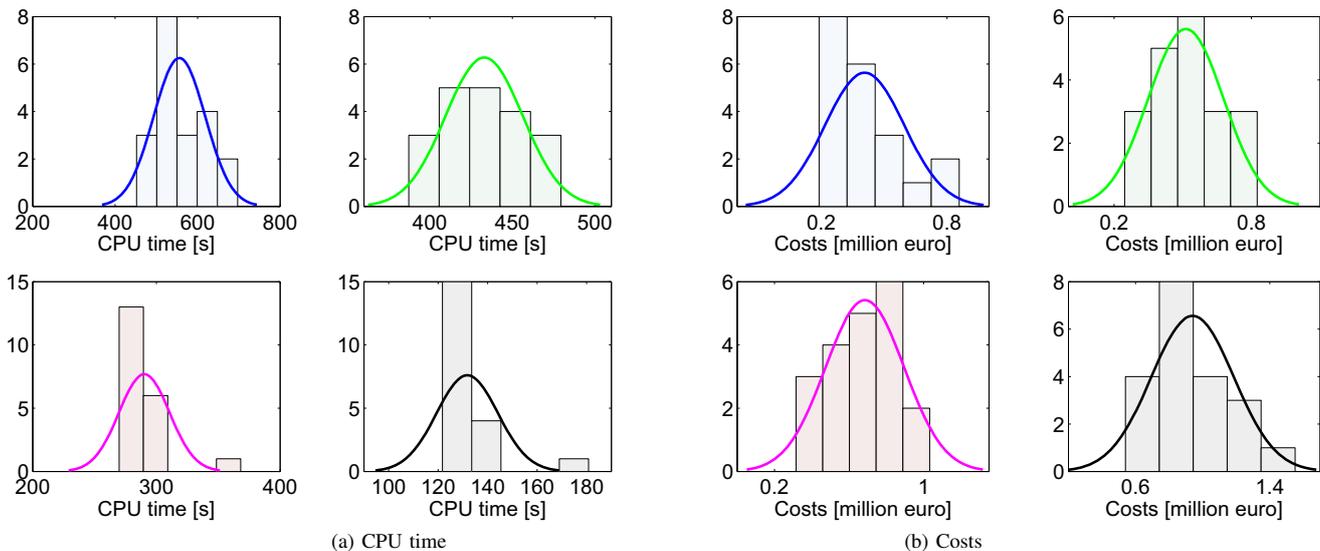


Fig. 3: CPU time and cost distributions based on 20 stochastic simulations. Blue distribution is for Economic MPC solved via exact Dantzig-Wolfe algorithm. Green distribution is for an early termination strategy with 15 vertices. Magenta distribution is for an early termination strategy with 10 vertices. Black distribution is for an early termination strategy with five vertices.

5. Figure 3a reports the CPU time for closed-loop simulations: the early termination technique substantially reduces the computational time by comparison with the implementation of the exact Dantzig-Wolfe algorithm. It should, however, be noted that the decrease in computational time is linked to an increase in the production costs, see Figure 3b.

## VI. CONCLUSION

In this work we have proposed a suboptimal Economic MPC to operate power systems. We have introduced power systems and their independent power units; consequently, we have defined the control problem as a linear program tailored for the Dantzig-Wolfe decomposition technique. After the description of the Dantzig-Wolfe algorithm, we have introduced the early termination strategy that provides a suboptimal solution of the control problem. Closed loop simulations have demonstrated that the algorithm developed noticeably decreases computational times. On the other hand, such reductions cause unavoidable extra costs. This finding should be explored in future work.

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**A Decomposition Algorithm for Optimal Control of  
Distributed Energy System**

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# A Decomposition Algorithm for Optimal Control of Distributed Energy System

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**Abstract**—In economic model predictive control of distributed energy systems, the constrained optimal control problem can be expressed as a linear program with a block-angular structure. In this paper, we present an efficient Dantzig-Wolfe decomposition algorithm specifically tailored to problems of this type. Simulations show that a MATLAB implementation of the algorithm is significantly faster than several state-of-the-art linear programming solvers and that it scales in a favorable way.

## I. INTRODUCTION

Due to global concerns related to environmental issues and security of supply, an increasing share of electricity is being produced by renewable energy sources. Accordingly, methods for power production planning that can handle the volatile and unpredictable power generation associated with technologies such as wind, solar and wave power are required. For this reason, energy systems management has emerged as a promising application area for economic model predictive control (MPC).

In economic MPC of energy systems, the power production planning is handled in real-time by an optimization algorithm that computes an optimal production plan based on the most recent information available such as forecasts of energy prices, wind power production, and district heating consumption. Examples of economic MPC in energy systems management include cost-efficient control of refrigeration systems [1], building climate control [2], [3], and optimal charging strategies for batteries in electric vehicles [4].

Economic MPC requires the solution of a linear program at every sampling instant. In energy systems management, the solution to this linear problem, known as the optimal control problem, provides a sequence of control moves that yields the most cost-efficient power generation, with respect to a process model of the power system. To compensate for non-predictable disturbances and discrepancies between the process model and the true system, only the first input in the sequence of control moves is applied to the system, and the optimization procedure is repeated using updated information at the following sampling instant.

As the control moves are computed in real-time, one of the key challenges in economic MPC is to solve the optimal control problem in an efficient and reliable way. The main contribution of this paper is an algorithm for control of distributed energy systems that satisfies these criteria. Our algorithm exploits that the units in a distributed energy system

are dynamically decoupled. This gives rise to a block-angular structure in the optimal control problem, that allows it to be decomposed into a master problem and a number of subproblems, using Dantzig-Wolfe decomposition [5], [6]. To solve the decomposed problem efficiently we use a column generation procedure, which is warm-started by a strategy that utilizes problem specific features. Similar algorithms have been applied to coordinate the target calculation in set-point based MPC [7], [8], building climate control [9], and hierarchical MPC-based control [10].

## A. Paper Organization

This paper is organized as follows. In Section II, we introduce the optimal control problem solved in economic MPC and a compact problem formulation is derived. We decompose the problem using Dantzig-Wolfe decomposition in Section III, and optimality conditions are derived in Section IV. In this section we also present a warm-started column generation procedure for solving the optimization problem. Performance benchmarks for the proposed algorithm based on a conceptual energy systems management case study are provided in Section V. We give concluding remarks in Section VI.

## II. PROBLEM DEFINITION

We consider an electrical grid with  $M$  dynamically decoupled power generating units. The units are modelled as discrete state space systems in the form

$$x_{j,k+1} = A_j x_{j,k} + B_j u_{j,k}, \quad j \in \mathcal{M}, \quad (1a)$$

$$y_{j,k} = C_j x_{j,k}, \quad j \in \mathcal{M}, \quad (1b)$$

where  $\mathcal{M} = \{1, 2, \dots, M\}$ . The state space matrices are denoted as  $(A_j, B_j, C_j)$ , the states as  $x_{j,k} \in \mathbb{R}^{n_x(j)}$ , the inputs as  $u_{j,k} \in \mathbb{R}^{n_u(j)}$ , and the outputs as  $y_{j,k} \in \mathbb{R}^{n_y(j)}$ .

Assuming that the power production is available as a linear combination of the outputs in (1), the total power production can be written as

$$y_{T,k} = \sum_{j \in \mathcal{M}} \Upsilon_j y_{j,k} = \sum_{j \in \mathcal{M}} \Upsilon_j C_j x_{j,k}, \quad (2)$$

in which  $\Upsilon_j \in \mathbb{R}^{1 \times n_y(j)}$  is a row vector such that  $\Upsilon_j C_j x_{j,k}$  is the power production of unit  $j$  at time step  $k$ .

Economic MPC defines a control law for the generating units (1), that optimizes the inputs (control moves) with

respect to an economic objective function, input limits, input rate limits and soft output limits. Evaluating this control law requires the solution to the minimization problem

$$\min_{u,x,y_T,\rho,\gamma} \sum_{k \in \mathcal{N}_0} q_{k+1}^T \rho_{k+1} + \sum_{j \in \mathcal{M}} p_{j,k}^T u_{j,k} + r_{j,k+1}^T \gamma_{j,k+1}, \quad (3a)$$

subject to the constraints

$$x_{j,k+1} = A_j x_{j,k} + B_j u_{j,k}, \quad k \in \mathcal{N}_0, \quad j \in \mathcal{M}, \quad (3b)$$

$$y_{j,k} = C_j x_{j,k}, \quad k \in \mathcal{N}_1, \quad j \in \mathcal{M}, \quad (3c)$$

$$y_{T,k} = \sum_{j \in \mathcal{M}} \Upsilon_j C_j x_{j,k}, \quad k \in \mathcal{N}_1, \quad (3d)$$

$$\underline{u}_{j,k} \leq u_{j,k} \leq \bar{u}_{j,k}, \quad k \in \mathcal{N}_0, \quad j \in \mathcal{M}, \quad (3e)$$

$$\Delta \underline{u}_{j,k} \leq u_{j,k} - u_{j,k-1} \leq \Delta \bar{u}_{j,k}, \quad k \in \mathcal{N}_0, \quad j \in \mathcal{M}, \quad (3f)$$

$$\underline{y}_{j,k} - \gamma_{j,k} \leq y_{j,k} \leq \bar{y}_{j,k} + \gamma_{j,k}, \quad k \in \mathcal{N}_1, \quad j \in \mathcal{M}, \quad (3g)$$

$$0 \leq \gamma_{j,k} \leq \bar{\gamma}_{j,k}, \quad k \in \mathcal{N}_1, \quad j \in \mathcal{M}, \quad (3h)$$

$$\underline{y}_{T,k} - \rho_k \leq y_{T,k} \leq \bar{y}_{T,k} + \rho_k, \quad k \in \mathcal{N}_1, \quad (3i)$$

$$0 \leq \rho_k \leq \bar{\rho}_k, \quad k \in \mathcal{N}_1, \quad (3j)$$

where  $\mathcal{N}_i = \{0 + i, 1 + i, \dots, N - 1 + i\}$ , with  $N$  being the length of the prediction horizon. The input data are the input limits,  $(\underline{u}_{j,k}, \bar{u}_{j,k})$ , the input rate limits,  $(\Delta \underline{u}_{j,k}, \Delta \bar{u}_{j,k})$ , the output limits associated with the generating units,  $(\underline{y}_{j,k}, \bar{y}_{j,k})$ , the output limits associated with the total power production,  $(\underline{y}_{T,k}, \bar{y}_{T,k})$ , the input prices,  $p_{j,k}$ , the price for violating the output limits associated with the generating units,  $r_{j,k}$ , and the price for violating the output limits associated with the total power production  $q_k$ . The slack variables  $\gamma_{j,k}$  and  $\rho_k$  represent the violation of the output constraints. We include upper limits,  $(\bar{\gamma}_{j,k}, \bar{\rho}_k)$ , on these variables, as this simplifies later computations considerably.

#### A. Compact Formulation

By eliminating the states using equation (1a), we can write the output equation, (1b), as

$$y_{j,k} = C_j A_j^k x_{j,0} + \sum_{i \in \mathcal{N}_0} H_{j,k-i} u_{j,i}, \quad j \in \mathcal{M},$$

where the impulse response coefficients are given by

$$H_{j,k} = C_j A_j^{k-1} B_j, \quad j \in \mathcal{M}.$$

Consequently

$$y_{T,k} = \sum_{j \in \mathcal{M}} \left( \Upsilon_j C_j A_j^k x_{j,0} + \sum_{i \in \mathcal{N}_0} \Upsilon_j H_{j,k-i} u_{j,i} \right).$$

Define the vectors

$$y_j = [y_{j,1}^T \quad y_{j,2}^T \quad \cdots \quad y_{j,N}^T]^T, \quad j \in \mathcal{M}, \quad (4a)$$

$$u_j = [u_{j,0}^T \quad u_{j,1}^T \quad \cdots \quad u_{j,N-1}^T]^T, \quad j \in \mathcal{M}, \quad (4b)$$

and the matrices

$$\Gamma_j = \begin{bmatrix} H_{j,1} & 0 & \cdots & 0 \\ H_{j,2} & H_{j,1} & & \\ \vdots & \vdots & \ddots & \\ H_{j,N} & H_{j,N-1} & \cdots & H_{j,1} \end{bmatrix}, \quad \Phi_j = \begin{bmatrix} C_j A_j \\ C_j A_j^2 \\ \vdots \\ C_j A_j^{N-1} \end{bmatrix},$$

for  $j \in \mathcal{M}$ .

We can then write the outputs, (4a), for each of the generating units as

$$y_j = \Gamma_j u_j + \Phi_j x_{j,0}, \quad j \in \mathcal{M}. \quad (5)$$

Moreover, by introducing  $\tilde{\Gamma}_j$  and  $\tilde{\Phi}_j$  accordingly, it follows that  $y_T = \sum_{j \in \mathcal{M}} \tilde{\Gamma}_j u_j + \tilde{\Phi}_j x_{j,0}$ . We simplify the notation further by introducing

$$\underline{u}_j = [\underline{u}_{j,0}^T \quad \underline{u}_{j,1}^T \quad \cdots \quad \underline{u}_{j,N-1}^T]^T, \quad j \in \mathcal{M},$$

$$\bar{u}_j = [\bar{u}_{j,0}^T \quad \bar{u}_{j,1}^T \quad \cdots \quad \bar{u}_{j,N-1}^T]^T, \quad j \in \mathcal{M},$$

and similarly we define  $\Delta \underline{u}_j$ ,  $\Delta \bar{u}_j$ ,  $\underline{y}_j$ ,  $\bar{y}_j$ ,  $\underline{y}_T$ ,  $\bar{y}_T$ ,  $\bar{\gamma}_j$ ,  $\bar{\rho}$ ,  $\rho$ ,  $q$ ,  $p_j$ ,  $r_j$  and  $\gamma_j$ . Using this notation, the optimal control problem, (3), can be written as

$$\min_{u,\rho,\gamma} q^T \rho + \sum_{j \in \mathcal{M}} p_j^T u_j + r_j^T \gamma_j, \quad (6a)$$

subject to a set of decoupled constraints

$$\underline{u}_j \leq u_j \leq \bar{u}_j, \quad j \in \mathcal{M}, \quad (6b)$$

$$\Delta \underline{u}_j \leq \Delta u_j \leq \Delta \bar{u}_j, \quad j \in \mathcal{M}, \quad (6c)$$

$$\underline{y}_j - \gamma_j \leq \Gamma_j u_j + \Phi_j x_{j,0} \leq \bar{y}_j + \gamma_j, \quad j \in \mathcal{M}, \quad (6d)$$

$$0 \leq \gamma_j \leq \bar{\gamma}_j, \quad j \in \mathcal{M}, \quad (6e)$$

$$0 \leq \rho \leq \bar{\rho}, \quad (6f)$$

and a set of linking constraints

$$\underline{y}_T - \rho \leq \sum_{j \in \mathcal{M}} \tilde{\Gamma}_j u_j + \tilde{\Phi}_j x_{j,0} \leq \bar{y}_T + \rho. \quad (6g)$$

In a compact form, (6) can be stated by

$$\min_z \sum_{j \in \mathcal{M}} c_j^T z_j, \quad (7a)$$

$$\text{s.t. } G_j z_j \geq g_j, \quad j \in \mathcal{M}, \quad (7b)$$

$$\sum_{j \in \mathcal{M}} H_j z_j \geq h, \quad (7c)$$

where  $\mathcal{M} = 1, 2, \dots, M+1$  and

$$z_j = [u_j^T \quad \gamma_j^T]^T, \quad c_j = [p_j^T \quad r_j^T]^T, \quad j \in \mathcal{M}$$

$$z_{M+1} = \rho^T, \quad c_{M+1} = q^T.$$

In (7), (7b) represents the decoupled constraints (6b)-(6f), and (7c) represents the linking constraints (6g). The data structures in (7) are defined as

$$G_j = \begin{bmatrix} \bar{G}_j \\ -\bar{G}_j \end{bmatrix}, \quad g_j = \begin{bmatrix} \underline{g}_j \\ -\bar{g}_j \end{bmatrix}, \quad H_j = \begin{bmatrix} \bar{H}_j \\ -\bar{H}_j \end{bmatrix}, \quad h = \begin{bmatrix} \underline{h} \\ -\bar{h} \end{bmatrix},$$

where

$$\left[ \bar{G}_j \mid \underline{g}_j \mid \bar{g}_j \right] = \left[ \begin{array}{c|c|c} I & 0 & \underline{u}_j \\ \Lambda & 0 & \Delta \underline{u}_j \\ \Gamma_j & I & \underline{y}_j \\ \Gamma_j & -I & -\infty \\ 0 & I & 0 \end{array} \middle| \begin{array}{c} \bar{u}_j \\ \infty \\ \bar{y}_j \\ \bar{\gamma}_j \end{array} \right],$$

$$\left[ \bar{H}_j \mid \underline{h} \mid \bar{h} \right] = \left[ \begin{array}{c|c|c} \tilde{\Gamma}_j & 0 & \underline{y}_T \\ \tilde{\Gamma}_j & 0 & -\infty \end{array} \middle| \begin{array}{c} \infty \\ \bar{y}_T \end{array} \right],$$

for  $j \in \mathcal{M}$ , with

$$\begin{aligned} \underline{y}_T &= \underline{y}_T - \sum_{j \in \mathcal{M}} \tilde{\Phi}_j x_{j,0}, & \bar{y}_T &= \bar{y}_T - \sum_{j \in \mathcal{M}} \tilde{\Phi}_j x_{j,0}, \\ \underline{y}_j &= \underline{y}_j - \Phi_j x_{j,0}, & \bar{y}_j &= \bar{y}_j - \Phi_j x_{j,0}, & j \in \mathcal{M}, \\ \Delta \underline{u}_j &= \Delta \underline{u}_j + I_0 u_{j,-1}, & \Delta \bar{u}_j &= \Delta \bar{u}_j + I_0 u_{j,-1}, & j \in \mathcal{M}, \end{aligned}$$

and  $\Lambda$  and  $I_0$  defined as

$$\Lambda_j = \begin{bmatrix} I & & & & \\ -I & I & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & -I & I \end{bmatrix}, \quad I_0 = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

In the special case  $j = M+1$ ,  $\bar{H}_{M+1} = [I \quad -I]^T$  and

$$\left[ \bar{G}_{M+1} \mid \underline{g}_{M+1} \mid \bar{g}_{M+1} \right] = \left[ I \mid 0 \mid \bar{p} \right].$$

### III. DANTZIG-WOLFE DECOMPOSITION

Dantzig-Wolfe decomposition exploits that a convex set can be characterized by its extreme points and its extreme rays [5], [6]. For each  $j \in \mathcal{M}$ , the set of points satisfying the decoupled constraints (7b),  $\mathcal{G}_j = \{z_j \mid G_j z_j \geq g_j\}$ , may be written as

$$\mathcal{G}_j = \left\{ z_j \mid z_j = \sum_{i \in \mathcal{P}} \lambda_j^i z_j^i, \sum_{i \in \mathcal{P}} \lambda_j^i = 1, \lambda_j^i \geq 0 \quad \forall i \in \mathcal{P} \right\},$$

where  $z_j^i$  are the extreme points of  $\mathcal{G}_j$ , and  $\lambda_j^i$  are convex combination multipliers. Notice that since each of the sets  $\mathcal{G}_j$  are bounded, extreme rays are not needed in their representation.

By replacing the decision variables in (7) by convex combination multipliers, we obtain the master problem formulation

$$\min_{\lambda \geq 0} \phi = \sum_{j \in \mathcal{M}} \sum_{i \in \mathcal{P}} c_j^i \lambda_j^i, \quad (8a)$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{M}} \sum_{i \in \mathcal{P}} H_j^i \lambda_j^i \geq h, \quad (8b)$$

$$\sum_{i \in \mathcal{P}} \lambda_j^i = 1, \quad j \in \mathcal{M}, \quad (8c)$$

where we have defined  $H_j^i = H_j z_j^i$  and  $c_j^i = c_j^T z_j^i$  for each  $j \in \mathcal{M}$  and  $i \in \mathcal{P}$ .

Given a solution,  $\lambda^*$ , to the master problem (8), a solution to the original problem, (7), can be obtained as

$$z_j^* = \sum_{i \in \mathcal{P}} (\lambda^*)^i z_j^i, \quad j \in \mathcal{M}.$$

Since the number of extreme points,  $|\mathcal{P}|$ , can increase exponentially with the size of the original problem, solving the master problem directly is inefficient. As demonstrated in the following section however, the problem can be solved efficiently using a column generation procedure that replaces  $\mathcal{P}$  by a subset  $\tilde{\mathcal{P}}$ .

### IV. COLUMN GENERATION PROCEDURE

The dual linear program of (8) can be stated as

$$\max_{\alpha \geq 0, \beta} h^T \alpha + \sum_{j \in \mathcal{M}} \beta_j, \quad (9a)$$

$$\text{s.t.} \quad (H_j^i)^T \alpha + \beta_j \leq c_j^i, \quad j \in \mathcal{M}, i \in \mathcal{P}, \quad (9b)$$

in which  $\alpha \in \mathbb{R}^{4N}$  and  $\beta \in \mathbb{R}^{M+1}$  are the Lagrange multipliers associated with the linking constraints, (8b), and the convexity constraints, (8c), respectively. The necessary and sufficient optimality conditions for (8) and (9) are

$$\sum_{j \in \mathcal{M}} \sum_{i \in \mathcal{P}} H_j^i \lambda_j^i \geq h, \quad (10a)$$

$$\sum_{i \in \mathcal{P}} \lambda_j^i = 1, \quad j \in \mathcal{M}, \quad (10b)$$

$$\lambda_j^i \geq 0, \quad j \in \mathcal{M}, i \in \mathcal{P}, \quad (10c)$$

$$c_j^i - (H_j^i)^T \alpha - \beta_j \geq 0, \quad j \in \mathcal{M}, i \in \mathcal{P}, \quad (10d)$$

$$\alpha \geq 0, \quad (10e)$$

$$\lambda_j^i (c_j^i - (H_j^i)^T \alpha - \beta_j) = 0, \quad j \in \mathcal{M}, i \in \mathcal{P}, \quad (10f)$$

In Proposition 1 we derive conditions for which a solution satisfying this set of optimality conditions, can be obtained by solving the master problem (8) over a subset of the original variables.

*Proposition 1:* Let  $\tilde{\mathcal{P}} \subseteq \mathcal{P}$ , and define  $(\tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$  as a primal-dual solution to (8) and (9) restricted to the subset  $\tilde{\mathcal{P}}$ . Then the solution

$$\begin{aligned} \alpha^* &= \tilde{\alpha}, \\ \beta_j^* &= \tilde{\beta}_j, & j \in \mathcal{M}, \\ (\lambda^*)^i_j &= \begin{cases} \tilde{\lambda}_j^i & \text{if } i \in \tilde{\mathcal{P}} \\ 0 & \text{if } i \in \mathcal{P} \setminus \tilde{\mathcal{P}} \end{cases}, & j \in \mathcal{M}, i \in \mathcal{P}, \end{aligned}$$

satisfies the conditions, (10), if the optimal objective value of the subproblem

$$\phi_j = \min_{\tilde{z}_j} \{(c_j - H_j^T \alpha^*)^T \tilde{z}_j - \beta_j^* \mid G_j \tilde{z}_j \geq g_j\}, \quad (11)$$

is non-negative for each  $j \in \mathcal{M}$ .

**Proof** The solution  $(\lambda^*, \alpha^*, \beta^*)$  satisfies (10a) since

$$\sum_{j \in \mathcal{M}} \sum_{i \in \mathcal{P}} H_j^i (\lambda^*)^i_j = \sum_{j \in \mathcal{M}} \sum_{i \in \tilde{\mathcal{P}}} H_j^i \tilde{\lambda}_j^i \geq h,$$

which follows from the definition of  $(\tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$ . Similarly, it is easy to verify that the conditions (10c), (10b), (10e) and (10f) are fulfilled.

Provided that  $(\lambda^*, \alpha^*, \beta^*)$  is optimal, (10d) yields

$$c_j^i - (H_j^i)^T \alpha^* - \beta_j^* = (c_j - H_j^T \alpha^*)^T z_j^i - \beta_j^* \geq 0, \quad (12)$$

for all  $j \in \mathcal{M}$  and  $i \in \mathcal{P}$ . By construction of the solution, (12) is satisfied for all  $i \in \tilde{\mathcal{P}}$ . To check that the condition holds for all  $i \in \mathcal{P} \setminus \tilde{\mathcal{P}}$ , we consider the optimization problem (11). Since this linear program minimizes the left hand side of (12) over all possible extreme points,  $\tilde{z}_j$ , of  $\mathcal{G}_j$ , the solution

$(\lambda^*, \alpha^*, \beta^*)$  also satisfies the remaining optimality condition (12) if  $\varphi_j$  is non-negative for all  $j \in \mathcal{M}$ . ■

In Algorithm 1, we have outlined a column generation procedure based on Proposition 1. The algorithm exploits that if (12) is violated, then the solution to the subproblems, (11), provides a set of extreme points that can be added to the master problem. Notice that when  $\mathcal{P}$  is restricted to the subset  $\tilde{\mathcal{P}}$ , the master problem (8) is much smaller than the original problem. Therefore, the column generation procedure requires less memory than conventional linear programming methods. Moreover, solving the subproblems is computationally inexpensive as they do not grow with the number of units  $M$ . We remark that this step may be performed in parallel.

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**Algorithm 1** Column generation procedure for solving (8).

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**Require:**  $\{z_j^0\}_{j=1}^{\mathcal{M}}$   
 $i = 0$ , converged = false  
**while not converged do**  
 $\tilde{\mathcal{P}} = \{0, 1, \dots, i\}$   
**for**  $j \in \mathcal{M}, i \in \tilde{\mathcal{P}}$  **do**  
 $H_j^i = H_j z_j^i, c_j^i = c_j^T z_j^i$   
**end for**  
 $(\phi^*, \lambda^*, \alpha^*, \beta^*) \leftarrow$  solve (8) with  $\mathcal{P} = \tilde{\mathcal{P}}$   
**for**  $j \in \mathcal{M}$  **do**  
 $(\varphi_j^*, z_j^*) \leftarrow$  solve (11)  
**end for**  
**if**  $\varphi_j \geq 0 \forall j \in \mathcal{M}$  **then**  
converged = true  
**else**  
**for**  $j \in \mathcal{M}$  **do**  
 $z_j^{i+1} = z_j^*$   
**end for**  
 $i = i + 1$   
**end if**  
**end while**

---

#### A. Warm-Starting

Algorithm 1 requires a set of initial points  $\{z_j^0\}_{j=1}^{\mathcal{M}}$  that are feasible for both the subproblems (11) and the original problem (7). As economic MPC is a receding horizon strategy, we can generate such a set of points by exploiting the solution from a previous time step.

Given the solution to (11)

$$z_j^* = [u_{j,0}^{*T} \quad \dots \quad u_{j,N-1}^{*T} \quad \gamma_{j,1}^{*T} \quad \dots \quad \gamma_{j,N}^{*T}]^T,$$

$$z_{M+1}^* = [\rho_1^{*T} \quad \dots \quad \rho_N^{*T}]^T,$$

we build a set of initial points in the following sampling instant as

$$z_j^0 = [u_{j,1}^{*T} \quad \dots \quad u_{j,N-1}^{*T} \quad \check{u}_j^T \quad \gamma_{j,2}^{*T} \quad \dots \quad \gamma_{j,N}^{*T} \quad \check{\gamma}_j^T]^T,$$

$$z_{M+1}^0 = [\rho_2^{*T} \quad \dots \quad \rho_N^{*T} \quad \check{\rho}^T]^T,$$

for each  $j \in \mathcal{M}$ . Hence, the original solution values are shifted forward in time, and the variables  $\check{u}_j$ ,  $\check{\gamma}_j$  and  $\check{\rho}$  are appended

to the initial points. In our implementation, we let

$$\check{u}_j = u_{j,N-1}^*, \quad j \in \mathcal{M}, \quad (13)$$

which leads to an initial input sequence with constant input in the two final sampling intervals. Using the state space equations (1)-(2), we compute the outputs  $\check{y}_{j,N}$  and  $\check{y}_{T,N}$  associated with this input sequence. Based on these values we let

$$\check{\gamma}_j = \max(\underline{y}_{j,N} - \check{y}_{j,N}, 0) + \max(\check{y}_{j,N} - \bar{y}_{j,N}, 0),$$

$$\check{\rho} = \max(\underline{y}_{T,N} - \check{y}_{T,N}, 0) + \max(\check{y}_{T,N} - \bar{y}_{T,N}, 0),$$

where the maximum function is evaluated element-wise.

Assuming that the inputs (13) satisfy the input constraints for the updated problem data, and that the upper limits on  $\gamma_j$  and  $\rho$  are sufficiently large, the strategy above yields a set of feasible initial points for Algorithm 1,  $\{z_j^0\}_{j=1}^{\mathcal{M}}$ , which exploits the solution obtained in the previous time step. As the solution in successive time steps are closely related in MPC applications, this approach provides a warm-start for Algorithm 1. In case no previous solution is available, a similar strategy can be used to adjust the slack variables for an arbitrary feasible input sequence.

## V. RESULTS

In this section, we compare a MATLAB implementation of Algorithm 1, denoted DWempc, to linear programming solvers from the following software packages: CPLEX, Gurobi and MOSEK. The algorithms are run on an Intel(R) Core(TM) i5-2520M CPU @ 2.50GHz with 4 GB RAM running a 64-bit Windows 7 Enterprise operating system. In DWempc, the restricted master problem and the subproblems are solved using CPLEX.

As a conceptual case study, we consider a collection of power generating units in the form

$$Y_j(s) = 1/(\tau_j s + 1)^3 U_j(s), \quad j \in \mathcal{M}, \quad (14)$$

where  $U_j(s)$  is the fuel input and the  $Y_j(s)$  is the power production. The third order model, (14), has been validated against actual measurement data in [11]. In our study, we vary the time constant,  $\tau_j$ , to represent different types of power generating units. Time constants in the range 80-120 are associated with slow units, such as centralized thermal power plants, while time constants in the range 20-60 represent units with faster dynamics such as diesel generators and gas turbines. To control the units, (14), using economic MPC, we realize the system in the discrete state space form (1)-(2) using a sampling time of  $T_s = 5$  seconds. In the resulting model structure,  $u_{j,k} \in \mathbb{R}$  is fuel input,  $y_{j,k} \in \mathbb{R}$  is the power production, and  $y_{T,k} \in \mathbb{R}$  is the total power production. Thus,  $\Upsilon_j = 1$ , for all  $j \in \mathcal{M}$ . Fig. 1 demonstrates the production plan obtained using economic MPC in a case study with  $M = 3$  power generating units. The graphs show the individual outputs, as well as the output limits for the total power production. The case study parameters are listed in Table I. All parameters listed in the table, are kept constant over the entire

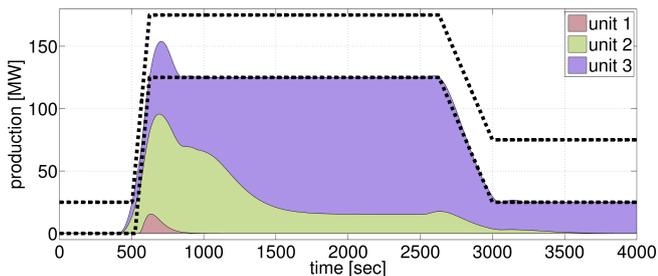


Fig. 1. Closed-loop simulation study of economic MPC.

TABLE I  
CASE STUDY PARAMETERS

	$\tau_j$	$p_{j,k}$	$u_{j,k}$	$\bar{u}_{j,k}$	$\Delta u_{j,k}$	$\Delta \bar{u}_{j,k}$
Generating Unit 1	40	24	0	50	-30	30
Generating Unit 2	90	12	0	100	-20	20
Generating Unit 3	100	6	0	200	-5	5

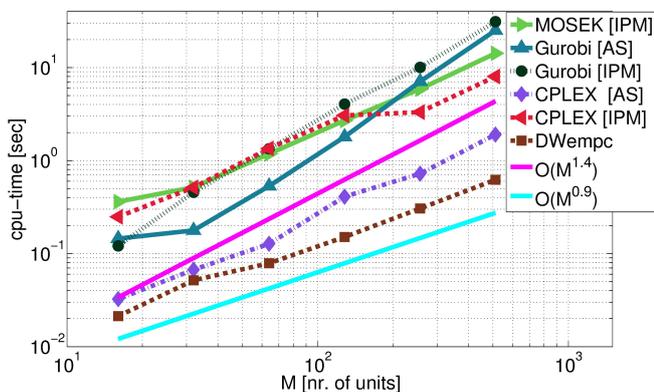


Fig. 2. CPU-time for solving (3) as a function of the number of power generating units, and fixed  $N = 50$ . Active-set methods are denoted by (AS) and interior-point methods are denoted by (IPM).

horizon. The values,  $p_{j,k}$ , are the prices pr. unit of fuel (e.g oil, natural gas or coal). We have defined these parameters such that the fuel price for fast units is higher than the fuel price for slow units. The price for imbalances is fixed to  $q_k = 10000$ .

Based on a similar case study as above, we have solved the constrained optimal control problem, (3), for an increasing number of generating units. The computation time is depicted in Fig. 2. In the simulations  $DWempc$  outperforms conventional linear programming solvers with a significant margin, and the difference in computing time grows with the number of units controlled. As the subproblems are solved sequentially in our implementation, the performance of  $DWempc$  can be improved even further using parallel computing. To make a fair comparison, we have initialized all algorithms using their default cold-starting point. For  $DWempc$  we define such a cold-starting point by setting all inputs,  $u_{j,k}$  to zero, and increase the slack variables,  $\gamma_{j,k}$  and  $\rho_k$ , to their upper bounds. In Table II, we have listed the iteration numbers associated with Figure 2. For  $DWempc$ , increasing the number of generating

TABLE II  
COMPARISON OF THE NUMBER OF ITERATIONS

M	MOS [IPM]	GUR [AS]	GUR [IPM]	CPL [AS]	CPL [IPM]	DWempc
16	19	692	22	590	20	30
32	33	700	49	486	44	31
64	34	902	55	559	43	25
128	36	1452	46	1058	40	24
256	36	2221	60	1168	47	23
512	42	3087	65	1737	62	20

units decreases the number of iterations, while for all other solvers the number of iterations increases. We expect that the number of iterations can be reduced additionally in closed-loop by employing the warm-starting strategy proposed in Section IV-A.

## VI. CONCLUSIONS

In this paper, we have presented a warm-started Dantzig-Wolfe decomposition algorithm for economic MPC of distributed energy systems. Our results show that a MATLAB implementation of the algorithm is significantly faster than both active-set methods and interior-point methods, provided by MOSEK, CPLEX and Gurobi. Moreover,  $DWempc$  has several desirable features, such as low memory costs and parallelization capabilities, which makes it favorable for real-time applications such as economic MPC.

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APPENDIX

E

# Paper E - DTU International Energy Conference, 2013

**Economic MPC to Operate Power Systems via Dantzig-Wolfe  
Decomposition**

**Authors:**

Laura Standardi, Leo Emil Sokoler, Niels Kjølstad Poulsen and John Bagterp  
Jørgensen.

**Published in:**

*DTU International Energy Conference, 2013.*



# Economic MPC to Operate Power Systems via Dantzig-Wolfe Decomposition

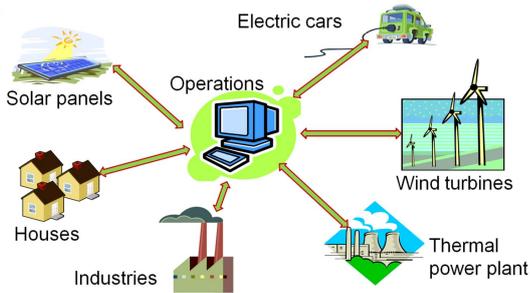
Laura Standardi<sup>1</sup>, Leo Emil Sokoler<sup>2</sup>, Niels Kjølstad Poulsen<sup>1</sup> and

John Bagterp Jørgensen<sup>1</sup>

**DTU International Energy Conference 2013**

## Introduction

- Future power systems will consist of a large number of independent power units.
- New control algorithms are necessary to control and coordinate the power production in order to satisfy the customers' demand, minimizing production costs.



## Economic Model Predictive Control

- Consider a power system consisting of  $P$  power units.
- The regulator computes the optimal production plan for each power unit.
- Therefore, the total power production satisfy the customers' demand and minimizes the production costs.
- The production costs and the power demand are assumed to be available, provided by external forecasting systems.
- The control linear problem is, therefore, formulated as

$$\min \phi_k = \sum_{i=1}^P \sum_{j=0}^{N-1} c'_{i,k} u_{i,k} + \rho'_{i,k} s_{i,k} + \tilde{\rho}'_k \tilde{s}_k \quad (1a)$$

$$s.t. \quad x_{i,k+1} = A_i x_{i,k} + B_i u_{i,k} + E_i d_{i,k} \quad (1b)$$

$$z_{i,k} = C_{z,i} x_{i,k} \quad (1c)$$

$$u_{\min,i} \leq u_{i,k} \leq u_{\max,i} \quad (1d)$$

$$\Delta u_{\min,i} \leq \Delta u_{i,k} \leq \Delta u_{\max,i} \quad (1e)$$

$$z_{i,k} + s_{i,k} \geq r_{\min,i,k}, z_{i,k} - s_{i,k} \leq r_{\max,i,k}, s_{i,k} \geq 0 \quad (1f)$$

$$\tilde{z}_k = \sum_{i=1}^P \tilde{C}_{z,i} x_{i,k} + \tilde{D}_{z,i} u_{i,k} \quad (1g)$$

$$\tilde{z}_k + \tilde{s}_k \geq \tilde{r}_{\min,k}, \tilde{z}_k - \tilde{s}_k \leq \tilde{r}_{\max,k}, \tilde{s}_k \geq 0. \quad (1h)$$

## Dantzig-Wolfe Decomposition

- The control linear problem has the following block-angular structure

$$\min_{\{w_{i,k}\}_{i=1}^M} \varphi = \sum_{i=1}^M c'_i w_{i,k} \quad (2a)$$

$$s.t. \quad \begin{bmatrix} F_1 & F_2 & \dots & F_M \\ G_1 & & & \\ & G_2 & & \\ & & \dots & \\ & & & G_M \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_M \end{bmatrix} \geq \begin{bmatrix} g \\ h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix} \quad (2b)$$

where  $M = 1, \dots, P, P + 1$  as the global slack variables  $\tilde{s}_k$  are considered as an independent unit.

- **Early Termination:**

- This strategy stops the Dantzig-Wolfe algorithm after a fixed number of iteration: in this way the solution is not optimal but still feasible.
- Furthermore, the number of iterations is reduced.

## Case Study

$$Z_i(s) = \frac{1}{(\tau_i s + 1)^3} U_i(s) \quad (3)$$

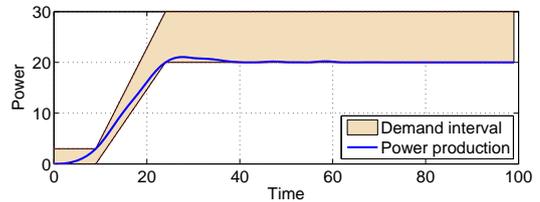
- The case study is a power system including power units modelled as (3), representing
  - central thermal,
  - diesel generators,
  - power plants,
  - gas turbines.

## Power Production & Customers' Demand Interval [2]

- The control algorithm coordinates and controls the power units keeping the overall power production inside the demand interval.

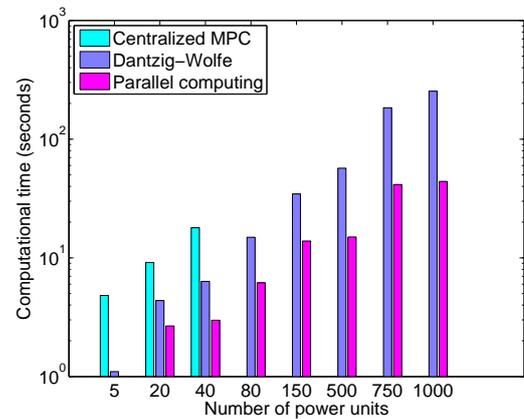
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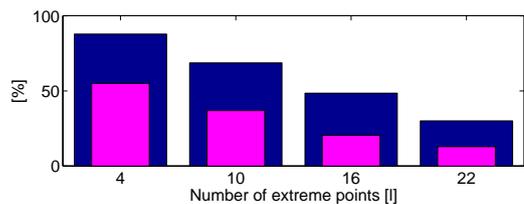
## Computational Times [1]

- The Dantzig-Wolfe decomposition provides a decentralized controller that computes the optimal trajectories even when the centralized one fails.
- Parallel computing highly reduces computation times.



## Early Termination Results [2]

- The early termination technique substantially reduces computation times (blue bars). However, it yields to unavoidable extra costs (magenta bars).



## Conclusions

- The algorithm proposed operates large-scale power systems.
- Our strategy overcomes the computation time issues by the implementation of parallel computing and early termination strategy.

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APPENDIX

F



# Paper F - 18<sup>th</sup> Nordic Process Control Workshop, 2013

Early Termination of Dantzig-Wolfe Algorithm for Economic  
MPC

**Authors:**

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**Published in:**

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# Early Termination of Dantzig-Wolfe Algorithm for Economic MPC

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**Abstract:** In this paper we apply the Economic Model Predictive Control (MPC) for balancing the power supply and demand in the future power systems in the most economic way. The control problem is formulated as a linear program, having a block-angular structure solved by the implementation of the Dantzig-Wolfe decomposition. For real-time applications we introduce an early termination technique. Simulations demonstrate that the algorithm developed operates efficiently a power system, reducing significantly computational time.

*Keywords:* Economic Model Predictive Control, Linear Programming, Distributed Optimization, Power Systems

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## 1. INTRODUCTION

During the past decades climate has dramatically changed. Scientists define the recent global warming as unprecedented and emphasize the need of accelerated and urgent actions (Vidal, 2013). CO<sub>2</sub> emissions and other pollutants are collecting in the atmosphere like a thickening blanket, trapping the sun's heat and causing the planet to warm up. The combustion of fossil fuels to generate electricity is one of the largest source of CO<sub>2</sub> emissions (United States Environmental Protection Agency, 2013). Hence, a transition from fossil to non-fossil fuels plays a key role in our future, leading to a new electricity system. Future electric grids will consist of independent energy sources and customers; these are characteristics of Smart Grids (European Technology Platform SmartGrids, 2012; The Danish Energy Agreement of March 2012, Ministry of Climate, Energy and Building, 2012). Renewable energy sources (RES) take part in the Smart Grids with their intermittent energy production. This innovative scenario requires control actions so as to ensure the total energy production satisfies customers' demands.

We propose an optimization-based controller to balance power production and consumption in an economically efficient way. As a case study we consider a large scale system in which multiple power generators that are dynamically decoupled, operate in a coordinated way to serve as a single power portfolio. We address two issues related to the power management in a large scale scenario. The first issue concerns minimizing the cost of producing enough power to meet the market demand. The second issue concerns providing supply security. Our control strategy is an Economic MPC applied to a power portfolio in a large scale scenario. The optimization problem of the proposed controller shows a block-angular constraints matrix; because of this, we solve the control problem by using Dantzig-Wolfe decomposition. However, real-time applications re-

quire fast computation of the optimal control sequence: because of this, an early termination strategy is applied on the Dantzig-Wolfe decomposition algorithm. Such early termination provides a suboptimal solution of MPC and reduces significantly computational times.

Recent applications for energy systems have included the Economic MPC: refrigeration systems (Hovgaard et al., 2010, 2011, 2012a,b), heat pumps for residential buildings (Halvgaard et al., 2012c), solar-heated water tanks (Halvgaard et al., 2012a), and batteries in electrical vehicles (Halvgaard et al., 2012b). Due to computational complexity and the communication bandwidth limitation, distributed control structures have been developed for large-scale systems (Scattolini, 2009). The interest in distributed MPC has led to the use of decomposition techniques applied to large-scale linear programs, (Lasdon, 1970; Chvatal, 1983; Nazareth, 1987; Dantzig and Thapa, 2003; Conejo et al., 2006). The Dantzig-Wolfe decomposition algorithm for large linear programs was first introduced in 1960 (Dantzig and Wolfe, 1960, 1961). However, recently, the Dantzig-Wolfe algorithm has been used in a number application connected to the MPC: in an oil field by (Gunnerud and Foss, 2010; Gunnerud et al., 2010), control of building temperature (Morsan et al., 2011) and power balancing (Edlund et al., 2011). Suboptimal MPC controllers are stabilizing and guarantee feasibility and stability of the controller (Pannocchia et al., 2010). However, often real-time suboptimal MPC is a combination of offline and online optimization (Scokaert et al., 1999; Zeilinger et al., 2008). Other strategies involve online active set and bounds on the CPU time (Ferreau et al., 2008) and early termination approach for interior point methods (Wang and Boyd, 2010).

The outline of the paper is as follows: Section 2 introduces power systems. Section 3 formulates a linear Economic MPC for linear power systems. Section 4 describes the Dantzig-Wolfe decomposition algorithm. The early ter-

mination strategy is explained in Section 5. Section 6.1 proposes a model for the power generators included in the portfolio; Section 6.2 reports simulation results and, finally, the conclusion and suggestions for future work are presented in Section 7.

## 2. POWER SYSTEMS

Power system consists of a number of independent power units, such as power producers and consumers. Figure 1 depicts a generic power system, where power units are connected only with *operation* center. The total power supply includes the production from each of these independent power producers. Such power systems are also called Distributed Energy Sources (DES). Moreover, power units are independent and dynamically decoupled systems; such decoupled models are ubiquitous in power systems. Accordingly, the energy units considered in this paper can be described as a linear discrete time state space model

$$x_{k+1} = Ax_k + Bu_k, \quad (1a)$$

$$y_k = Cx_k, \quad (1b)$$

$$z_k = C_z x_k. \quad (1c)$$

$x_k$  denotes the states,  $u_k$  the manipulated variables (MVs),  $y_k$  denotes the measurement used for feedback, and  $z_k$  is output variables.

The manipulated variable,  $u_k$ , is subject to bounds and rate-of-movements constraints

$$u_{min} \leq u_k \leq u_{max} \quad (2a)$$

$$\Delta u_{min} \leq \Delta u_k \leq \Delta u_{max} \quad (2b)$$

These are hard constraints and not mean-value constraints.

The system output  $z_k$  denotes the power produced by the generator and it must satisfy the customers' demand,  $r$ . Often the electricity demand is forecast in advance and defined by an interval as  $[r_{min,k}, r_{max,k}]$ ; we assume to have such demand interval from external forecasts. However, due to the manifold power units involved, it might be impossible to have the total power production  $z_k$  within the demand interval; because of this, the constraints on the power produced include slack variables  $s_k$ . The slack variables,  $s_k$ , may represent selling or buying power from the short-term market, violation of temperature limits, or violation of state-of-charge limits. Every time  $s_k$  is non-zero, a penalty cost, e.g. the cost of buying or selling power on the short-term market must be paid.

$$r_{min,k} - s_k \leq z_k \leq r_{max,k} + s_k \quad (3a)$$

$$s_k \geq 0 \quad (3b)$$

The cost of producing power over a period of time, is  $\phi_k$ . This economic cost,  $\phi_k$ , consists of the cost of operating a power generator,  $c_k$ , and the penalties,  $\rho_k$ , related to the use of slack variables,  $s_k$

$$\phi_k = \sum_{j=0}^{N-1} c'_k u_k + \sum_{j=0}^{N-1} \rho'_k s_k. \quad (4)$$

## 3. ECONOMIC MPC FOR OPERATIONS

Figure 1 illustrates a power system where the *operations* center has the task to coordinate and control power units.

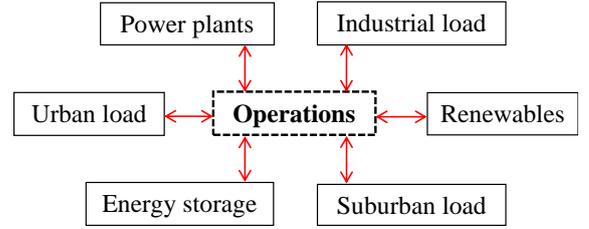


Fig. 1. A generic future power system. Power producers and consumers are independent units, and *Operations* coordinates and controls these power units to guarantee power supply in response to the customers' demand.

Operating such power system means making real-time decisions as planning the power production in response to the customers' demand. This section introduces Economic Model Predictive Control (MPC) to operate a power system as the one in Figure 1 balancing power supply and demand in the most economic way.

Consider a power system, as described in Section 2, which consists of  $P$  power producers. These power generators collectively produce the total portfolio power production  $\hat{z}_{k+j+1|k}$  subject to the following connecting constraints

$$\hat{z}_{k+j+1|k} = \sum_{i=1}^P \tilde{C}_i \hat{x}_{i,k+j+1|k}, \quad (5a)$$

$$\hat{z}_{k+j+1|k} + s_{k+j+1|k} \geq \hat{r}_{min,k+j+1|k}, \quad (5b)$$

$$\hat{z}_{k+j+1|k} - s_{k+j+1|k} \leq \hat{r}_{max,k+j+1|k}, \quad (5c)$$

$$s_{k+j+1|k} \geq 0. \quad (5d)$$

Constraints (5b)-(5d) are equivalent to the constraints (3) but referring to the total power produced by the power system.

The Economic MPC is formulated as a linear program because of the linear dynamics of the power units (1), linear cost functions (4), and linear constraints (2)-(3) and (5). In addition, a Kalman filter predicts  $\hat{x}_{k+1+j|k}$ . Accordingly, the Linear Economic MPC to operate a power system of  $P$  power units, is formulated as

$$\min \phi_k = \sum_{i=1}^P \phi_{i,k} + \sum_{j=0}^{N-1} \rho'_{k+j+1|k} s_{k+j+1|k} \quad (6)$$

subject to the local constraints  $\forall i \in \mathcal{P}$  and  $\forall j \in \mathcal{N}$

$$\hat{x}_{i,k+j+1|k} = A_i \hat{x}_{i,k+j|k} + B_i u_{i,k+j|k} \quad (7a)$$

$$\hat{z}_{i,k+j+1|k} = C_{z,i} \hat{x}_{i,k+j+1|k} \quad (7b)$$

$$u_{min,i} \leq u_{i,k+j|k} \leq u_{max,i} \quad (7c)$$

$$\Delta u_{min,i} \leq \Delta u_{i,k+j|k} \leq \Delta u_{max,i} \quad (7d)$$

$$\hat{z}_{i,k+j+1|k} + s_{i,k+j+1|k} \geq \hat{r}_{min,i,k+j+1|k} \quad (7e)$$

$$\hat{z}_{i,k+j+1|k} - s_{i,k+j+1|k} \leq \hat{r}_{max,i,k+j+1|k} \quad (7f)$$

$$s_{i,k+j+1|k} \geq 0 \quad (7g)$$

and subject to the connecting constraints  $\forall j \in \mathcal{N}$  in (5).

The optimization control problem (5)-(7) has a block-angular structure that is suitable for the implementation of Dantzig-Wolfe decomposition to solve efficiently the control linear program.

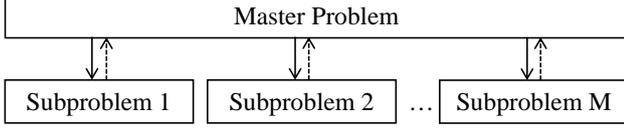


Fig. 2. Dantzig-Wolfe structure. Each subproblem communicates exclusively with the master problem that must coordinate such units.

#### 4. DANTZIG-WOLFE DECOMPOSITION TECHNIQUE

The Dantzig-Wolfe decomposition algorithm is a decomposition technique to solve efficiently linear programs having a block-angular structure, as (5)-(7), (Dantzig and Wolfe, 1960, 1961). The Economic MPC expressed as a linear program in (5)-(7), can be formulated as

$$\min_{q_{i,k}} \varphi = \sum_{i=1}^M e'_i q_{i,k} \quad (8a)$$

$$s.t. \begin{bmatrix} F_1 & F_2 & \dots & F_M \\ G_1 & & & \\ & G_2 & & \\ & & \ddots & \\ & & & G_M \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_M \end{bmatrix} \geq \begin{bmatrix} g \\ h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix} \quad (8b)$$

where  $i \in \mathcal{M} = \{1, \dots, P, P+1\}$  as the slack variables  $s_{k+j+1|k}$  in (5) and (6) are considered as an extra unit. Therefore,

$$e'_j = [p'_{j,0} \dots p'_{j,N} \hat{\rho}'_N], q'_j = [\bar{u}'_{j,0} \dots \bar{u}'_{j,N} s'_N]$$

where the variables  $p'_k$  and  $\bar{u}'_k$  are from the objective function (4)

$$\phi_k = \sum_{j=0}^{N-1} c'_k u_k + \sum_{j=0}^{N-1} \rho'_k s_k = \sum_{j=0}^{N-1} p'_k \bar{u}_k$$

The  $M$  diagonal blocks in the linear program (8) denotes  $M$  subproblems having their own set of constraints. Moreover, a master problem coordinates such subproblems as Figure 2 shows. From here on we assume that the feasible region of each subproblems is closed and bounded.

In a view of describing the Dantzig-Wolfe decomposition technique, it is necessary to introduce the *convex combination* theorem (Dantzig and Thapa, 2003).

*Theorem 1.* (Convex Combination). Consider  $\mathcal{Q} = \{q \mid Gq \geq h\}$  be nonempty, bounded and closed set, i.e. a polytope.  $v^j$  denotes the extreme point of  $\mathcal{Q}$  with  $j \in \{1, 2, \dots, V\}$ .

Then any point  $q$  in the polytope  $\mathcal{Q}$  can be written as a convex combination of its extreme points

$$q = \sum_{j=1}^V \lambda_j v^j \quad (9a)$$

$$s.t. \lambda_j \geq 0, \quad j = 1, 2, \dots, V \quad (9b)$$

$$\sum_{j=1}^V \lambda_j = 1 \quad (9c)$$

**Proof.** See (Dantzig and Thapa, 2003).

Substituting (9) into (8) yields to the following linear program

$$\min_{\lambda} \varphi = \sum_{i=1}^M \sum_{j=1}^{V_i} f_{ij} \lambda_{ij} \quad (10a)$$

$$s.t. \sum_{i=1}^M \sum_{j=1}^{V_i} p_{ij} \lambda_{ij} \geq g \quad (10b)$$

$$\sum_{j=1}^{V_i} \lambda_{ij} = 1, \quad i = 1, 2, \dots, M \quad (10c)$$

$$\lambda_{ij} \geq 0, \quad i = 1, 2, \dots, M; j = 1, 2, \dots, V_i \quad (10d)$$

where the coefficients are

$$f_{ij} = e'_i v_i^j, \quad p_{ij} = F_i v_i^j \quad (11)$$

The linear program (10), known as *Master Problem (MP)*, is equivalent to the block-angular linear problem (8). It is worth noting that (10) has fewer constraints than the original problem (8). However the MP considers the extreme points of each subproblem, thus the number of variables is larger than in the original problem (8). The Dantzig-Wolfe decomposition algorithm overcomes this problem by including a reduced number of extreme points, and adding new vertices when needed. As a result, the *Reduced Master Problem (RMP)* is defined as

$$\min_{\lambda} \varphi = \sum_{i=1}^M \sum_{j=1}^l f_{ij} \lambda_{ij} \quad (12a)$$

$$s.t. \sum_{i=1}^M \sum_{j=1}^l p_{ij} \lambda_{ij} \geq g \quad (12b)$$

$$\sum_{j=1}^l \lambda_{ij} = 1, \quad i = 1, 2, \dots, M \quad (12c)$$

$$\lambda_{ij} \geq 0, \quad i = 1, 2, \dots, M; j = 1, 2, \dots, l \quad (12d)$$

where  $l \leq V_i$  for all  $i \in \{1, 2, \dots, M\}$ . Solving the RMP provides the Lagrangian multipliers  $\pi$  associated with the inequality constraint (12b), the Lagrangian multipliers  $\rho$ , associated with equalities (12c), and the Lagrange multipliers  $\kappa$  for the positivity constraints (12d). These are playing a key role in the Dantzig-Wolfe algorithm as they represent the information sent from the Master Problem to each subproblem. The Lagrangian associated to the Master Problem (10) yields to the following necessary and sufficient optimality conditions, for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, V_i$

$$\nabla_{\lambda_{ij}} \mathcal{L} = f_{ij} - p'_{ij} \pi - \rho_i - \kappa_{ij} = 0 \quad (13a)$$

$$\sum_{i=1}^M \sum_{j=1}^{V_i} p_{ij} \lambda_{ij} - g \geq 0 \quad \perp \quad \pi \geq 0 \quad (13b)$$

$$\sum_{j=1}^{V_i} \lambda_{ij} - 1 = 0 \quad (13c)$$

$$\lambda_{ij} \geq 0 \quad \perp \quad \kappa_{ij} \geq 0 \quad (13d)$$

We notice that the conditions (13a) and (13d) imply

$$\kappa_{ij} = f_{ij} - p'_{ij} \pi - \rho_i = [e_i - F_i^t \pi]^t v_i^j - \rho_i \geq 0 \quad (14)$$

such that the KKT-conditions for (10) may be stated as for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, V_i$

$$\sum_{i=1}^M \sum_{j=1}^{V_i} p_{ij} \lambda_{ij} - g \geq 0 \quad \perp \quad \pi \geq 0 \quad (15a)$$

$$\sum_{j=1}^{V_i} \lambda_{ij} - 1 = 0 \quad (15b)$$

$$\lambda_{ij} \geq 0 \quad \perp \quad \kappa_{ij} = [e_i - F'_i \pi]' v_i^j - \rho_i \geq 0 \quad (15c)$$

An optimal solution must satisfy the KKT conditions (15). We denote  $\lambda_{ij}^{RMP}$  a solution of RMP, such that a feasible solution to Master Problem (10) is

$$\lambda_{ij} = \lambda_{ij}^{RMP} \quad i = 1, 2, \dots, M; j = 1, 2, \dots, l \quad (16a)$$

$$\lambda_{ij} = 0 \quad i = 1, 2, \dots, M; j = l+1, l+2, \dots, V_i \quad (16b)$$

This solution satisfies (15a) and (15b). To be optimal it also needs to satisfy (15c). These conditions are already satisfied for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, l$ . We need to verify whether they are satisfied for all  $i = 1, 2, \dots, M$  and  $j = l+1, l+2, \dots, V_i$ . This is complicated by the fact that we only know the extreme points,  $v_i^j$  for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, l$ . An efficient initialization technique is introduced in (Standardi et al., 2012). Condition (15c) is satisfied for all  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, V_i$  if  $\min_i \psi_i - \rho_i \geq 0$  where

$$\psi_i = \min_{v_i^j} [e_i - F'_i \pi]' v_i^j \quad (17)$$

$v_i^j$  is an extreme point of the polytope  $\mathcal{Q}_i = \{q_i \mid G_i q_i \geq h_i\}$ . Therefore, using the Simplex Algorithm we compute the solution of (17) as a solution of the following linear program

$$\psi_i = \min_{q_i} \varphi = [e_i - F'_i \pi]' q_i \quad (18a)$$

$$s.t. \quad G_i q_i \geq h_i \quad (18b)$$

These linear programs are called *subproblems* and can be solved via either parallel or sequential computation; this possible parallel computation of the subproblems represents one of the advantages of the Dantzig-Wolfe decomposition algorithm. Let  $(\psi_i, q_i)$  be the optimal value-minimizer pair for the linear problem (18); then if

$$\psi_i - \rho_i \geq 0 \quad \forall i \in \{1, 2, \dots, M\} \quad (19)$$

is satisfied, then the solution computed from the RMP is optimal. Therefore the solution of the original control problem (8) is given by

$$q_i^* = \sum_{j=1}^l v_i^j \lambda_{ij} \quad i \in \{1, 2, \dots, M\} \quad (20)$$

Otherwise, if (19) is not satisfied, then the number of extreme points considered,  $l$ , is not enough and a new vertex  $v_i^{l+1}$  needs to be included.

The Dantzig-Wolfe algorithm needs an initial feasible solution. As this decomposition algorithm solves an Economic MPC, the previous solution is available and utilized as initial value at the next sampling time. To initialize the slack variables in the control problem (5)-(7) the output constraints (3) are utilized.

## 5. EARLY TERMINATION

The Dantzig-Wolfe decomposition solves the control problem reducing computational times (Standardi et al., 2012).

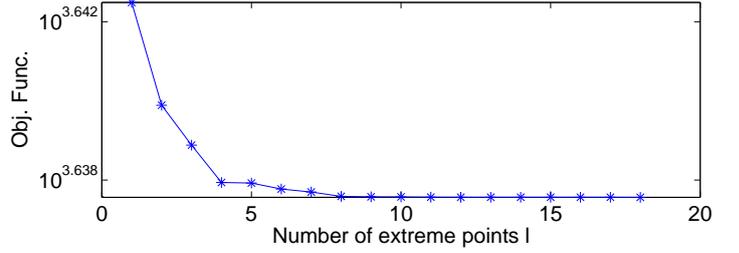


Fig. 3. Master Problem (10) objective function Vs. number of extreme points of the polytope (9).

However, many real-time applications include limits on the computational time that restrict the applicability of the MPC; real-time constraints and high-speed application may prevent the computation of the optimal controller as well. Early termination strategy and suboptimal MPC maintain feasibility and stability, as demonstrated in (Zeilinger et al., 2008; Scokaert et al., 1999; Pannocchia et al., 2010; Wang and Boyd, 2010). Section 4 illustrates that  $l$  extreme points of the feasible polytope are necessary to compute the optimal solution  $q_i^*$  (20); the Dantzig-Wolfe algorithm includes one vertex of the polytope at each iteration until the stopping criteria (19) is not satisfied. However, a smaller number of vertices can compute a solution that is not optimal but feasible though. The computation of such suboptimal solution reduces the number of iterations in the Dantzig-Wolfe algorithm, hence, reduces the computational time.

## 6. APPLICATION TO A POWER SYSTEM

In this section we apply the Economic MPC controller to a power system consisting of power plants, and the Dantzig-Wolfe decomposition computes the optimal control trajectory. In addition, we implement the early termination strategy in order to reduce computational times.

### 6.1 Boiler Load Generators

Section 2 introduces power units as independent and dynamically decoupled systems; these power units are coupled only through the objective to follow the customers' demand. This work includes boiler load units as power unit and the models are (Edlund et al., 2009)

$$Z_i(s) = G_i(s)U_i(s) \quad G_i(s) = \frac{1}{(\tau_i s + 1)} \quad (21)$$

where  $z_i(t)$  is the produced power at unit  $i$ , while  $u_i(t)$  is the corresponding reference signal.

### 6.2 Simulations Results

We apply the algorithm developed in this paper on a power system consisting of five power plants as described in Section 6.1. Open-loop simulation provides Figure 3 that illustrates the reason of early termination effectiveness. Section 4 describes that the Dantzig-Wolfe algorithm computes the optimal solution considering a certain number of extreme points of the feasible polytope (9). With reference to the number of extreme points necessary to compute the optimal solution, Section 5 introduces the

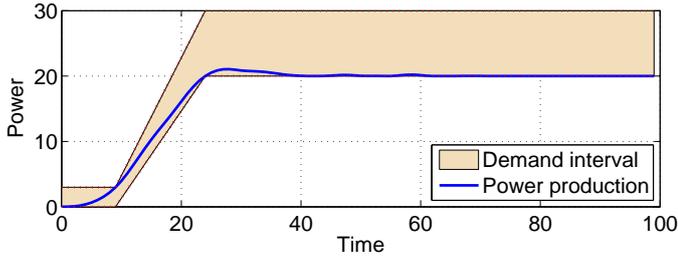


Fig. 4. Closed-loop simulations results. The total power production is within the customers' demand interval.

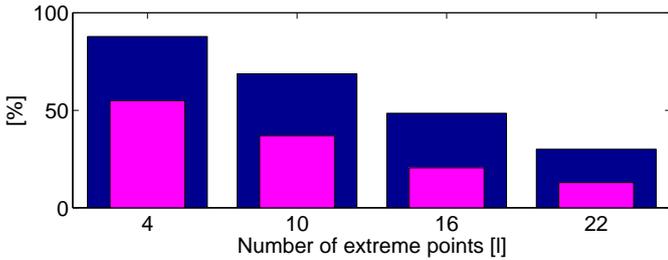


Fig. 5. CPU time vs. Extra Costs. The early termination strategy decreases CPU time, blu bars. However, such strategy leads to extra costs shown in the magenta bars. Consequently, a smaller number of extreme points of the feasible polytope yields to a decrease of the CPU time and extra costs to pay. For instance, if we set 16 as upper bound on the number of extreme points, then there is a decrease of 50% on the CPU time and 20% of extra costs to pay.

early termination strategy. Accordingly, Figure 3 shows that the master problem objective function  $\varphi(10)$  reaches its optimal value before the stopping criteria (19) of the Dantzig-Wolfe algorithm is satisfied.

The Economic MPC strategy controls a power system, and the Dantzig-Wolfe decomposition solves efficiently the control linear problem. The controller performances are in Figure 4, where the power system output is kept within the interval demand for the entire closed-loop simulation. Figure 5 reports the early termination effects. In closed-loop simulation the Dantzig-Wolfe algorithm computes the optimal control trajectory at each sampling time; in average, the decomposition algorithm takes 25 extreme points of the feasible polytope. The early termination utilizes fewer extreme points by setting bounds on these vertices. Such strategy reduces the computational time appreciably even higher than 50%. Whereas, the early termination leads to extra costs upwards of 10%.

## 7. CONCLUSION

Future power systems need new control algorithms to balance power supply and demand efficiently. The Economic MPC can operate power systems efficiently. The work of this paper differs from the recent applications of Economic MPC to energy systems because we compute the optimal control trajectory implementing a decomposition technique, known as Dantzig-Wolfe. Moreover, the early termination approach provides valuable results reducing substantially computational times. The controller developed coordinates the power production of a power sys-

tem consisting of several power generators, i.e. boiler load units. Future work should focus on the early termination in order to minimize the associated extra costs.

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# A Dantzig-Wolfe Decomposition Algorithm for Economic MPC of Distributed Energy Systems

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**Abstract:** In economic model predictive control of energy systems with a large share of distributed generation and consumption, the constrained optimal control problem can be expressed as a linear program with a block-angular structure. In this paper, we present an efficient Dantzig-Wolfe decomposition algorithm specifically tailored to problems of this type. Simulations show that a MATLAB implementation of the algorithm is significantly faster than several state-of-the-art linear programming solvers and that it scales in a favorable way.

*Keywords:* Predictive control, Decomposition methods, Renewable energy systems

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## 1. INTRODUCTION

Due to global concerns related to environmental issues and security of supply, an increasing amount of renewable energy sources is being integrated in the power grid. Accordingly, methods for power production planning that can handle the volatile and unpredictable power generation associated with technologies such as wind, solar and hydropower are required. For this reason, energy systems management has emerged as a promising application area for economic model predictive control (MPC). In economic MPC of energy systems, the power production planning is handled in real-time by an optimization algorithm that computes an optimal production plan based on the most recent information available such as forecasts of energy prices, wind power production, and district heating consumption. Examples of economic MPC in energy systems management include cost-efficient control of refrigeration systems (Hovgaard et al., 2011), building climate control (Ma et al., 2011; Halvgaard et al., 2012a), and optimal charging strategies for batteries in electric vehicles (Halvgaard et al., 2012b).

Economic MPC is a receding horizon control strategy, and requires the solution of a linear program in every sampling instant. In energy systems management, the solution to this linear problem, known as the optimal control problem, provides a sequence of control moves that yields the most cost-efficient power generation respecting system dynamics, capacity constraints and electricity demand, with respect to a process model of the power system. To compensate for non-predictable disturbances and discrepancies between the process model and the true system only the first input in the sequence of control moves is applied to the system, and the optimization procedure is repeated using updated information at the following sampling instant. As the control moves are computed in real-time, one of the key challenges in economic MPC is to solve the optimal control

problem in an efficient and reliable way. The main contribution of this paper is an algorithm for control of distributed energy systems that satisfies these criteria. Our algorithm exploits that the units in a distributed energy system are dynamically decoupled. This gives rise to a block-angular structure in the optimal control problem that allows it to be decomposed, into a master problem and a number of subproblems, using Dantzig-Wolfe decomposition (Dantzig and Wolfe, 1960, 1961). To solve the decomposed problem efficiently, we use a column generation procedure, which is warm-started by a strategy that utilizes problem specific features.

Previously, Dantzig-Wolfe decomposition algorithms have been applied to MPC applications in Edlund et al. (2011); Cheng et al. (2008, 2007); Morosan et al. (2011). The work Cheng et al. (2008) uses Dantzig-Wolfe decomposition to coordinate the target calculation in set-point based MPC with  $\ell_1$ -penalty, and similar work for  $\ell_2$ -penalty is conducted in Cheng et al. (2007). Examples in energy systems management are provided in e.g. Edlund et al. (2011) in which a hierarchical control structure based on Dantzig-Wolfe decomposition is proposed, and in Morosan et al. (2011) that applies a Dantzig-Wolfe decomposition algorithm for building climate control.

### 1.1 Paper Organization

This paper is organized as follows. In Section 2, we introduce the optimal control problem solved in economic MPC, and a compact problem formulation is derived. We apply Dantzig-Wolfe decomposition to this problem in Section 3. In Section 4, we present optimality conditions for the decomposed problem, and we propose a warm-started column generation procedure for solving the problem. Performance benchmarks for the proposed algorithm, based on a conceptual energy systems management case

study, is provided in Section 5. We give concluding remarks in Section 6.

## 2. PROBLEM DEFINITION

We consider an electrical grid with  $M$  distributed power generating units. The units are modelled as discrete state space systems in the form

$$x_{j,k+1} = A_j x_{j,k} + B_j u_{j,k}, \quad j \in \mathcal{M}, \quad (1a)$$

$$y_{j,k} = C_j x_{j,k}, \quad j \in \mathcal{M}, \quad (1b)$$

where  $\mathcal{M} = \{1, 2, \dots, M\}$ . The state space matrices are denoted as  $(A_j, B_j, C_j)$ , the states as  $x_{j,k} \in \mathbb{R}^{n_x(j)}$ , the inputs as  $u_{j,k} \in \mathbb{R}^{n_u(j)}$ , and the outputs as  $y_{j,k} \in \mathbb{R}^{n_y(j)}$ . For non-linear systems, the description (1) can be obtained by linearization about an equilibrium point.

Assuming that the power production is available as a linear combination of the outputs in (1), the total power production can be written as

$$y_{T,k} = \sum_{j \in \mathcal{M}} \Upsilon_j y_{j,k} = \sum_{j \in \mathcal{M}} \Upsilon_j C_j x_{j,k}, \quad (2)$$

in which  $\Upsilon_j \in \mathbb{R}^{1 \times n_y(j)}$  is a row vector such that  $\Upsilon_j C_j x_{j,k}$  is the power production of unit  $j$  at time step  $k$ .

Economic MPC defines a control law for the generating units (1), that optimizes the inputs (control moves) with respect to an economic objective function, input limits, input rate limits and soft output limits. Evaluating this control law requires the solution to the minimization problem

$$\min_{u, x, y, y_T, \rho, \gamma} \sum_{k \in \mathcal{N}_0} q_{k+1}^T \rho_{k+1} + \sum_{j \in \mathcal{M}} p_{j,k}^T u_{j,k} + r_{j,k+1}^T \gamma_{j,k+1}, \quad (3a)$$

subject to the constraints

$$x_{j,k+1} = A_j x_{j,k} + B_j u_{j,k}, \quad k \in \mathcal{N}_0, \quad j \in \mathcal{M}, \quad (3b)$$

$$y_{j,k} = C_j x_{j,k}, \quad k \in \mathcal{N}_1, \quad j \in \mathcal{M}, \quad (3c)$$

$$y_{T,k} = \sum_{j \in \mathcal{M}} \Upsilon_j C_j x_{j,k}, \quad k \in \mathcal{N}_1, \quad (3d)$$

$$\underline{u}_{j,k} \leq u_{j,k} \leq \bar{u}_{j,k}, \quad k \in \mathcal{N}_0, \quad j \in \mathcal{M}, \quad (3e)$$

$$\Delta \underline{u}_{j,k} \leq u_{j,k} - u_{j,k-1} \leq \Delta \bar{u}_{j,k}, \quad k \in \mathcal{N}_0, \quad j \in \mathcal{M}, \quad (3f)$$

$$\underline{y}_{j,k} - \gamma_{j,k} \leq y_{j,k} \leq \bar{y}_{j,k} + \gamma_{j,k}, \quad k \in \mathcal{N}_1, \quad j \in \mathcal{M}, \quad (3g)$$

$$0 \leq \gamma_{j,k} \leq \bar{\gamma}_{j,k}, \quad k \in \mathcal{N}_1, \quad j \in \mathcal{M}, \quad (3h)$$

$$\underline{y}_{T,k} - \rho_k \leq y_{T,k} \leq \bar{y}_{T,k} + \rho_k, \quad k \in \mathcal{N}_1, \quad (3i)$$

$$0 \leq \rho_k \leq \bar{\rho}, \quad k \in \mathcal{N}_1, \quad (3j)$$

where  $\mathcal{N}_i = \{0+i, 1+i, \dots, N-1+i\}$ , with  $N$  being the length of the prediction horizon. The input data are the input limits,  $(\underline{u}_{j,k}, \bar{u}_{j,k})$ , the input rate limits,  $(\Delta \underline{u}_{j,k}, \Delta \bar{u}_{j,k})$ , the output limits associated with the generating units,  $(\underline{y}_{j,k}, \bar{y}_{j,k})$ , the output limits associated with the total power production,  $(\underline{y}_{T,k}, \bar{y}_{T,k})$ , the input prices,  $p_{j,k}$ , the price for violating the output constraints associated with the generating units,  $r_{j,k}$ , and the price for violating the output constraints associated with the total power production  $q_k$ . We also include upper limits on the variables  $\gamma_{j,k}$  and  $\rho_k$ , as this simplifies later computations considerably.

Notice that if process noise or measurement noise is present in the model (1), an optimization problem in the

form (3) can be derived using the Kalman filter under the certainty equivalence assumption.

### 2.1 Compact Formulation

By eliminating the states using equation (1a), we can write the output equation, (1b), as

$$y_{j,k} = C_j A_j^k x_{j,0} + \sum_{i \in \mathcal{N}_0} H_{j,k-i} u_{j,i}, \quad j \in \mathcal{M},$$

where the impulse response coefficients are given by

$$H_{j,k} = C_j A_j^{k-1} B_j, \quad j \in \mathcal{M}.$$

Consequently

$$y_{T,k} = \sum_{j \in \mathcal{M}} \left( \Upsilon_j C_j A_j^k x_{j,0} + \sum_{i \in \mathcal{N}_0} \Upsilon_j H_{j,k-i} u_{j,i} \right).$$

Define the vectors

$$y_j = [y_{j,1}^T \ y_{j,2}^T \ \dots \ y_{j,N}^T]^T, \quad j \in \mathcal{M}, \quad (4a)$$

$$u_j = [u_{j,0}^T \ u_{j,1}^T \ \dots \ u_{j,N-1}^T]^T, \quad j \in \mathcal{M}, \quad (4b)$$

and the matrices

$$\Gamma_j = \begin{bmatrix} H_{j,1} & 0 & \dots & 0 \\ H_{j,2} & H_{j,1} & & \\ \vdots & \vdots & \ddots & \\ H_{j,N} & H_{j,N-1} & \dots & H_{j,1} \end{bmatrix}, \quad \Phi_j = \begin{bmatrix} C_j A_j \\ C_j A_j^2 \\ \vdots \\ C_j A_j^{N-1} \end{bmatrix},$$

for  $j \in \mathcal{M}$ .

We can then write the outputs, (4a), for each of the generating units as

$$y_j = \Gamma_j u_j + \Phi_j x_{j,0}, \quad j \in \mathcal{M}. \quad (5)$$

Moreover, by introducing  $\tilde{\Gamma}_j$  and  $\tilde{\Phi}_j$  accordingly, it follows that

$$y_T = \sum_{j \in \mathcal{M}} \tilde{\Gamma}_j u_j + \tilde{\Phi}_j x_{j,0}. \quad (6)$$

We simplify the notation further by introducing

$$\underline{u}_j = \begin{bmatrix} \underline{u}_{j,0} \\ \underline{u}_{j,1} \\ \vdots \\ \underline{u}_{j,N-1} \end{bmatrix}, \quad \bar{u}_j = \begin{bmatrix} \bar{u}_{j,0} \\ \bar{u}_{j,1} \\ \vdots \\ \bar{u}_{j,N-1} \end{bmatrix}, \quad j \in \mathcal{M},$$

and similarly we define  $\Delta \underline{u}_j$ ,  $\Delta \bar{u}_j$ ,  $\underline{y}_j$ ,  $\bar{y}_j$ ,  $\underline{y}_T$ ,  $\bar{y}_T$ ,  $\bar{\gamma}_j$ ,  $\bar{\rho}$ ,  $\rho$ ,  $q$ ,  $p_j$ ,  $r_j$  and  $\gamma_j$ . Using this notation, the optimal control problem, (3), can be written as

$$\min_{u, \rho, \gamma} q^T \rho + \sum_{j \in \mathcal{M}} p_j^T u_j + r_j^T \gamma_j, \quad (7a)$$

subject to a set of decoupled constraints

$$\underline{u}_j \leq u_j \leq \bar{u}_j, \quad j \in \mathcal{M}, \quad (7b)$$

$$\Delta \underline{u}_j \leq \Delta u_j \leq \Delta \bar{u}_j, \quad j \in \mathcal{M}, \quad (7c)$$

$$\underline{y}_j - \gamma_j \leq \Gamma_j u_j + \Phi_j x_{j,0} \leq \bar{y}_j + \gamma_j, \quad j \in \mathcal{M}, \quad (7d)$$

$$0 \leq \gamma_j \leq \bar{\gamma}_j, \quad j \in \mathcal{M}, \quad (7e)$$

$$0 \leq \rho \leq \bar{\rho}, \quad (7f)$$

and a set of linking constraints

$$\underline{y}_T - \rho \leq \sum_{j \in \mathcal{M}} \tilde{\Gamma}_j u_j + \tilde{\Phi}_j x_{j,0} \leq \bar{y}_T + \rho. \quad (7g)$$

In a compact form, (7) can be stated by

$$\min_z \sum_{j \in \bar{\mathcal{M}}} c_j^T z_j, \quad (8a)$$

$$\text{s.t. } G_j z_j \geq g_j, \quad j \in \bar{\mathcal{M}}, \quad (8b)$$

$$\sum_{j \in \bar{\mathcal{M}}} H_j z_j \geq h, \quad (8c)$$

where  $\bar{\mathcal{M}} = 1, 2, \dots, M+1$  and

$$c = [c_1^T | \dots | c_M^T | c_{M+1}^T]^T = [p_1^T \ r_1^T | \dots | p_M^T \ r_M^T | q^T]^T,$$

$$z = [z_1^T | \dots | z_M^T | z_{M+1}^T]^T = [u_1^T \ \gamma_1^T | \dots | u_M^T \ \gamma_M^T | \rho^T]^T.$$

In (8), (8b) represents the decoupled constraints (7b)-(7f), and (8c) represents the linking constraints (7g). The data structures in (8) are defined as

$$G_j = \begin{bmatrix} \bar{G}_j \\ -\bar{G}_j \end{bmatrix}, \quad g_j = \begin{bmatrix} \underline{g}_j \\ -\bar{g}_j \end{bmatrix}, \quad H_j = \begin{bmatrix} \bar{H}_j \\ -\bar{H}_j \end{bmatrix}, \quad h = \begin{bmatrix} \underline{h} \\ -\bar{h} \end{bmatrix},$$

where

$$\left[ \bar{G}_j \mid \underline{g}_j \mid \bar{g}_j \right] = \left[ \begin{array}{c|cc} I & 0 & \Delta \underline{u}_j + I_0 u_{j,-1} \\ \Lambda & 0 & \Delta \bar{u}_j + I_0 u_{j,-1} \\ \Gamma_j & I & \infty \\ \Gamma_j & -I & \bar{y}_j - \Phi_j x_{j,0} \\ 0 & I & 0 \\ & & \bar{\gamma}_j \end{array} \right],$$

$$\left[ \bar{H}_j \mid \underline{h} \mid \bar{h} \right] = \left[ \begin{array}{c|cc} \tilde{\Gamma}_j & 0 & \infty \\ \tilde{\Gamma}_j & 0 & \bar{y}_T - \sum_{j \in \bar{\mathcal{M}}} \tilde{\Phi}_j x_{j,0} \\ & & -\infty \\ & & \bar{y}_T - \sum_{j \in \bar{\mathcal{M}}} \tilde{\Phi}_j x_{j,0} \end{array} \right],$$

for  $j \in \mathcal{M}$ , with  $\Lambda$  and  $I_0$  defined as

$$\Lambda_j = \begin{bmatrix} I & & & & \\ -I & I & & & \\ & & \ddots & \ddots & \\ & & & & -I & I \end{bmatrix}, \quad I_0 = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

In the special case  $j = M+1$

$$\left[ \bar{G}_{M+1} \mid \underline{g}_{M+1} \mid \bar{g}_{M+1} \right] = [I \mid 0 \mid \bar{\rho}],$$

$$\bar{H}_{M+1} = [I \ -I]^T.$$

### 3. DANTZIG WOLFE DECOMPOSITION

Dantzig-Wolfe decomposition (Dantzig and Wolfe, 1960, 1961; Nemhauser and Wolsey, 1988; Martin, 1999; Ladson, 2002) exploits that a convex set can be characterized by its extreme points and its extreme rays. In particular, for each  $j \in \mathcal{M}$ , the set of points satisfying the decoupled constraints (8b) may be written as

$$\mathcal{G}_j = \{z_j \mid G_j z_j \geq g_j\},$$

$$= \left\{ z_j \mid z_j = \sum_{i \in \mathcal{P}} \lambda_j^i z_j^i, \sum_{i \in \mathcal{P}} \lambda_j^i = 1, \lambda_j^i \geq 0 \ \forall i \in \mathcal{P} \right\},$$

where  $z_j^i$  are the extreme points of  $\mathcal{G}_j$ , and  $\lambda_j^i$  are convex combination multipliers. Notice that since each of the sets,  $\mathcal{G}_j$ , are bounded, extreme rays are not needed to characterize the sets.

By replacing the decision variables in (8) by convex combination multipliers, we obtain the master problem formulation

	Original Problem
#constraints	$6N + N \sum_{j \in \mathcal{M}} (4n_u(j) + 6n_y(j))$
#variables	$N + N \sum_{j \in \mathcal{M}} (n_u(j) + n_y(j))$
	Master Problem
#constraints	$4N + M + 1 + \sum_{j \in \bar{\mathcal{M}}}  \mathcal{P} $
#variables	$\sum_{j \in \bar{\mathcal{M}}}  \mathcal{P} $

Table 1. Dimensions of the original problem, (8), and the master problem (9).

$$\min_{\lambda} \phi = \sum_{j \in \bar{\mathcal{M}}} \sum_{i \in \mathcal{P}} c_j^i \lambda_j^i, \quad (9a)$$

$$\text{s.t. } \sum_{j \in \bar{\mathcal{M}}} \sum_{i \in \mathcal{P}} H_j^i \lambda_j^i \geq h, \quad (9b)$$

$$\sum_{i \in \mathcal{P}} \lambda_j^i = 1, \quad j \in \bar{\mathcal{M}}, \quad (9c)$$

$$\lambda_j^i \geq 0, \quad j \in \bar{\mathcal{M}}, \ i \in \mathcal{P}, \quad (9d)$$

where we have defined

$$H_j^i = H_j z_j^i, \quad j \in \bar{\mathcal{M}}, \ i \in \mathcal{P}, \quad (10a)$$

$$c_j^i = c_j^T z_j^i, \quad j \in \bar{\mathcal{M}}, \ i \in \mathcal{P}. \quad (10b)$$

Given a solution,  $\lambda^*$ , to the master problem (9), a solution to the original problem, (8), can be obtained as

$$z_j^* = \sum_{i \in \mathcal{P}} (\lambda^*)^i z_j^i, \quad j \in \bar{\mathcal{M}}.$$

In Table 1, we have compared the dimensions of the original problem, (8), and the master problem (9). Since the number of extreme points,  $|\mathcal{P}|$ , can increase exponentially with the size of the original problem, solving the master problem directly is inefficient. As demonstrated in the following section, however, the problem can be solved in an attractive way using a column generation procedure that replaces  $\mathcal{P}$  by a subset  $\tilde{\mathcal{P}}$ .

### 4. COLUMN GENERATION

The dual linear program of (9) can be stated as

$$\max_{\alpha, \beta} \alpha^T h + \sum_{j \in \bar{\mathcal{M}}} \beta_j, \quad (11a)$$

$$\text{s.t. } (H_j^i)^T \alpha + \beta_j \leq c_j^i, \quad j \in \bar{\mathcal{M}}, \ i \in \mathcal{P}, \quad (11b)$$

$$\alpha \geq 0, \quad (11c)$$

in which  $\alpha \in \mathbb{R}^{4N}$  and  $\beta \in \mathbb{R}^{M+1}$  are the Lagrange multipliers associated with the linking constraints, (9b), and the convexity constraints, (9c), respectively. The necessary and sufficient optimality conditions for (9) and (11) are

$$\sum_{j \in \bar{\mathcal{M}}} \sum_{i \in \mathcal{P}} H_j^i \lambda_j^i \geq h, \quad (12a)$$

$$\sum_{i \in \mathcal{P}} \lambda_j^i = 1, \quad j \in \bar{\mathcal{M}}, \quad (12b)$$

$$\lambda_j^i \geq 0, \quad j \in \bar{\mathcal{M}}, \ i \in \mathcal{P}, \quad (12c)$$

$$c_j^i - (H_j^i)^T \alpha - \beta_j \geq 0, \quad j \in \bar{\mathcal{M}}, \ i \in \mathcal{P}, \quad (12d)$$

$$\alpha \geq 0, \quad (12e)$$

$$\lambda_j^i (c_j^i - (H_j^i)^T \alpha - \beta_j) = 0, \quad j \in \bar{\mathcal{M}}, \ i \in \mathcal{P}, \quad (12f)$$

In Proposition 1 we derive conditions for which a solution satisfying this set of optimality conditions, can be obtained by solving the master problem (9) over a subset of the original variables.

*Proposition 1.* Let  $\tilde{\mathcal{P}} \subseteq \mathcal{P}$  for all  $j \in \bar{\mathcal{M}}$ , and define  $(\tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$  as a primal-dual solution to (9) and (11) restricted to the subset  $\tilde{\mathcal{P}}$ . Then the solution

$$\begin{aligned} \alpha^* &= \alpha, \\ \beta_j^* &= \beta_j, & j \in \bar{\mathcal{M}}, \\ (\lambda^*)_j &= \begin{cases} \tilde{\lambda}_j^i & \text{if } i \in \tilde{\mathcal{P}} \\ 0 & \text{if } i \in \mathcal{P} \setminus \tilde{\mathcal{P}}, \end{cases} & j \in \bar{\mathcal{M}}, i \in \mathcal{P}, \end{aligned}$$

satisfies the conditions, (12), if the optimal objective value of the subproblem

$$\min_{\tilde{z}_j} \varphi_j = (c_j - H_j^T \alpha^*)^T \tilde{z}_j - \beta_j^* \quad (13a)$$

$$\text{s.t. } G_j \tilde{z}_j \geq g_j, \quad (13b)$$

is non-negative for each  $j \in \bar{\mathcal{M}}$ .

**Proof** The solution  $(\lambda^*, \alpha^*, \beta^*)$  satisfies (12a) since

$$\sum_{j \in \bar{\mathcal{M}}} \sum_{i \in \mathcal{P}} H_j^i (\lambda^*)_j^i = \sum_{j \in \bar{\mathcal{M}}} \sum_{i \in \tilde{\mathcal{P}}} H_j^i \tilde{\lambda}_j^i \geq h,$$

which follows from the definition of  $(\tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$ . Similarly, it is easy to verify that the conditions (12c), (12b), (12e) and (12f) are fulfilled.

Provided that  $(\lambda^*, \alpha^*, \beta^*)$  is optimal, (12d) yields

$$c_j^i - (H_j^i)^T \alpha^* - \beta_j^* = (c_j - H_j^T \alpha^*)^T z_j^i - \beta_j^* \geq 0, \quad (14)$$

for all  $j \in \bar{\mathcal{M}}$  and  $i \in \mathcal{P}$ . By construction of the solution, (14) is satisfied for all  $i \in \tilde{\mathcal{P}}$ . To check that the condition also holds for all  $i \in \mathcal{P} \setminus \tilde{\mathcal{P}}$ , we consider the linear program (13) which determines the extreme point,  $\tilde{z}_j$ , of  $\mathcal{G}_j$  that minimizes the left hand side of the inequality in (14). Therefore, if  $\varphi_j$  is non-negative for all  $j \in \bar{\mathcal{M}}$ , then  $(\lambda^*, \alpha^*, \beta^*)$  satisfies (14) for all  $i \in \mathcal{P}$ , and it is therefore optimal for the original problem. ■

In Algorithm 1, we have outlined a column generation procedure based on Proposition 1. The algorithm exploits that if (14) is violated, then the solution to the subproblems, (13), provides a set of extreme points that can be added to the master problem. As can be read in Table 1, the master problem is much smaller than the original problem when  $\mathcal{P}$  is restricted to the subset  $\tilde{\mathcal{P}}$ , as long as  $|\tilde{\mathcal{P}}|$  is small. Moreover, solving the subproblem is computational inexpensive as it does not grow with the number of units  $M$ . For large systems, we therefore expect Algorithm 1 to outperform conventional linear programming solvers with a significant margin.

#### 4.1 Warm-Starting

Algorithm 1 requires a set of initial points  $\{z_j^0\}_{j=1}^{\bar{\mathcal{M}}}$  that are feasible for both the subproblems, (13), and the original problem (8). As economic MPC requires running the algorithm in a closed-loop fashion, we can generate such a set of points by exploiting the solution from a previous time step.

Given the solution to (13)

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**Algorithm 1** Column generation procedure for the solution of the master problem (9).

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**Require:**  $\{z_j^0\}_{j=1}^{\bar{\mathcal{M}}}$   
 $i = 0, \text{ converged} = 0$   
**while not converged do**  
 $\tilde{\mathcal{P}} = \{0, 1, \dots, i\}$   
 COMPUTE PROBLEM DATA  
**for**  $j \in \bar{\mathcal{M}}$  **do**  
**for**  $i \in \tilde{\mathcal{P}}$  **do**  
 $H_j^i = H_j z_j^i$   
 $c_j^i = c_j^T z_j^i$   
**end for**  
**end for**  
 SOLVE RESTRICTED MASTER PROBLEM  
 $(\phi^*, \lambda^*, \alpha^*, \beta^*) \leftarrow \text{solve (9) with } \mathcal{P} = \tilde{\mathcal{P}}$   
 SOLVE SUBPROBLEMS  
**for**  $j \in \bar{\mathcal{M}}$  **do**  
 $(\varphi_j^*, \tilde{z}_j^*) \leftarrow \text{solve (13)}$   
**end for**  
 CHECK IF CONVERGED  
**if**  $\varphi_j \geq 0 \forall j \in \bar{\mathcal{M}}$  **then**  
 $\text{converged} = 1$   
**else**  
 UPDATE EXTREME POINTS  
**for**  $j \in \bar{\mathcal{M}}$  **do**  
 $z_j^{i+1} = \tilde{z}_j^*$   
 $i = i + 1$   
**end for**  
**end if**  
**end while**

---

$$z_j^* = \left[ u_{j,0}^{*T} \cdots u_{j,N-1}^{*T} \gamma_{j,1}^{*T} \cdots \gamma_{j,N}^{*T} \right]^T, \quad j \in \mathcal{M},$$

$$z_{M+1}^* = \left[ \rho_1^{*T} \cdots \rho_N^{*T} \right]^T,$$

we build a set of initial points in the following sampling instant as

$$z_j^0 = \left[ u_{j,1}^{*T} \cdots u_{j,N-1}^{*T} \check{u}_j^T \gamma_{j,2}^{*T} \cdots \gamma_{j,N}^{*T} \check{\gamma}_j^T \right]^T, \quad j \in \mathcal{M}$$

$$z_{M+1}^0 = \left[ \rho_2^{*T} \cdots \rho_N^{*T} \check{\rho}^T \right]^T.$$

Hence, the strategy shifts the original solution values forward in time, and append the variables  $\check{u}_j$ ,  $\check{\gamma}_j$  and  $\check{\rho}$  to the initial point. In our implementation, we have define  $\check{u}_j$  as

$$\check{u}_j = u_{j,N-1}^*, \quad j \in \mathcal{M}. \quad (15)$$

which leads to an initial input sequence with constant input in the two final sampling intervals. Using the state space equations (1)-(2), we compute the outputs  $\check{y}_{j,N}$  and  $\check{y}_{T,N}$  associated with this input sequence. Based on these values we let

$$\check{\gamma}_j = \max(\underline{y}_{j,N} - \check{y}_{j,N}, 0) + \max(\check{y}_{j,N} - \bar{y}_{j,N}, 0),$$

$$\check{\rho} = \max(\underline{y}_{T,N} - \check{y}_{T,N}, 0) + \max(\check{y}_{T,N} - \bar{y}_{T,N}, 0).$$

where the maximum function is evaluated element-wise.

Assuming that the inputs (15) satisfy the input constraints for the updated problem data, and that the upper limits on  $\gamma_j$  and  $\rho$  are sufficiently large, the strategy above yields a set of feasible initial points for Algorithm 1,  $\{z_j^0\}_{j=1}^{\bar{\mathcal{M}}}$ , which exploits the solution obtained in the previous time step. As the solution in successive time steps are closely related

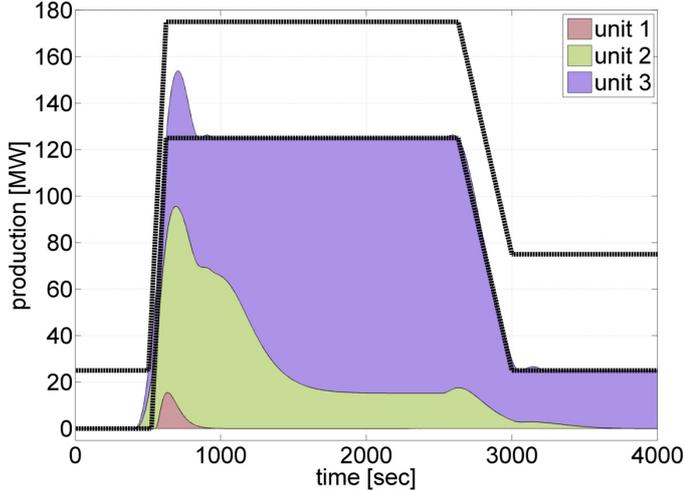


Fig. 1. Closed-loop simulation study of economic MPC. The marginal price of using the units is decreasing with the unit number.

in MPC applications, this approach provides a warm-start for Algorithm 1. In case no previous solution is available, a similar strategy can be used to adjust the slack variables for an arbitrary feasible input sequence.

## 5. RESULTS

In this section, we compare a MATLAB implementation of Algorithm 1, denoted `DWempc`, to linear programming solvers from the following software packages: `CPLEX`, `Gurobi` and `MOSEK`. For each solver, the computation time of solving the optimal control problem (3) is measured. The algorithms are run on an Intel(R) Core(TM) i5-2520M CPU @ 2.50GHz with 4 GB RAM running a 64-bit Windows 7 Enterprise operating system. In `DWempc`, the restricted master problem and the subproblems are solved using `CPLEX`.

As a conceptual case study, we consider a collection of power generating units in the form

$$Y_j(s) = \frac{1}{(\tau_j s + 1)^3} U_j(s), \quad j \in \mathcal{M}, \quad (16)$$

where  $U_j(s)$  is the fuel input and the  $Y_j(s)$  is the power production. The third order model, (16), has been validated against actual measurement data in Edlund et al. (2010). In our study, we vary the time constant,  $\tau_j$ , to represent different types of power generating units. Time constants in the range 80-120 are associated with slow units such as centralized thermal power plants, while time constants in the range 20-60 represent units with faster dynamics such as diesel generators and gas turbines. Although this system description is too simplified for many practical purposes, it is convenient for computational benchmarks.

To control the units, (16), using economic MPC we realize the system in the discrete state-space form (1)-(2). In the resulting model structure,  $u_{j,k} \in \mathbb{R}$  is fuel input,  $y_{j,k} \in \mathbb{R}$  is the power production, and  $y_{T,k} \in \mathbb{R}$  is the total power production. Thus, in this case,  $\Upsilon_j = 1$ , for all  $j \in \mathcal{M}$ . Fig. 1 depicts a closed-loop simulation with  $M = 3$  power generating units. The graphs show the individual outputs, as well as the output limits for the total production.

Table 2. Case study parameters.

	$\tau_j$	$p_j$	$\underline{u}_j$	$\bar{u}_j$	$\Delta \underline{u}_j$	$\Delta \bar{u}_j$
Unit 1	40	24	0	50	-30	30
Unit 2	90	12	0	100	-20	20
Unit 3	100	6	0	200	-5	5

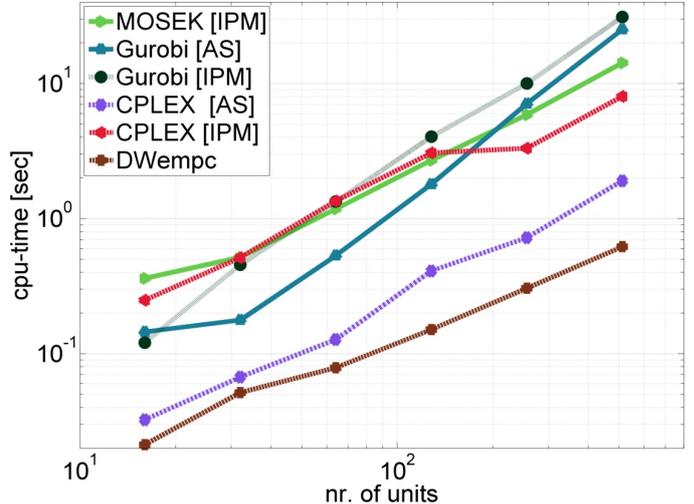


Fig. 2. CPU-time for solving (3) as a function of the number of power generating units, and fixed  $N = 50$ . Active-set methods are denoted by (AS) and interior-point methods are denoted by (IPM).

The case study parameters are listed in Table 2. All the parameters listed, are kept constant over the entire horizon  $\mathcal{N}_0$ .

The values,  $p_j$ , are the prices pr. unit of fuel (e.g oil, natural gas or coal). We have defined these parameters such that the fuel price for fast units is higher than the fuel price for slow units. The price for imbalances is fixed to  $q_k = 10000$ . It can be read from Fig. 1, that in the production plan obtained using economic MPC, the cheapest plant accounts for the main load whereas the more expensive plants compensate for its slow dynamics. This represents a common situation in the power industry, where large thermal power plants typically produce a majority of the electricity, while units with faster dynamics such as diesel generators are used only in critical peak periods.

In Fig. 2 we have compared the computation time of solving the constrained optimal control problem, (3), using `DWempc`, `CPLEX`, `Gurobi` and `DWempc` for an increasing number of generating units. The problem data was generated in a similar way as in the simulation presented above. Fig. 2 shows that `DWempc` is up to an order of magnitude faster than all other solvers in the comparison, and that the difference grows as the number of units is increased. This demonstrates that the column generation method for economic MPC of distributed energy systems. We also notice that, as an additional advantage over conventional algorithms, the subproblems, (13), can be solved in parallel. Finally, `DWempc` requires less memory, as the necessary data for solving the master problem, (9), are generated on the fly.

## 6. CONCLUSION

In this paper, we have presented a detailed description of a warm-started Dantzig-Wolfe decomposition algorithm for economic MPC of distributed energy system. Our results show that a MATLAB implementation of the algorithm, denoted `DWempc`, is significantly faster than both active-set methods and interior-point methods provided by MOSEK, CPLEX and Gurobi. Moreover, `DWempc`, has several desirable features such as low memory costs and parallelization capabilities, which makes it well suited for economic MPC applications with a decentralized structure such as the control of distributed energy systems.

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# A Dantzig-Wolfe Decomposition Algorithm for Linear Economic MPC of a Power Plant Portfolio

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**Abstract:** Future power systems will consist of a large number of decentralized power producers and a large number of controllable power consumers in addition to stochastic power producers such as wind turbines and solar power plants. Control of such large scale systems requires new control algorithms. In this paper, we formulate the control of such a system as an Economic Model Predictive Control (MPC) problem. When the power producers and controllable power consumers have linear dynamics, the Economic MPC may be expressed as a linear program and we apply Dantzig-Wolfe decomposition for solution of this linear program. The Dantzig-Wolfe decomposition algorithm for Economic MPC is tested on a simulated case study with a large number of power producers. The Dantzig-Wolfe algorithm is compared to a standard linear programming (LP) solver for the Economic MPC. Simulation results reveal that the Dantzig-Wolfe algorithm is faster than the standard LP solver and enables solution of larger problems.

*Keywords:* Economic Model Predictive Control, Linear Programming, Distributed Optimization, Power Systems

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## 1. INTRODUCTION

Increasing prices of fossil fuels and climate concerns related to CO<sub>2</sub> have stimulated an increased political and technical interest in power systems that are not based on fossil fuels (Gore, 2006, 2009; Friedman, 2009; Danish Energy Agency, 2010; Danish Commission on Climate Change Policy, 2010; European Technology Platform SmartGrids, 2012). The predominant renewable energy sources in such a system are wind and solar energy. The power productions from sources such as the sun and the wind are stochastic. Inclusion of large shares of stochastic power producers in the power system requires that the existing power system is restructured such that they can quickly compensate for variations in energy production from the stochastic generators. Consequently, future power systems must include a large number of decentralized agile controllable power producers and consumers to compensate for the stochastic power production from wind turbines and solar plants. Such an integration of a large number of new power producers and consumers in the power system requires new control algorithms for balancing their power production and consumption.

In this paper, we present an optimization based controller for balancing the power production and consumption in an economic efficient way. The optimization based controller is obtained by formulating the power balancing problem as an Economic Model Predictive Control (MPC) problem. Many energy system components can be approximated well by linear models (Edlund et al., 2009). Accordingly, the Economic MPC for power systems with a large number

of linear components results in large scale linear programs that must be solved efficiently and reliably in real time. Due to the decoupled dynamics of the energy components, the linear program representing the Economic MPC has a block angular structure that is utilized in the Dantzig-Wolfe algorithm. The key contributions of this paper is a Dantzig-Wolfe decomposition algorithm for Economic MPC of linear systems and demonstration of this Economic MPC to power systems with many power producers.

Previously, Economic MPC has been applied to smart energy systems such as refrigeration systems (Hovgaard et al., 2010, 2011, 2012a,b), heat pumps for residential buildings (Halvgaard et al., 2012c), solar heated water tanks (Halvgaard et al., 2012a), and batteries in electrical vehicles (Halvgaard et al., 2012b). Scattolini (2009) reviewed model predictive control for distributed systems. Using the terminology in Scattolini (2009), the Dantzig-Wolfe decomposition method presented in this paper is a distributed optimization method for solution of the centralized MPC. Other well-known techniques for distributed optimization that have been applied for MPC are Lagrange dual decomposition (Rantzer, 2009) and Bender's decomposition (Morsan et al., 2011a). Dantzig and Wolfe (1960, 1961) introduced a decomposition algorithm for large linear programs. This decomposition algorithm is known as the Dantzig-Wolfe algorithm. Like Lagrange dual decomposition, the Dantzig-Wolfe algorithm uses Lagrange relaxation to decompose the large scale linear program into smaller linear programs. However, in Dantzig-Wolfe decomposition another linear program (the master problem) is used to compute the Lagrange multipliers, while

Lagrange dual decomposition uses heuristics for updating the Lagrange multipliers. Lasdon (1970), Chvatal (1983), Nazareth (1987), Dantzig and Thapa (2003), and Conejo et al. (2006) discuss decomposition algorithms for large scale linear programs in a tutorial manner. Recently, the Dantzig-Wolfe algorithm has been used in a number MPC-like applications. Cheng et al. (2007, 2008) used Dantzig-Wolfe decomposition in the steady-state target LP for a two-layer MPC algorithm. Gunnerud and Foss (2010) and Gunnerud et al. (2010) applied a Dantzig-Wolfe algorithm for production optimization in an oil field. Morsan et al. (2011b) studied a building temperature control problem and solved the MPC for that problem using a Dantzig-Wolfe decomposition algorithm. Based on an  $\ell_1$ -penalty function, Edlund et al. (2011) formulated the power balancing problem as an MPC and solved the resulting large scale linear program using a Dantzig-Wolfe algorithm. Bürger et al. (2012) proposed an iterative algorithm related to the Dantzig-Wolfe algorithm that do not need a master problem.

This paper is organized as follows. Section 2 introduces Economic MPC for linear stochastic systems and show that such problems can be solved by solution of linear programs. Section 4 describes the Dantzig-Wolfe decomposition algorithm for linear programs with a block-angular structure. A power plant case study is introduced in Section 5 to illustrate the Economic MPC based on Dantzig-Wolfe decomposition. Conclusions are provided in Section 6.

## 2. ECONOMIC MPC FOR LINEAR SYSTEMS

This section is about the Economic Model Predictive Control (MPC) stated for linear programs, where the optimal solution is found minimizing an economic cost. In this paper, the control problem is stated as a linear program (Hovgaard et al., 2010). The slack variables are introduced in the optimization problem to adjust in case that the portfolio output fails in following the reference.

### 2.1 The Stochastic System

Consider the stochastic system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k + G\mathbf{w}_k + E\mathbf{d}_k \quad (1a)$$

$$\mathbf{y}_k = C\mathbf{x}_k + \mathbf{v}_k \quad (1b)$$

$$\mathbf{z}_k = C_z\mathbf{x}_k \quad (1c)$$

The initial state is distributed as  $\mathbf{x}_0 \sim N(\hat{x}_{0|-1}, P_{0|-1})$ , the process noise is distributed as  $\mathbf{w}_k \sim N_{iid}(0, R_{ww,k})$ , and the measurement noise is distributed as  $\mathbf{v}_k \sim N_{iid}(0, R_{vv})$ .  $\mathbf{x}_k$  denotes the states,  $\mathbf{u}_k$  denotes the manipulated variables (MVs),  $\mathbf{y}_k$  denotes the measurement used for feedback, and  $\mathbf{z}_k$  denotes output variables.  $\mathbf{d}_k \sim N(\bar{d}_k, R_{dd,k})$  denotes process noise that can be predicted by a prognosis system and are predicted independently of the measurements  $y$ . Accordingly, we consider a non-standard situation, in which the process disturbance  $\mathbf{d}_k$  can be predicted by some realization  $I_k^d$  of a stochastic information vector  $\mathcal{I}_k^d$ . We assume that the conditional variable has the distribution

$$\mathbf{d}_{k+j|k} = (\mathbf{d}_{k+j} | \mathcal{I}_k^d = I_k^d) \sim N(\hat{d}_{k+j|k}, R_{dd,k+j|k}) \quad (2)$$

In many situations in smart energy systems,  $d$  involves variables such as wind speed, temperature and sun radi-

ation. Accordingly, the forecast  $\mathbf{d}_{k+j|k}$  is the result of a weather prognosis. We denote the mean of these forecasts as

$$\mathcal{D}_k = \left\{ \hat{d}_{k+j|k} \right\}_{j=0}^{N-1} \quad (3)$$

The manipulated variable,  $\mathbf{u}_k$ , is a stochastic variable. For the systems we consider, it is given by a function of the form  $\mathbf{u}_k = \mu(\hat{x}_{k|k}, u_{k-1}, \mathcal{D}_k, \mathcal{F}_k, \mathcal{R}_k)$  with  $\hat{x}_{k|k}$  being a filtered state estimate depending on the current measurement  $\mathbf{y}_k$  as well as the history of the system summarized by the previous filtered state estimate,  $\hat{x}_{k-1|k-1}$ , and its covariance,  $P_{k-1|k-1}$ .  $\mathcal{F}_k$  and  $\mathcal{R}_k$  are some forecasts to be defined later. The fact that  $\mathbf{u}_k$  is a stochastic variable implies that it is a function  $\mathbf{u}_k : \Omega \mapsto \mathbb{R}^{n_u}$ , i.e.  $u_k = \mathbf{u}_k(\omega)$  for  $\omega \in \Omega$  and  $(\Omega, \mathcal{G}, P)$  is an associated probability field (Billingsley, 1995). The manipulated variables are limited by bounds and rate-of-movement constraints

$$u_{\min} \leq \mathbf{u}_k \leq u_{\max} \quad (4a)$$

$$\Delta u_{\min} \leq \Delta \mathbf{u}_k \leq \Delta u_{\max} \quad (4b)$$

These constraints says, that  $u_k = \mathbf{u}_k(\omega)$  must satisfy the constraints. Accordingly,  $\mathbf{u}_k$  cannot be normally distributed as the tails are removed by the constraints. It should also be noticed that these constraints are different from similar mean-value constraints and probabilistic constraints.

The outputs,  $\mathbf{z}_k$ , should be in some interval  $[\mathbf{r}_{\min,k}, \mathbf{r}_{\max,k}]$  where  $\mathbf{r}_{\min,k} \sim \mathbb{F}(\bar{r}_{\min,k}, R_{(r_{\min}, r_{\min}),k})$  and  $\mathbf{r}_{\max,k} \sim \mathbb{F}(\bar{r}_{\max,k}, R_{(r_{\max}, r_{\max}),k})$  are stochastic variables stemming from some distribution. Forecasts,  $\mathcal{R}_k$ , of the interval  $[\mathbf{r}_{\min,k}, \mathbf{r}_{\max,k}]$  are available and used by the controller. Let

$$\mathbf{r}_{\min,k+j|k} = (\mathbf{r}_{\min,k+j} | \mathcal{I}_k^r = I_k^r) \sim \mathbb{F}(\hat{r}_{\min,k+j|k}, R_{(r_{\min}, r_{\min}),k+j|k}) \quad (5a)$$

$$\mathbf{r}_{\max,k+j|k} = (\mathbf{r}_{\max,k+j} | \mathcal{I}_k^r = I_k^r) \sim \mathbb{F}(\hat{r}_{\max,k+j|k}, R_{(r_{\max}, r_{\max}),k+j|k}) \quad (5b)$$

such that the mean of the forecast,  $\mathcal{R}_k$ , may be denoted as

$$\mathcal{R}_k = \left\{ \hat{r}_{\min,k+j|k}, \hat{r}_{\max,k+j|k} \right\}_{j=1}^N \quad (6)$$

In energy systems, the interval  $[\mathbf{r}_{\min,k}, \mathbf{r}_{\max,k}]$  can be related to the power consumption, indoor temperature in a building, temperatures in a refrigeration system or some desired state-of-charge of a battery. For some scenarios or disturbances, it may be very expensive or even impossible to keep the outputs  $\mathbf{z}_k$  in the interval  $[\mathbf{r}_{\min,k}, \mathbf{r}_{\max,k}]$ . For such situations, we introduce slack variables defined by

$$\mathbf{s}_k = \max \{0, \mathbf{r}_{\min,k} - \mathbf{z}_k, \mathbf{z}_k - \mathbf{r}_{\max,k}\} \quad (7)$$

such that the possible interval for the outputs is expanded to

$$\mathbf{r}_{\min,k} - \mathbf{s}_k \leq \mathbf{z}_k \leq \mathbf{r}_{\max,k} + \mathbf{s}_k \quad (8)$$

with  $\mathbf{s}_k \geq 0$ . The slack variables,  $\mathbf{s}_k$ , may represent selling or buying power from the short-term market, violation of temperature limits, or violation of state-of-charge limits. Every time  $\mathbf{s}_k$  is non-zero, a penalty cost, e.g. the cost of buying or selling power on the short-term market, must be paid.

The average cost of operating the system in a period is the stochastic variable

$$\psi = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^N \mathbf{b}'_k \mathbf{z}_k + \mathbf{c}'_k \mathbf{u}_k + \boldsymbol{\rho}'_k \mathbf{s}_k \quad (9)$$

with  $\mathbf{b}_k \sim \mathbb{F}(\bar{\mathbf{b}}_k, R_{bb,k})$ ,  $\mathbf{c}_k \sim \mathbb{F}(\bar{\mathbf{c}}_k, R_{cc,k})$ , and  $\boldsymbol{\rho}_k \sim \mathbb{F}(\bar{\boldsymbol{\rho}}_k, R_{\rho\rho,k})$  being unit costs. These unit costs are predicted by yet another forecasting system. The unit price forecasts are the conditional stochastic variables

$$\mathbf{b}_{k+j|k} = (\mathbf{b}_{k+j} | \mathcal{I}_k^p = I_k^p) \sim \mathbb{F}(\hat{\mathbf{b}}_{k+j|k}, R_{bb,k+j|k}) \quad (10a)$$

$$\mathbf{c}_{k+j|k} = (\mathbf{c}_{k+j} | \mathcal{I}_k^p = I_k^p) \sim \mathbb{F}(\hat{\mathbf{c}}_{k+j|k}, R_{cc,k+j|k}) \quad (10b)$$

$$\boldsymbol{\rho}_{k+j|k} = (\boldsymbol{\rho}_{k+j} | \mathcal{I}_k^p = I_k^p) \sim \mathbb{F}(\hat{\boldsymbol{\rho}}_{k+j|k}, R_{\rho\rho,k+j|k}) \quad (10c)$$

and we denote the unit price forecast as

$$\mathcal{F}_k = \left\{ \hat{\mathbf{b}}_{k+j+1|k}, \hat{\mathbf{c}}_{k+j|k}, \hat{\boldsymbol{\rho}}_{k+j+1|k} \right\}_{j=0}^{N-1} \quad (11)$$

## 2.2 Filtering and Prediction

The filtered estimate,  $\hat{\mathbf{x}}_{k|k} = E\{\mathbf{x}_k | \mathcal{Y}_k = Y_k\}$ , of a system governed by (1) is computed using the Kalman filter (Jazwinski, 1970; Kailath et al., 2000; Jørgensen and Jørgensen, 2007; Jørgensen et al., 2011). The innovation is computed as

$$e_k = y_k - \hat{y}_{k|k-1} = y_k - C\hat{\mathbf{x}}_{k|k-1} \quad (12)$$

The innovation covariance,  $R_{e,k}$ , the filter gain,  $K_{f,x,k}$ , and the filtered state covariance,  $P_{k|k}$ , are computed as

$$R_{e,k} = R_{vv} + CP_{k|k-1}C' \quad (13a)$$

$$K_{f,x,k} = P_{k|k-1}C'R_{e,k}^{-1} \quad (13b)$$

$$P_{k|k} = P_{k|k-1} - K_{f,x,k}R_{e,k}K'_{f,x,k} \quad (13c)$$

such that the filtered state can be computed by

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + K_{f,x,k}e_k \quad (14)$$

Equations (12)-(14) are standard Kalman filter operations for the measurement update. The predictions are slightly different than the standard Kalman prediction due to the forecasts of  $d_k$ . Given the conditional predictions of the exogenous variables,  $\hat{d}_{k+j|k}$ , and the manipulated variables,  $\hat{u}_{k+j|k}$ , the conditional predictions of the states and the outputs are

$$\hat{\mathbf{x}}_{k+1+j|k} = A\hat{\mathbf{x}}_{k+j|k} + B\hat{u}_{k+j|k} + E\hat{d}_{k+j|k} \quad (15a)$$

$$\hat{z}_{k+j+1|k} = C_z\hat{\mathbf{x}}_{k+1+j|k} \quad (15b)$$

for  $j = 0, 1, \dots, N-1$  and all  $k \geq 0$ . The corresponding covariances of the predicted states are

$$P_{k+j+1|k} = AP_{k+j|k}A' + GR_{ww,k+j}G' + ER_{dd,k+j|k}E' \quad (16)$$

## 2.3 A Certainty Equivalent Regulator for Economic MPC

By now we have defined the stochastic system and established the optimal filtering and prediction in this system. Next we will describe our method for computing the manipulated variables,  $u_k$ . We use a certainty equivalence assumption such that the regulator uses mean value predictions for all variables. Consequently, at time  $k$ , the predicted operating cost looking  $N$  periods ahead is

$$\begin{aligned} \phi &= \sum_{j=0}^{N-1} \hat{\mathbf{b}}'_{k+j+1|k} \hat{z}_{k+j+1|k} + \hat{\mathbf{c}}'_{k+j|k} \hat{u}_{k+j|k} \\ &+ \sum_{j=0}^{N-1} \hat{\boldsymbol{\rho}}'_{k+j+1|k} \hat{\mathbf{s}}_{k+j+1|k} \end{aligned} \quad (17)$$

This cost function is linear in  $\hat{z}_{k+j+1|k}$ ,  $\hat{u}_{k+j|k}$ , and  $\hat{\mathbf{s}}_{k+j+1|k}$ . This objective function is not necessarily an exact penalty function that selects the slack variables as defined by (7). It will be exact if the prices,  $\hat{\boldsymbol{\rho}}_{k+j+1|k}$ , are larger than the corresponding Lagrange-multipliers for output constraints of the form

$$r_{\min,k+j+1|k} \leq \hat{z}_{k+j+1|k} \leq r_{\max,k+j+1|k} \quad (18)$$

In the case with an exact penalty function for the output constraints (18),  $\hat{\mathbf{s}}_{k+j+1|k}$  will only be non-zero if (18) cannot be met.

Given the mean value of the forecasts, i.e.  $\mathcal{D}_k$ ,  $\mathcal{R}_k$ , and  $\mathcal{F}_k$ , the filtered state,  $\hat{\mathbf{x}}_{k|k}$ , from (14), the previous input,  $u_{k-1}$ , as well as the predictions (15) and the objective function (17), the optimal trajectory of the predicted manipulated variables and slack variables,  $\{\hat{u}_{k+j|k}, \hat{\mathbf{s}}_{k+j+1|k}\}_{j=0}^{N-1}$ , may be computed by solution of the linear program

$$\min_{\hat{u}, \hat{\mathbf{s}}} \phi = \phi(\{\hat{u}_{k+j|k}, \hat{\mathbf{s}}_{k+j+1|k}\}_{j=0}^{N-1}) \quad (19a)$$

$$s.t. \quad \hat{\mathbf{x}}_{k+1+j|k} = A\hat{\mathbf{x}}_{k+j|k} + B\hat{u}_{k+j|k} + E\hat{d}_{k+j|k} \quad (19b)$$

$$\hat{z}_{k+j+1|k} = C_z\hat{\mathbf{x}}_{k+1+j|k} \quad (19c)$$

$$u_{\min} \leq \hat{u}_{k+j|k} \leq u_{\max} \quad (19d)$$

$$\Delta u_{\min} \leq \Delta \hat{u}_{k+j|k} \leq \Delta u_{\max} \quad (19e)$$

$$\hat{z}_{k+j+1|k} + \hat{\mathbf{s}}_{k+j+1|k} \geq \hat{r}_{\min,k+j+1|k} \quad (19f)$$

$$\hat{z}_{k+j+1|k} - \hat{\mathbf{s}}_{k+j+1|k} \leq \hat{r}_{\max,k+j+1|k} \quad (19g)$$

$$\hat{\mathbf{s}}_{k+j+1|k} \geq 0 \quad (19h)$$

This linear program is based on the certainty equivalence assumption. Only the first input,  $\hat{u}_{k|k}$ , of this sequence is implemented. The function involving solution of (19) and selecting  $\hat{u}_{k|k}$  is denoted as

$$u_k = \hat{u}_{k|k} = \mu(\hat{\mathbf{x}}_{k|k}, u_{k-1}, \mathcal{D}_k, \mathcal{R}_k, \mathcal{F}_k) \quad (20)$$

## 2.4 Forecast based Certainty Equivalent MPC Algorithm

The certainty equivalent Economic MPC developed in this section is listed in Algorithm 1. It computes the manipulated variable,  $u_k$ , based on the current measurement,  $y_k$ , the previous input,  $u_{k-1}$ , the forecasts  $(\mathcal{D}_k, \mathcal{R}_k, \mathcal{F}_k)$ , and the smoothed mean-covariance estimate  $(\hat{d}_{k-1|k}, R_{dd,k-1|k})$ . The smoothed estimate,  $(\hat{d}_{k-1|k}, R_{dd,k-1|k})$ , is needed because we do the one-step prediction of the states,  $\hat{\mathbf{x}}_{k|k-1} = E\{\mathbf{x}_k | \mathcal{Y}_{k-1} = Y_{k-1}\}$  at time  $k$  when the information vector  $\mathcal{I}_k^d = I_k^d$  has been realized and is known. These information availability considerations are the reason that the one-step predictions in Algorithm 1 must be expressed as (22a) and (24a).

The main computational load in Algorithm 1 is solution of the linear program (19).

## 3. DYNAMICALLY DECOUPLED SYSTEMS

In this section, we specialize the stochastic system (1) to a dynamically decoupled system. Such decoupled models are ubiquitous in energy systems. Furthermore, we demonstrate how the linear program (19) for dynamically decoupled systems have a block angular structure. This block-angular structure may be utilized for efficient solution of (19) using decomposition algorithms such as the Dantzig-Wolfe algorithm.

**Algorithm 1** Certainty equivalent Economic MPC with external forecasts

**Require:**

Input:  $y_k, u_{k-1}$

Memory:  $\hat{x}_{k-1|k-1}, P_{k-1|k-1}$ .

Forecasts:

$$\mathcal{D}_k = \left\{ \hat{d}_{k+j|k} \right\}_{j=0}^{N-1}, (\hat{d}_{k-1|k}, R_{dd,k-1|k}) \quad (21a)$$

$$\mathcal{R}_k = \left\{ \hat{r}_{\min,k+j+1|k}, \hat{r}_{\max,k+j+1|k} \right\}_{j=0}^{N-1} \quad (21b)$$

$$\mathcal{F}_k = \left\{ \hat{b}_{k+j+1|k}, \hat{c}_{k+j|k}, \hat{\rho}_{k+j+1|k} \right\}_{j=0}^{N-1} \quad (21c)$$

**One-step predictor and filter:**

Compute the one-step prediction

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + Bu_{k-1} + E\hat{d}_{k-1|k} \quad (22a)$$

$$\hat{y}_{k|k-1} = C\hat{x}_{k|k-1} \quad (22b)$$

Compute the innovation

$$e_k = y_k - \hat{y}_{k|k-1} \quad (23)$$

Compute

$$P_{k|k-1} = AP_{k-1|k-1}A' + GR_{ww,k-1}G' + ER_{dd,k-1|k}E' \quad (24a)$$

$$R_{e,k} = R_{vv} + CP_{k|k-1}C' \quad (24b)$$

$$K_{fx,k} = P_{k|k-1}C'R_{e,k}^{-1} \quad (24c)$$

$$P_{k|k} = P_{k|k-1} - K_{fx,k}R_{e,k}K_{fx,k}' \quad (24d)$$

Compute the filtered state

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{fx,k}e_k \quad (25)$$

**Regulator:**

Compute  $u_k = \mu(\hat{x}_{k|k}, u_{k-1}, \mathcal{D}_k, \mathcal{R}_k, \mathcal{F}_k)$  by solution of the linear program (19).

**Return:**

Manipulated variable:  $u_k$

Update the memory with:  $\hat{x}_{k|k}, P_{k|k}$

Consider a set,  $\mathcal{P} = \{1, \dots, P\}$ , of dynamically decoupled systems

$$\mathbf{x}_{i,k+1} = A_i \mathbf{x}_{i,k} + B_i \mathbf{u}_{i,k} + G_i \mathbf{w}_k + E_i \mathbf{d}_k \quad i \in \mathcal{P} \quad (26a)$$

$$\mathbf{y}_{i,k} = C_i \mathbf{x}_{i,k} + \mathbf{v}_{i,k} \quad i \in \mathcal{P} \quad (26b)$$

$$\mathbf{z}_{i,k} = C_{z,i} \mathbf{x}_{i,k} \quad i \in \mathcal{P} \quad (26c)$$

that jointly create the following measurement,  $\tilde{\mathbf{y}}_k$ , and output,  $\tilde{\mathbf{z}}_k$ :

$$\tilde{\mathbf{y}}_k = \sum_{i=1}^P \tilde{C}_i \mathbf{x}_{i,k} + \tilde{\mathbf{v}}_k \quad (27a)$$

$$\tilde{\mathbf{z}}_k = \sum_{i=1}^P \tilde{C}_{z,i} \mathbf{x}_{i,k} \quad (27b)$$

The dynamically decoupled system (26)-(27) is a special case of (1) with the variables defined as

$$\begin{aligned} \mathbf{x}_k &= [\mathbf{x}_{1,k}; \mathbf{x}_{2,k}; \dots; \mathbf{x}_{P,k}] \\ \mathbf{u}_k &= [\mathbf{u}_{1,k}; \mathbf{u}_{2,k}; \dots; \mathbf{u}_{P,k}] \\ \mathbf{y}_k &= [\mathbf{y}_{1,k}; \mathbf{y}_{2,k}; \dots; \mathbf{y}_{P,k}; \tilde{\mathbf{y}}_k] \\ \mathbf{z}_k &= [\mathbf{z}_{1,k}; \mathbf{z}_{2,k}; \dots; \mathbf{z}_{P,k}; \tilde{\mathbf{z}}_k] \\ \mathbf{r}_{\min,k} &= [\mathbf{r}_{\min,1,k}; \mathbf{r}_{\min,2,k}; \dots; \mathbf{r}_{\min,P,k}; \tilde{\mathbf{r}}_{\min,k}] \\ \mathbf{r}_{\max,k} &= [\mathbf{r}_{\max,1,k}; \mathbf{r}_{\max,2,k}; \dots; \mathbf{r}_{\max,P,k}; \tilde{\mathbf{r}}_{\max,k}] \\ \mathbf{v}_k &= [\mathbf{v}_{1,k}; \mathbf{v}_{2,k}; \dots; \mathbf{v}_{P,k}; \tilde{\mathbf{v}}_k] \end{aligned}$$

and the corresponding state space matrices defined as

$$A = \text{block diag}([A_1 \ A_2 \ \dots \ A_P])$$

$$B = \text{block diag}([B_1 \ B_2 \ \dots \ B_P])$$

$$G = [G_1; G_2; \dots; G_P]$$

$$E = [E_1; E_2; \dots; E_P]$$

$$C = \begin{bmatrix} C_1 & & & & \\ & C_2 & & & \\ & & \ddots & & \\ & & & C_P & \\ \tilde{C}_1 & \tilde{C}_2 & \dots & \tilde{C}_P & \end{bmatrix} \quad C_z = \begin{bmatrix} C_{z,1} & & & & \\ & C_{z,2} & & & \\ & & \ddots & & \\ & & & C_{z,P} & \\ \tilde{C}_{z,1} & \tilde{C}_{z,2} & \dots & \tilde{C}_{z,P} & \end{bmatrix}$$

Eqs. (26)-(27) may be used to model the dynamics of a system of  $P$  individual linear plants with local measurements,  $\mathbf{y}_{i,k}$ , and outputs,  $\mathbf{z}_{i,k}$ . Collectively the system generate the output signal,  $\tilde{\mathbf{z}}_k$ , and the measurement,  $\tilde{\mathbf{y}}_k$ . For energy systems, the output signal  $\tilde{\mathbf{z}}_k$  may represent the total net power generated by the  $P$  controllable plants.

For the dynamically decoupled system, the predicted cost at time  $k$  (17) may be specialized to

$$\begin{aligned} \phi_k &= \sum_{i=1}^P \sum_{j=0}^{N-1} \hat{b}'_{i,k+j+1|k} \hat{z}_{i,k+j+1|k} + \hat{c}'_{i,k+j|k} \hat{u}_{i,k+j|k} \\ &\quad + \sum_{i=1}^P \sum_{j=0}^{N-1} \hat{\rho}'_{i,k+j+1|k} \hat{s}_{i,k+j+1|k} \\ &\quad + \sum_{j=0}^{N-1} \hat{\rho}'_{k+j+1|k} \hat{s}_{k+j+1|k} \\ &= \sum_{i=1}^P \phi_{i,k} + \sum_{j=0}^{N-1} \hat{\rho}'_{k+j+1|k} \hat{s}_{k+j+1|k} \end{aligned} \quad (30)$$

with the local objective functions for  $i \in \mathcal{P}$

$$\begin{aligned} \phi_{i,k} &= \sum_{j=0}^{N-1} \hat{b}'_{i,k+j+1|k} \hat{z}_{i,k+j+1|k} + \hat{c}'_{i,k+j|k} \hat{u}_{i,k+j|k} \\ &\quad + \sum_{j=0}^{N-1} \hat{\rho}'_{i,k+j+1|k} \hat{s}_{i,k+j+1|k} \end{aligned} \quad (31)$$

Consequently, the linear program (19) may be formulated as the following linear program

$$\min \phi_k = \sum_{i=1}^P \phi_{i,k} + \sum_{j=0}^{N-1} \hat{\rho}'_{k+j+1|k} \hat{s}_{k+j+1|k} \quad (32)$$

subject to the local constraints  $\forall i \in \mathcal{P}$  and  $\forall j \in \mathcal{N}$

$$\hat{x}_{i,k+j+1|k} = A_i \hat{x}_{i,k+j|k} + B_i \hat{u}_{i,k+j|k} + E_i \hat{d}_{k+j|k} \quad (33a)$$

$$\hat{z}_{i,k+j+1|k} = C_{z,i} \hat{x}_{i,k+j+1|k} \quad (33b)$$

$$u_{\min,i} \leq \hat{u}_{i,k+j|k} \leq u_{\max,i} \quad (33c)$$

$$\Delta u_{\min,i} \leq \Delta \hat{u}_{i,k+j|k} \leq \Delta u_{\max,i} \quad (33d)$$

$$\hat{z}_{i,k+j+1|k} + \hat{s}_{i,k+j+1|k} \geq \hat{r}_{\min,i,k+j+1|k} \quad (33e)$$

$$\hat{z}_{i,k+j+1|k} - \hat{s}_{i,k+j+1|k} \leq \hat{r}_{\max,i,k+j+1|k} \quad (33f)$$

$$\hat{s}_{i,k+j+1|k} \geq 0 \quad (33g)$$

as well as the connecting constraints  $\forall j \in \mathcal{N}$

$$\hat{z}_{k+j+1|k} = \sum_{i=1}^P \tilde{C}_i \hat{x}_{i,k+j+1|k} \quad (34a)$$

$$\hat{z}_{k+j+1|k} + \hat{s}_{k+j+1|k} \geq \hat{r}_{\min,k+j+1|k} \quad (34b)$$

$$\hat{z}_{k+j+1|k} - \hat{s}_{k+j+1|k} \leq \hat{r}_{\max,k+j+1|k} \quad (34c)$$

$$\hat{s}_{k+j+1|k} \geq 0 \quad (34d)$$

The linear program (32)-(34) has a block-angular structure that may be used for its efficient solution using a Dantzig-Wolfe decomposition algorithm.

#### 4. THE DANTZIG-WOLFE ALGORITHM

The Dantzig-Wolfe algorithm is a decomposition algorithm to solve large dimensions linear programming problems which have a block diagonal structure, (Dantzig and Wolfe, 1961). This decomposition technique breaks the problem into independent subproblems, which are coordinated by a master problem (MP). The units communicate only with the MP, exchanging Lagrange multipliers.

The optimization problem we investigate is a block-angular structured linear problem (35), (Hovgaard et al., 2010; Edlund et al., 2011), obtained from (32)-(33) and (34)

$$\min_{\{q_i\}_{i=1}^P, s} \phi = \sum_{i=1}^P c'_i q_i + d' s \quad (35a)$$

$$s.t. \quad \begin{bmatrix} F_1 & F_2 & \dots & F_P & E \\ G_1 & & & & \\ & G_2 & & & \\ & & \ddots & & \\ & & & G_P & \\ & & & & I \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_P \\ s \end{bmatrix} \geq \begin{bmatrix} g \\ h_1 \\ h_2 \\ \vdots \\ h_P \\ 0 \end{bmatrix} \quad (35b)$$

where  $F_i$  stands for the connecting constraints and  $G_i$  for the decoupled constraints of each subsystem. The  $F_i$  are obtained introducing the impulse response matrices into (33).

Our optimization variables,  $q_i$  and  $s$ , are related to (32). It has to be noticed that while in the previous sections we consider  $P$  subsystems, here the slack variables are considered as an independent unit as well; therefore the Dantzig-Wolfe decomposition splits the control problem into  $P + 1$  subsystems.

The key of this decomposition technique is the theorem of convex combinations.

*Theorem 1.* Let  $\mathcal{Q} = \{q | Gq \geq h\}$  be nonempty, closed and bounded, i.e. a polytope. The extreme points of  $\mathcal{Q}$  are denoted  $v^j$  with  $j \in \{1, 2, \dots, M\}$ .

Then any point  $q$  in the polytopic set  $\mathcal{Q}$  can be written as a convex combination of extreme points

$$q = \sum_{j=1}^M \lambda_j v^j \quad (36a)$$

$$s.t. \quad \lambda_j \geq 0, \quad j = 1, 2, \dots, M \quad (36b)$$

$$\sum_{j=1}^M \lambda_j = 1 \quad (36c)$$

**Proof.** See Dantzig and Thapa (2003).

As a decomposition algorithm, the first step is defining the Master Problem, (Ho and Loute, 1981). Using the theorem of convex combinations, the polytopes are defined by (37) where  $v_i^j$  are the vertices of  $\mathcal{Q}_i$ , (Dantzig and Wolfe, 1961).

$$\mathcal{Q}_i = \{q_i | G_i q_i \geq h_i\} \quad (37a)$$

$$q_i = \sum_{j=1}^{M_i} \lambda_{ij} v_i^j \quad (37b)$$

$$\sum_{j=1}^{M_i} \lambda_{ij} = 1 \quad (37c)$$

$$\lambda_{ij} \geq 0 \quad j = 1, 2, \dots, M_i \quad (37d)$$

The linear block angular problem (35) can be then rewritten as the equivalent Master Problem (38) for  $P + 1$  subproblems

$$\min_{\lambda} \phi = \sum_{i=1}^{P+1} \sum_{j=1}^{M_i} f_{ij} \lambda_{ij} \quad (38a)$$

$$s.t. \quad \sum_{i=1}^{P+1} \sum_{j=1}^{M_i} p_{ij} \lambda_{ij} \geq g \quad (38b)$$

$$\sum_{j=1}^{M_i} \lambda_{ij} = 1, \quad i = 1, 2, \dots, P + 1 \quad (38c)$$

$$\lambda_{ij} \geq 0, \quad i = 1, 2, \dots, P + 1; j = 1, 2, \dots, M_i \quad (38d)$$

$f_{ij}$  and  $p_{ij}$  are defined as

$$f_{ij} = c'_i v_i^j \quad (39a)$$

$$p_{ij} = F_i v_i^j \quad (39b)$$

The (38) defines as well the Lagrange multipliers  $\pi$  from the coupling constraints (38b),  $\rho$  for (38c) and  $\kappa_{ij}$  from (38d). The Master Problem (38) has fewer constraints than the original optimization problem (35), but more variables as the vertices of each polytope are included. For this reason the Reduced Master Problem is introduced as a MP but with  $l$  number of vertices, where  $l \leq M_i$ :

$$\min_{\lambda} \phi = \sum_{i=1}^{P+1} \sum_{j=1}^l f_{ij} \lambda_{ij} \quad (40a)$$

$$s.t. \quad \sum_{i=1}^{P+1} \sum_{j=1}^l p_{ij} \lambda_{ij} \geq g \quad (40b)$$

$$\sum_{j=1}^l \lambda_{ij} = 1, \quad i = 1, 2, \dots, P+1 \quad (40c)$$

$$\lambda_{ij} \geq 0, \quad i = 1, 2, \dots, P+1; j = 1, 2, \dots, l \quad (40d)$$

in which  $l \leq M_i$  for all  $i \in \{1, 2, \dots, P+1\}$ . Obviously, the Reduced Master Problem can be regarded as the Master Problem with  $\lambda_{i,j} = 0$  for  $j = l+1, l+2, \dots, M_i$  and all  $i \in \{1, 2, \dots, P+1\}$ . Initially, a feasible extreme point to the Master Problem (38) is needed. (4.1) addresses this topic. We assume now that a feasible extreme point has been computed. We can use this feasible extreme point to form a Reduced Master Problem with  $l = 1$ .

We denote the solution to the Reduced Master Problem (40) as  $\lambda_{ij}^{RMP}$  such that a feasible solution to Master Problem (38) is

$$\lambda_{ij} = \lambda_{ij}^{RMP} \quad i = 1, 2, \dots, P; j = 1, 2, \dots, l \quad (41a)$$

$$\lambda_{ij} = 0 \quad i = 1, 2, \dots, P; j = l+1, l+2, \dots, M_i \quad (41b)$$

Analysing the Karush-Kuhn-Tucker conditions for the (38), these are stated as

$$\sum_{i=1}^{P+1} \sum_{j=1}^{M_i} p_{ij} \lambda_{ij} - g \geq 0 \quad \perp \quad \pi \geq 0 \quad (42a)$$

$$\sum_{j=1}^{M_i} \lambda_{ij} - 1 = 0 \quad (42b)$$

$$\lambda_{ij} \geq 0 \quad \perp \quad \kappa_{ij} = [c_i - F_i' \pi]' v_i^j - \rho_i \geq 0 \quad (42c)$$

They are already satisfied for  $i = 1, 2, \dots, P+1$  and  $j = 1, 2, \dots, l$ . For all  $i = 1, 2, \dots, P+1$  and  $j = l+1, l+2, \dots, M_i$  instead, they are satisfied if  $\min_i \psi_i - \rho_i \geq 0$  where

$$\psi_i = \min_{v_i^j} [c_i - F_i' \pi]' v_i^j \quad i = 1, 2, \dots, P+1 \quad (43)$$

$v_i^j$  is an extreme point of the polytope  $\mathcal{Q}_i = \{q_i : G_i q_i \geq h_i\}$ . Therefore, using the Simplex Algorithm we may compute the solution of (43) as the solution of the linear program

$$\psi_i = \min_{q_i} \phi = [c_i - F_i' \pi]' q_i \quad (44a)$$

$$s.t. \quad G_i q_i \geq h_i \quad (44b)$$

for  $i = 1, 2, \dots, P+1$ . The programs (44) are called subproblems.

If  $\psi_i - \rho_i \geq 0$  for all  $i = 1, 2, \dots, P+1$ , the solution generated by the Reduced Master Problem is optimal. We can compute the solution to original problem (35) by

$$q_i^* = \sum_{j=1}^l v_i^j \lambda_{ij} \quad i = 1, 2, \dots, P+1 \quad (45)$$

Instead if  $\psi_i - \rho_i < 0$  for some  $i \in \{1, 2, \dots, P+1\}$  then the KKT conditions are not satisfied and the solution generated by the Reduced Master Problem is not a solution to the Master Problem. In this case, we augment the Reduced Master Problem with the new extreme points,  $v_i^{l+1}$ , obtained by solution of the subproblems (44).

The next iteration of the algorithm starts with the solution of the new Reduced Master Problem. The algorithm terminates in a finite number of iterations as there is a finite number of extreme points in a polytope.

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#### Algorithm 2 Dantzig-Wolfe

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Compute the initial feasible vertex for the Master Problem (38).

If any points is found then stop.

$l=1$ . Converged=false

**while** Converged == false **do**

Solve the  $l$ -th RMP (40)

Solve all the subproblems ( $i = 1, 2, \dots, P+1$ ) (44) considering the  $\pi$  from (40b) and  $\rho_i$  from (40c).

**if**  $\phi_i - \rho_i \geq 0 \quad \forall i \in \{1, 2, \dots, P+1\}$  **then**

Converged = true

The optimal solution is given by (45)

**else**

Compute the coefficients for the new columns in the RMP

$$f_{i,l+1} = c_i' v_i^{l+1}$$

$$p_{i,l+1} = F_i v_i^{l+1}$$

$$l = l + 1$$

**end if**

**end while**

---

#### 4.1 Initial feasible vertex

In (Dantzig and Thapa, 2003) the initial feasible solution for the Master Problem is obtained by Phase I procedure. A feasible vertex of the block angular linear program (35) is identical to a feasible vertex of the Master Problem (38) as these two linear programs are different representations of the same problem. The initial feasible vertex of the Master Problem (38) may be computed solving the following linear program

$$\min_{\alpha, \{\beta_i\}_{i=1}^{P+1}} \phi_I = e'_{\alpha} \alpha + \sum_{i=1}^{P+1} e'_{\beta_i} \beta_i \quad (46a)$$

$$s.t. \quad \sum_{i=1}^{P+1} F_i q_i + R \alpha \geq g \quad (46b)$$

$$G_i q_i + S_i \beta_i \geq h_i \quad i = 1, 2, \dots, P+1 \quad (46c)$$

$$0 \leq \alpha \leq |g| \quad (46d)$$

$$0 \leq \beta_i \leq |h_i| \quad i = 1, 2, \dots, P+1 \quad (46e)$$

with  $R$  and  $S$  diagonal matrices defined for  $i = j$  and  $p = q$  as

$$R_{ij} = \begin{cases} 1 & g_i \geq 0 \\ -1 & g_i < 0 \end{cases} \quad (S_i)_{p,q} = \begin{cases} 1 & (h_i)_p \geq 0 \\ -1 & (h_i)_p < 0 \end{cases}$$

It should be noticed that the computation of a feasible vertex of (38), i.e. solution of (46) by the Dantzig-Wolfe algorithm, is of approximately the same computational complexity as the computation of the optimal solution when a feasible vertex is available. This means that we can utilise the block-angular structure efficiently in the computation of a feasible vertex. It also means that just finding a feasible vertex may be just as expensive as computing the optimal solution. Therefore, if a feasible vertex is readily available, it should be used directly instead of applying a phase I simplex procedure. A feasible initial vertex for our problem (35) may be defined as

$$\{q_i^0 = q_{i,\min}\}_{i=1}^P \quad s = \max \left\{ g - \sum_{i=1}^P F_i q_i^0, 0 \right\} \quad (47)$$

The Dantzig-Wolfe algorithm is a part of a MPC controller, so the previous solution is always available and it can be used to compute the initial vertex as well.

## 5. RESULTS

In this section we provide an example of a controller which implements the Dantzig-Wolfe algorithm for energy systems.

The algorithm developed is compared to a centralized MPC controller. We consider a scenario of distributed energy system (DES) with several power generators.

### 5.1 Closed-loop simulations

We implement the Dantzig-Wolfe decomposition in solving the linear program (19) as described in Algorithm (1). The simulation runs over an horizon of 100 time steps. Here the benchmark is an energy system with two power plants, where both process noise and measurements noise are affecting the system. In this case Figure (1) demonstrates that the total portfolio output follows the reference.

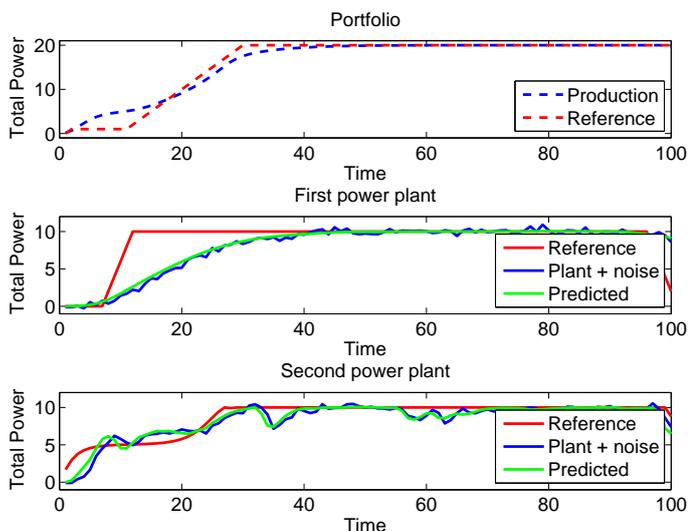


Fig. 1. Closed loop simulation.

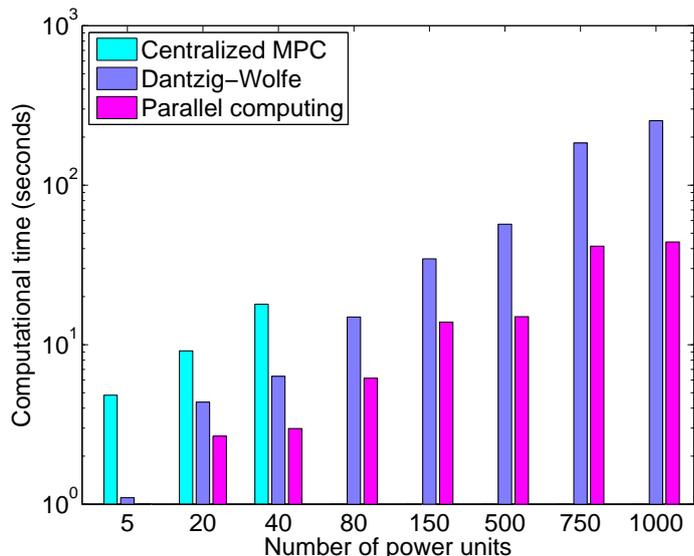


Fig. 2. Centralized MPC and Dantzig-Wolfe computational times.

### 5.2 Computational time

To investigate how the Dantzig-Wolfe perform in controlling large-scale energy systems, we compare it to a centralized MPC controller in open loop simulations. The latter fails in solving the control problem where the number of power units is high, i.e. more than 60 power generators due to the large size of the problem, as depicted in Figure 2. It appears that implementing the Dantzig-Wolfe algorithm, solves quicker the control problem compared to a centralized MPC controller. Furthermore in the Dantzig-Wolfe decomposition, the subproblems (44) can be solved in parallel; such way of computing reduces the computational time as Figure 2 demonstrates.

## 6. CONCLUSIONS

In this paper we have developed a controller for large scale energy systems. All the power units are dynamically decoupled. In this way, the control problem shows a block-angular structure which allows the implementation of decomposition techniques.

The controller obtained solves the control problem (35) implementing the Dantzig-Wolfe decomposition algorithm. Under such control action, the manipulated output follows the reference even when noises are affecting the system. This approach has potential in large-scale systems, as the computational time taken is lower compared to a centralized MPC controller. Furthermore the Dantzig-Wolfe algorithm allows parallel computing which improves speed of the algorithm.

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# Paper I - 17<sup>th</sup> Nordic Process Control Workshop, 2012

**A Dantzig-Wolfe Decomposition Algorithm for Linear  
Economic MPC of a Power Plant Portfolio**

**Authors:**

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## A Dantzig-Wolfe Decomposition Algorithm for Linear Economic MPC of a Power Plant Portfolio

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**Abstract:** Recently, the interest in renewable energy sources is increasing. In the short future, their penetration in the power systems will be significantly higher than today. Denmark is working on achieving its goal by 2020 of having 30% of the energy production provided by renewable sources. 50% of the total power consumption is expected to stem from wind turbines. Due to the inherent stochasticity in renewable energy systems (RES), their energy production is usually complicated to forecast and control. The aim of the smart grid in which consumers as well as producers are controlled is to allow for larger variation in the power production due to the significant amount of renewable energy. The multiple power generators and consumers must be coordinated to balance the supply and demand for power at all times.

The aim of this study is to examine a control technique for large scale distributed energy systems (DES), where a significant amount of renewable energy sources are present. Economic Model Predictive Control (MPC) is applied to control the power generators, minimizing the cost and producing the amount of energy required. We examine the large scale scenario, where multiple power generators and consumers such as e.g. electrical vehicles, heat pumps for domestic heating, and refrigeration and cooling systems must be controlled to balance the supply and demand for power. The system is very large scale. To address the large scale of the system and be able to compute the control decisions within a sample period, Dantzig-Wolfe decomposition is used for solution of the resulting linear program describing the Economic MPC of such systems. The controller obtained has been tested by simulations of a power portfolio system.

*Keywords:* Decoupled subsystems, Model based control, Predictive control, Optimization, Power system control, Decomposition

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# Paper L - Technical report DTU, 2014

**Partial Cycling in Dantzig-Wolfe Optimization Applied to  
Linear MPC for Power Plant Management**

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# Partial Cycling in Dantzig-Wolfe Optimization Applied to Linear MPC for Power Plant Management\*

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**Abstract**— Computation time and storage space are the main constraints for the applicability of the Model Predictive Control (MPC) strategy to real-time problems. In this work we suggest MPC as a strategy to control a large-scale energy system consisting of independent power units. These units are seen as linear models, subject to linear constraints and coupled through a linear objective function: therefore, the control problem is a linear program with a block-angular structure. The Dantzig-Wolfe decomposition technique efficiently computes the optimal control sequence. The novel features in this paper are the partial cycling strategies applied to the Dantzig-Wolfe algorithm. These strategies aim to speed up the Dantzig-Wolfe distributed optimization technique applied to LPs, thus reducing data handling. Simulations reveal that the computation time highly decreases and the amount of data storage lessens. However, these changes cause deterioration in the objective function.

## I. INTRODUCTION

The continued burning of fossil fuels to run our cars, factories and electricity plants is now certain to induce serious alterations to our climate and affect our lives. In regards to electricity plants, a revolution is about to happen due to the extensive use of renewable energy sources (RES) and to the introduction of smart grids. These new energy systems will connect power generators and customers to enhance the performance, economics, and sustainability of the production and distribution of electricity. However, re-engineering such large-scale energy systems requires reliable controllers that operate in real-time. Dynamic control of power plants is becoming highly important as power companies need to adapt their production at short notice to unavoidable and uncontrollable fluctuations in consumer demand and in the availability of production resources.

In our work we propose an Economic Model Predictive Control (MPC) strategy for balancing power supply and demand in future energy systems. The case study is a large-scale energy system consisting of multiple independent and dynamically decoupled power units that are coupled only through the overall power production. We address a power plant management problem, where the controller must coordinate power generators in the view of producing enough power to satisfy the customers' demand. Moreover, the power units are modelled as linear systems, subject to linear constraints and minimizing a linear objective function; hence, the control problem is linear and has a block-angular structure. The Dantzig-Wolfe decomposition applies to this

linear control problem and it efficiently computes the optimal control trajectory. This paper presents a novel approach to the Dantzig-Wolfe decomposition applied to linear programming problems (LPs) aimed to speed up the algorithm as well as to increase the applicability of such controller to real-time applications. We introduce partial cycling strategies that quicken the algorithm and reduce data storage.

The success of MPC as a control strategy is widely recognized and it has been applied to several industrial and smart-energy applications. Distributed control structures have been developed especially when the systems are large scale [1]; for this purpose, decomposition techniques are often applied to distributed MPC regulators, i.e. the Dantzig-Wolfe decomposition technique. Based on an  $\ell_1$ -penalty function, the power balancing problem is formulated as an MPC and the resulting large scale linear program is solved using a Dantzig-Wolfe algorithm [2]. The performances of a centralized MPC controller have been compared to the implementation of the Dantzig-Wolfe decomposition algorithm [3]. Nevertheless, the computational aspect of the Dantzig-Wolfe decomposition has been addressed mostly for mix integer and binary problems [4], [5], [6]. In addition, the computational aspects of Bender's decomposition and its dual Dantzig-Wolfe for stochastic programming are studied in [7]. Algorithms explicitly tailored for LP are rare and distributed simplex algorithms have been tailored for multi-agent assignments and degenerate LP [8]. Suboptimal MPC has been applied successfully in many applications ensuring feasibility and stability [9]; however, many works have introduced a combination of explicit MPC and online optimization as suboptimal MPC strategy [10], [11], [12], [13]. An early termination approach has been applied to the Dantzig-Wolfe decomposition algorithm yielding to highly computational time decreases [14].

The paper is structured as follows. Section II introduces a large-scale energy system and its power units and presents the MPC control problem. Section III describes the Dantzig-Wolfe decomposition algorithm. Section IV illustrates the novel partial cycling strategies and the suboptimal MPC approach. Section V shows the results obtained, while conclusions and future works are in Section VI.

## II. DISTRIBUTED LINEAR ECONOMIC MPC

In a large-scale energy system the overall power production includes the power produced by each independent power unit. The power plant management problem requires taking real-time decisions to plan power production in response to the customer demand. This section introduces the power

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generators units and the MPC controller that operates such an energy system. We propose an Economic MPC policy that balances power supply and demand, minimizing production costs.

### A. Power Systems

Let us consider a power system that consists of  $P$  power units, each of those is modelled as a linear state space model

$$x_{i,k+1} = A_i x_{i,k} + B_i u_{i,k} \quad (1a)$$

$$z_{i,k} = C_{z,i} x_{i,k} \quad (1b)$$

$$(1c)$$

where  $i = 1, \dots, P$ ,  $x_{i,k}$  denotes the states,  $u_{i,k}$  the manipulated variables (MVs) and  $z_{i,k}$  is output variables. The manipulated variable,  $u_{i,k}$ , is limited by bounds and rate-of-movement constraints

$$u_{i,min} \leq u_{i,k} \leq u_{i,max} \quad (2a)$$

$$\Delta u_{i,min} \leq \Delta u_{i,k} \leq \Delta u_{i,max} \quad (2b)$$

These are hard constraints and not mean-value constraints.

The system output  $z_{i,k}$  must be within a given interval  $[r_{i,min,k}, r_{i,max,k}]$  where the bounds are provided by external forecasting systems. However, for energy systems this interval might represent the state of charge of a battery in an electrical vehicle, or the temperature in a building, or the electricity demand forecast in advance.

The cost of producing the power over a period of time  $N$ , is

$$\phi_{i,k} = \sum_{k=0}^{N-1} c'_{i,k} u_{i,k} \quad (3)$$

where  $c_{i,k}$  is the cost of operating the power plan  $u_k$ .

### B. Linear Economic Model Predictive Control (MPC)

Consider an energy system consisting of independent and dynamically decoupled power units described in Section II-A. In a set of power units  $\mathcal{P} = \{1, \dots, P\}$ , these are coupled only through the overall portfolio power production  $\tilde{z}_k$  subject to the following coupling constraints

$$\tilde{z}_k = \sum_{i=1}^P C_{z,i} x_{i,k} \quad (4a)$$

$$\tilde{r}_{min,k} - s_k \leq \tilde{z}_k \leq \tilde{r}_{max,k} + s_k \quad (4b)$$

The interval  $[\tilde{r}_{min,k}, \tilde{r}_{max,k}]$  represents the customer demand interval and the limits are forecast by external systems. Slack variables  $s_k$  are often introduced to these constraints.

Overall, linear models describe the single power unit (1), which are subject to linear constraints (2),(4) and the objective function (3) to minimize is linear as well. Therefore, the Economic Model Predictive Control to operate an energy system consisting of  $P$  power units, for  $\forall k \in \mathcal{N}$ , can be formulated as

$$\min \quad \phi_k = \sum_{i=1}^P \phi_{i,k} + \rho'_k s_{i,k} \quad (5)$$

subject to the local constraints  $\forall i \in \mathcal{P}$  and  $\forall k \in \mathcal{N}$

$$x_{i,k+1} = A_i x_{i,k} + B_i u_{i,k} \quad (6a)$$

$$z_{i,k} = C_{z,i} x_{i,k} \quad (6b)$$

$$u_{min,i} \leq u_{i,k} \leq u_{max,i} \quad (6c)$$

$$\Delta u_{min,i} \leq \Delta u_{i,k} \leq \Delta u_{max,i} \quad (6d)$$

$$z_{i,k} + s_{i,k} \geq r_{min,i,k} \quad (6e)$$

$$z_{i,k} - s_{i,k} \leq r_{max,i,k} \quad (6f)$$

$$s_{i,k} \geq 0 \quad (6g)$$

and subject to the connecting constraints  $\forall k \in \mathcal{N}$  in (4).

$s_{i,k}$  are slack variables and  $\rho_k$  is the penalty that must be paid whenever the slack variables are non zero.

The linear program (4)-(6) has a block-angular structure: the coupling constraints (4) define the blocks on the first row, while the decoupling constraints (6) design the blocks on the main diagonal. The Dantzig-Wolfe decomposition technique efficiently solves linear programs having such block-angular structure.

### III. DANTZIG-WOLFE OPTIMIZATION

The Dantzig-Wolfe decomposition algorithm is an efficient computational solution of linear programs that have a block-angular structure. Such problems can be seen as generalized linear programs in which the column of coefficients may be freely chosen as any point from a convex set  $\mathcal{Q}$  [15], [16]. A generalized linear program can be seen as

$$\min_q \quad \phi = \sum_{i=1}^M d'_i q_i \quad (7a)$$

$$s.t. \quad \sum_{i=1}^M F_i q_i \geq g \quad (7b)$$

$$G_i q_i \geq h_i \quad i \in \mathcal{M}. \quad (7c)$$

where the slack variables in the global constraints (4) are considered as an independent unit, hence  $\mathcal{M} = \{1, \dots, P, P+1\}$ . This constitutes the set of *subproblems* that cooperate through the joint constraints (7b), the set of constraints (7c) is defined for each subproblem. Our MPC control problem (4)-(6) can easily be related to this specific block-angular problem (7).

The Dantzig-Wolfe decomposition hinges on the D-W transformation, which states that every feasible solution of a convex polyhedral set  $\mathcal{Q}_i$  in  $\mathcal{R}^n$  can be represented as a convex combination of a finite set of the extreme points,  $v_i^j$ , of  $\mathcal{Q}_i$  and a non-negative linear combination of the finite set of homogeneous solutions (extreme rays),  $u_i^k$ , of  $\mathcal{Q}_i$ . Therefore, any feasible solution to (7) can be written as

$$q_i = \sum_{j=1}^L \alpha_{ij} v_i^j + \sum_{k=1}^K \beta_{ik} u_i^k \quad (8a)$$

$$\sum_{j=1}^L \alpha_{i,j} = 1, \quad i \in \mathcal{M} \quad (8b)$$

$L$  is the number of possible extreme points (basic feasible solutions) and  $K$  denotes the possible normalized extreme

homogeneous solutions. In addition,  $\alpha_{ij}, \beta_{ik} \geq 0$ ,  $j = 1, \dots, L$ ,  $k = 1, \dots, K$  and  $i = 1, \dots, M$ .

We assume a feasible, non-empty and bounded polytope, therefore we can use the extreme points only; see [15] for including the extreme rays in the problem.

#### A. Master Problem or Extremal Problem

Substituting (8) into (7) transforms the original block-angular problem into an equivalent linear program with fewer constraints but more variables. Such a problem is called *master problem* (MP) or *extremal problem*, and it is defined as

$$\min_{\alpha} \quad \phi = \sum_{i=1}^M \sum_{j=1}^{L_i} f_{ij} \alpha_{ij} \quad (9a)$$

$$s.t. \quad \sum_{i=1}^M \sum_{j=1}^{L_i} p_{ij} \alpha_{ij} \geq g \quad (9b)$$

$$\sum_{j=1}^{L_i} \alpha_{ij} = 1 \quad i = 1, \dots, M \quad (9c)$$

$$\alpha_{ij} \geq 0 \quad i = 1, \dots, M, j = 1, \dots, L_i \quad (9d)$$

where

$$f_{ij} = d_i' v_i^j \quad p_{ij} = F_i v_i^j \quad (10)$$

The MP (9) is equivalent to the block-angular program (7). In practice it is impractical to generate a full MP: it is worth noting that all the extreme points of each polytope are included as variables. Through the *wait-and-see column generation procedure* the Dantzig-Wolfe algorithm generates only the column of the MP that is coming into the basis (see [15]); in this way, the algorithm generates only the column having the most negative reduced cost without having to generate and price out all the remaining columns of the MP. Therefore, at each iteration the algorithm solves a *reduced master problem* (RMP).

#### B. Simplex Multipliers and Subproblems

Solving the RMP provides the simplex multiplier,  $\pi$  and  $\rho_i$ , respectively associated to the coupling (9b) and convexity (9c) constraints. The Dantzig-Wolfe algorithm modifies the subproblems and solves to find the priced-out column associated with  $\alpha_{ij}$ ; each subproblem is defined as

$$\psi_i = \min_{q_i} \phi_i = [d_i - F_i \pi]' q_i \quad (11a)$$

$$s.t. \quad G_i q_i \geq h_i \quad (11b)$$

The objective functions in (11) include an augmented term for the information from the RMP; such functions are called adjust costs. Moreover, the subproblems (11) can be solved in parallel: this option might significantly affect the computation time.

#### C. Optimality Condition

The solution is assumed to be optimal if the following condition is satisfied

$$\psi_i - \rho_i \geq 0 \quad (12)$$

where  $\psi_i$  is the optimal objective function value for the  $i$  subproblem (11), while  $\rho_i$  is the simplex multipliers associated to the  $i$  column of the RMP.

Otherwise, if the optimality condition (12) is not satisfied, the Dantzig-Wolfe algorithm augments the RMP by the  $i$ , priced-out, column and reoptimizes the augmented RMP.

In summary, the Dantzig-Wolfe decomposition technique formulates the original block-angular program into an equivalent MP. In practice, the full MP is never generated and the technique utilizes the MP reduced form. Then, at each iteration, the RMP provides information to update the subproblems that are re-solved to provide the new incoming column of the RMP for the next iteration. To summarise, if the subproblem  $i$  does not satisfy the condition (12), then the column  $i$  comes into basis by adding it to the RMP: this process intends to include one extreme point to the polytope  $\mathcal{Q}_i$ . On the contrary, if the subproblem  $i$  satisfies the condition (12), then the number of extreme points of the polytope  $\mathcal{Q}_i$  is enough to compute the optimal solution.

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#### Algorithm 1 The Dantzig-Wolfe algorithm for block-angular LP

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Compute the coefficients (10) for the RMP using the initial feasible vertex available.

$l = 1$ . Converged=false

**while** Not Converged **do**

    Solve the  $l$ -th RMP.

    Solve all the subproblems ( $i = 1, 2, \dots, M$ ) (11) with  $\pi$  and  $\rho_i$ , respectively associated to the coupling (9b) and convexity (9c).

**if**  $\psi_i - \rho_i \geq 0 \quad \forall i \in \{1, 2, \dots, M\}$  **then**

        Converged = true

**else**

        Compute the coefficients for the new columns in the RMP

$$f_{i,l+1} = e_i' v_i^{l+1}$$

$$p_{i,l+1} = F_i v_i^{l+1}$$

$l = l + 1$

**end if**

**end while**

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#### D. Reduced Dantzig-Wolfe decomposition

In this work, we propose a reduced version of the Dantzig-Wolfe decomposition.

At each iteration of the Simplex algorithm, the Dantzig-Wolfe decomposition computes only the column of the RMP, which has to come into basis. This column has the most negative reduced cost. Moreover, let us assume that at

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**Algorithm 2** Reduced Dantzig-Wolfe algorithm for block-angular LP
 

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Initial feasible vertex for the RMP.  
**if** Any points are found **then**  
 Stop.  
**else**  
 $\mathcal{S} = \{\}$   
 $L=1$   
**while** *Converged* == *false* **do**  
 Solve the  $L$ -th RMP.  
 Solve subproblem  $i$  (15), for  $i \in \mathcal{M} \setminus \mathcal{S}$ .  
**if** optimality condition (13) is satisfied  $\forall i \in \mathcal{M}$   
**then**  
 $\text{Converged} == \text{true}$   
**else**  
**if** a subproblem  $s, s \in \mathcal{M}$ , satisfies the optimality condition (13) **then**  
 $\mathcal{S} = \{s\}, \mathcal{S} \subset \mathcal{M}$ .  
 Compute RMP coefficients  $\forall i \in \mathcal{M} \setminus \mathcal{S}$ .  
**else**  
 Compute RMP coefficients (14)  $\forall i \in \mathcal{M} \setminus \mathcal{S}$ .  
**end if**  
 $L = L + 1$   
**end while**  
**end if**

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iteration  $t$ , only a set of subproblems  $\mathcal{S} \subset \mathcal{M}$  satisfies the optimality condition (12)

$$\psi_s^* - \rho_s \geq 0 \quad s \in \mathcal{S} \subset \mathcal{M} \quad (13)$$

In such a scenario, the classic Dantzig-Wolfe brings variables into basis by adding columns to the RMP for every subproblem  $i \in \mathcal{M}$ , hence even for the set  $\mathcal{S}$  of subproblems.

In contrast, if condition (13) holds, then the reduced Dantzig-Wolfe does not add columns to the RMP for the set  $\mathcal{S}$  of subproblems; this yields to update the coefficients of the RMP as

$$f_{ij} = d_i^j v_i^j \quad p_{ij} = F_i v_i^j \quad i \in \mathcal{M} \setminus \mathcal{S} \quad (14)$$

At iteration  $t + 1$ , the reduced Dantzig-Wolfe solves the following subproblems

$$\psi_i = \min_{q_i} \phi_i = [d_i - F_i \pi]^T q_i \quad i \in \mathcal{M} \setminus \mathcal{S} \quad (15a)$$

$$s.t. \quad G_i q_i \geq h_i \quad (15b)$$

Consequently, the Dantzig-Wolfe applies the pricing problem on a reduced set of subproblems  $\mathcal{S}$ . As a result, by applying this reduced Dantzig-Wolfe decomposition, the number of iterations decreases. Algorithm 2 illustrates the reduced Dantzig-Wolfe decomposition.

#### IV. PARTIAL CYCLING STRATEGIES

The main issues in real-time problems are limits on the data storage or the computation time. In this work we

proposed three different approaches aimed at increasing the applicability of MPC controllers in many real-time problems; in particular, we address both issues, thus reducing the amount of data storage and computation time.

In accordance with Section III, the Dantzig-Wolfe decomposition technique includes new columns in the RMP at each iteration. This process involves two important issues: data storage and computational times. Adding a new column to the RMP clearly increases the amount of data storage. Moreover, a new column is added at each iteration, hence this process affects the computational time.

##### A. Partial Cycling A

The Dantzig-Wolfe algorithm adds a new column to the RMP if the optimality condition (12) is not satisfied, see Section III-A. Let us assume that in the set  $\mathcal{M}$  of the subproblems,  $\exists i \in \mathcal{M}$  such that the optimality condition (12) is not satisfied: accordingly, a new column has to be generated for the next iteration of the RMP. On the contrary,  $\exists h \in \mathcal{M}$  such that the optimality condition (12) is satisfied; then the  $h$  subproblem has enough extreme points of the polytope to compute the optimal solution, its objective function is already nonnegative, therefore, it does not have to add an extra column to the RMP.

In the following cycling strategies new columns of the RMP are added only for those subproblems that do not satisfy the optimal condition (12). This approach does not affect the finite termination of the algorithm because of the simplex property: the columns of any basis of the MP must be generated from a finite class as the convex sets are defined by a finite number of linear inequalities, so they can be represented by a finite number of extreme points [15].

##### B. Partial Cycling B

In addition to the adding columns strategy of Section IV-A, in this work we implement an heuristic in order to reduce the number of subproblems to solve at every iteration. Each subproblem (11) minimizes the objective function  $\phi_i$  adding extreme points in the polytope until the optimal condition (12) is not satisfied. Nonetheless, if the Dantzig-Wolfe algorithm provides a solution that does not satisfy the optimality condition (12), but it is still feasible for the control problem (7), then this solution is named to be suboptimal, as the following theorem states.

*Feasible and Optimal Solution 1:* Any  $\alpha_{ij}$  satisfying (9), determines a  $q_i$  via (8) which is a feasible solution to (7). Furthermore, if  $\phi_i$  has a minimum value for  $\alpha_{ij}^*$ , then the  $q_i^*$  related via (8) is a feasible optimal solution for (7).

*Proof:* See [15]. ■

We proposed a suboptimal MPC strategy based on the previous theorem. After a number of iterations, several extreme points have been added to each polytope; if a subproblem  $s \in \mathcal{M}$  has a decreasing objective function, then this problem will not be solved at the next iteration. Consequently, the solution obtained is feasible and suboptimal, as it does not satisfy the optimality condition.

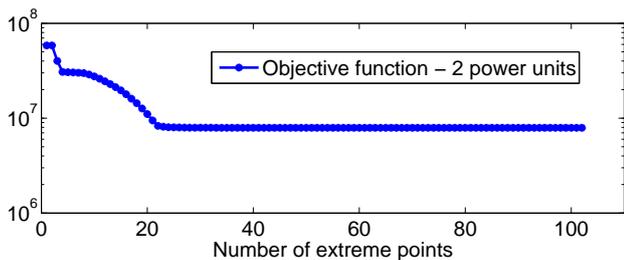


Fig. 1. Objective function for two power units

## V. RESULTS

We conclude by presenting some computational results of applying the partial cycling strategies to a case study. Simulations are run in MATLAB R2013a on a standard computing environment: an Intel® i7-2620 M Processor and 8 GB memory. We do not exploit either parallelism capabilities or specific solvers as the purpose of this work is not to show how fast the Dantzig-Wolfe algorithm can provide the optimal control trajectory; on the contrary we want to illustrate how the performances of the algorithm are improved by implementing the partial cycling strategies proposed in this paper.

The time horizon is  $T = 100$  and the simulations are open-loop. Moreover, a warm-start is used to provide a feasible initial vertex to the MP (9).

As a case study we proposed a large-scale energy system including multiple power generator units, which represent thermal power plants, diesel generator and gas turbines [17]. We include five large-scale energy systems using the following numbers of power generator units: 10, 50, 100, 150 and 200.

The production costs and the customer demand are assumed to be provided by external forecast systems, as well as the local output bounds  $r_{\min,i,k}$  and  $r_{\max,i,k}$  (6e)-(6f).

The number  $L$  in the partial strategy B is set equal to 15 due to the high number of constraints in the linear control problems. However, an high value of  $L$  might not yield to a substantial computational time decrease; on the contrary, a small value of  $L$  might cause a deterioration in the objective function optimal value. In this work, the value of  $L$  is set running a small case study, i.e. just two power units. Figure 1 shows that the objective function obtained reaches the minima after about 21 extreme points; therefore, we have set  $L$  equals to 15 extreme points corresponding to a decrease in the objective function.

### A. Computational Time

One of the purposes of this work is to reduce the computation times of the Dantzig-Wolfe distributed optimization algorithm to solve an MPC control linear problem. Figure 2(a) shows computation times related to the number of power units included in the case study. The partial strategy highly speed up the Dantzig-Wolfe algorithm compared to the traditional decomposition technique.

### B. Data storage

In the paper we focus on the idea that to increase the applicability of our controller to real-time problems, two issues need to be solved: the limits on computation times and on data storage. The previous results show the former issue, while now we address the latter one.

The Dantzig-Wolfe algorithm adds one new column to the RMP, as described in Section III and IV; therefore most of the data storage is related to this part of the algorithm. Table I reports the number of columns of the RMP related to the number of power generator units included in the case study. We note that the classic Dantzig-Wolfe implementation requires the highest number of columns, and hence, of variables.

TABLE I  
NUMBER OF COLUMNS IN RMP.

Number of Units	DW	Strategy A	Reduced DW	Strategy B
10	1365	334	266	294
50	4750	916	1139	785
100	9500	3015	3285	2792
150	13500	3038	3315	2488
200	18400	6629	8117	6556

### C. Performances

The purpose of this work is to increase the applicability of our controller for real-time problems without affecting the performance of the algorithm. Previous results show the efficiency of the algorithm and of the other strategies proposed in our work. However, such improvements must not affect the algorithm performance. We measure this possible influence considering the optimal value of the objective function.

Figure 2(b) shows the objective function optimal value for the four different implementations proposed: the classic and the reduced Dantzig-Wolfe algorithm, the two partial strategies.

## VI. CONCLUSION AND FUTURE WORKS

In this paper we have proposed two different strategies to reduce computational times and data storage. These strategies are tailored for the Dantzig-Wolfe distributed optimization algorithm applied to a linear economic MPC problem. We have introduced five case studies: large-scale energy systems consisting of multiple independent power generator units; consequently, we have defined the control strategy as a linear economic MPC problem. Because of the block-angular structure of such control linear problems, we have applied the Dantzig-Wolfe distributed optimization technique to efficiently solve it. Simulations have revealed that the strategies proposed in this paper significantly quicken the Dantzig-Wolfe decomposition algorithm and reduce the amount of data storage. Furthermore, simulations have demonstrated that such improvements cause deteriorations in the optimal values of the objective function. Future works will address

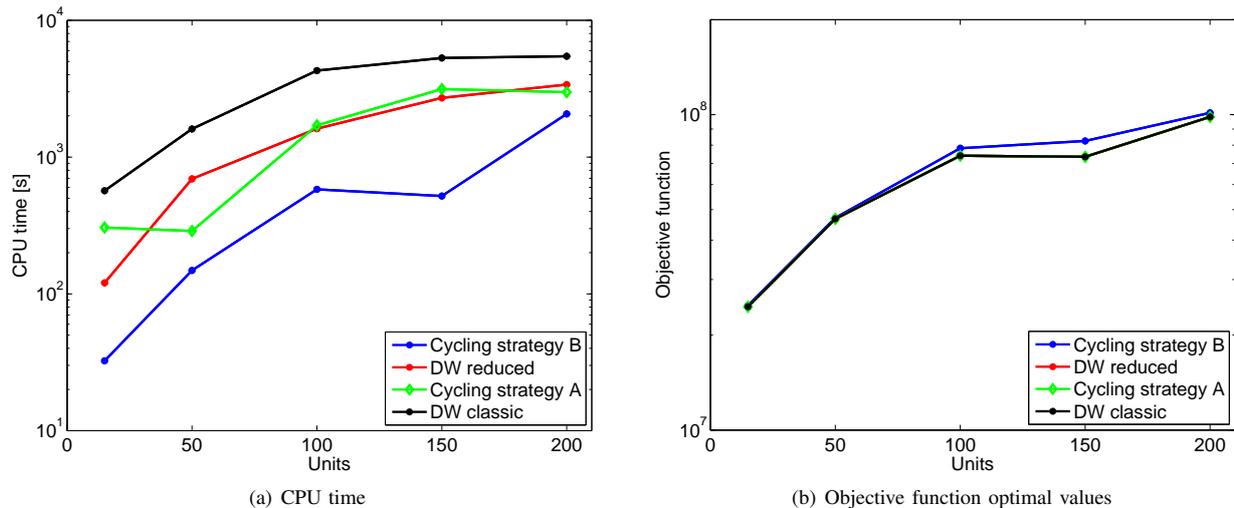


Fig. 2. CPU time and objective function optimal values vs. Number of power generator units in the energy system. The objective functions are provided by the implementation of: the classic Dantzig-Wolfe (black graph), the partial strategy A (green line), the reduced Dantzig-Wolfe (red graph) and the partial cycling B (blue line).

an implementation of the Dantzig-Wolfe algorithm to solve a linear economic MPC together with the strategies proposed in this work including parallel computing and specific LPs solvers.

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APPENDIX

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Operation of Power Systems via Dantzig-Wolfe Decomposition

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# Operation of Power Systems via Dantzig-Wolfe Decomposition

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**Abstract**— Future power systems will include a large number of decentralized and dynamically independent power producers and consumers. In this paper we propose an Economic MPC strategy for operation of such power systems. The dynamics are linear, the cost function is linear and the constraints are linear; as a result, the Economic MPC may be expressed as a linear program. This linear program is large scale that must be solved online with real-time constraints. To enable the solution of such large scale linear program, we propose a Dantzig-Wolfe algorithm tailored for the solution of Economic MPC of dynamically independent linear systems. The novel features of the Dantzig-Wolfe algorithm for solution of linear Economic MPC are: an initialization technique, that efficiently utilizes the slack variables of the power balancing problem, an early termination strategy to decrease the computational time and a computational efficient implementation that simulations demonstrate is significantly faster than state of the art general purpose LP solvers.

## I. INTRODUCTION

There is an unquestionable need for a change in the current landscape for energy production with the purpose of achieving a sustainable production of energy. Rising global temperatures have been accompanied by changes in weather and climate. As these changes become more pronounced in the coming decades, they will likely present challenges to our society and our environment. It is well-known that greenhouse gases are responsible for climate changes; however, it should be mentioned that the majority of greenhouse gases come from burning fossil fuels to produce energy. The rising prices of fossil fuels have led to the increase of energy prices as well. These issues suggest that the amount of renewable energy, such as solar power and wind power, in the mix of energy sources is increased. The future electricity delivery systems are so-called Smart Grids [1]. These networks connect from power plants, wind farms, solar panels all the way to the consumers of electricity in homes and businesses. The main purposes of Smart Grids is to control production and consumption of electrical power in an intelligent, reliable and cost efficient way [2], [3]. Future power producers will be mainly renewable energy sources (RES), characterized by stochastic predictions. Consequently, future power systems must include a large number of decentralized agile controllable stochastic power producers and consumers. Coordination and control are central for the operation of the smart grids; i.e. control and coordination are

needed to balance demand and supply of power in a robust and cost efficient manner.

In this paper we propose an Economic Model Predictive Control strategy for balancing the supply and demand of electrical power in a smart grid. We consider system with a very large number of dynamically independent energy components that must be operated in such a way that the power supply and demand is balanced in the most economic way. Power units dynamics are linear, costs functions are linear as well as constraints; because of this, the Economic MPC is expressed as a large scale linear program. The LP constituting the Economic MPC for dynamically independent energy components that must be operated to balance supply and demand has a block angular structure. Because of this property, the Dantzig-Wolfe decomposition technique is applied. Furthermore, we present a new technique to have an initial solution for the Dantzig-Wolfe algorithm. We investigate an early termination strategy to decrease the computational time. The contribution of this paper is to use Dantzig-Wolfe decomposition for computation of the dynamic calculations of MPC and thereby reduce the computational complexity compared to the centralized solution.

Model predictive Control (MPC) is widely recognized as the most successful methodology for control of industrial processes [4], [5]. Due to computational complexity and due to the communication bandwidth limitation, distributed control structures have been developed for large scale system [6]. The Dantzig-Wolfe decomposition algorithm for large linear programs was introduced [7], [8]. Recently, smart energy systems as refrigeration systems, heat pumps for residential buildings, solar heated water tanks and batteries in electrical vehicles implemented Economic MPC [9]–[15]. Based on an  $\ell_1$ -penalty function, the power balancing problem is formulated as an MPC and solved the resulting large scale linear program using a Dantzig-Wolfe algorithm [16]. However, in this paper an Economic MPC is applied to the power balancing problem, minimizing economic costs directly as opposed to minimizing the deviation from a set-point in some norm.

This paper is organized as follows. Section II introduces power systems and power units, these are modeled as stochastic systems. Section III formulates the operation problem for power systems as a linear Economic Model Predictive Control and Section IV describes the Dantzig-Wolfe decomposition algorithm. A new initialization technique is explained in Section V. Section VI proposes a case study and illustrates the computational efficiency obtained by early termination, while conclusions are provided in Section VII.

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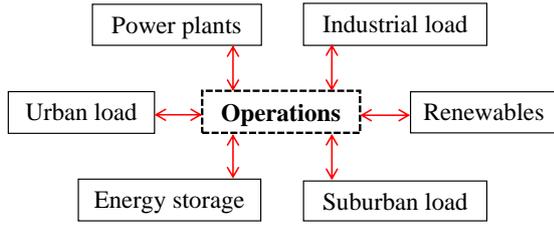


Fig. 1. Future power systems will connect power users and consumers as independent units; these power units can be expressed as stochastic systems. *Operations* coordinates and controls these power units, to guarantee power supply in response to the demand. Control of such large scale power systems requires new control algorithms.

## II. POWER SYSTEMS

Figure 1 shows a generic power system, where producers and consumers are independent units. The *operations* has the task to coordinate and control these power units, balancing power supply and demand. In a power system as in Figure 1, the total power supply includes the power produced by each independent producers. Such power systems are called Distributed Energy Sources (DES). In this section we introduce power units as independent stochastic and dynamically decoupled systems; such decoupled models are ubiquitous in power systems.

### A. Power Unit

The energy units considered in this paper can be described by the stochastic linear discrete time state space model

$$x_{k+1} = Ax_k + Bu_k + Ed_k \quad (1a)$$

$$y_k = Cx_k + v_k \quad (1b)$$

$$z_k = C_z x_k \quad (1c)$$

The measurement noise is distributed as  $v_k \sim N_{iid}(0, R_{vv})$ .  $x_k$  denotes the states,  $u_k$  the manipulated variables (MVs),  $y_k$  denotes the measurement used for feedback, and  $z_k$  is output variables.  $d_k \sim N(\bar{d}_k, R_{dd,k})$  denotes process noise that can be predicted by a prognosis system and independently of the measurements  $y_k$  [17]. In many power applications  $d_k$  might represents wind speed or sun radiation.

The manipulated variable,  $u_k$ , is stochastic limited by the bounds and rate-of-movements constraints

$$u_{min} \leq u_k \leq u_{max} \quad (2a)$$

$$\Delta u_{min} \leq \Delta u_k \leq \Delta u_{max} \quad (2b)$$

These are hard constraints and not probabilistic, not mean-value constraints.

The system output  $z_k$  represents the power produced by the generator; clearly the production has to satisfy the customers demand,  $r$ . Energy consumption is usually forecast and defined by an interval as  $[r_{min,k}, r_{max,k}]$ , where the bounds are stochastic variables stemming from some distribution. It is worth noting that for some scenarios or disturbances, it may be very expensive or even impossible to keep the outputs  $z_k$

within this interval. For such situations, we introduce slack variables

$$r_{min,k} - s_k \leq z_k \leq r_{max,k} + s_k \quad (3)$$

with  $s_k \geq 0$ . The slack variables,  $s_k$ , may represent selling or buying power from the short-term market, violation of temperature limits, or violation of state-of-charge limits. Every time  $s_k$  is non-zero, a penalty cost, e.g. the cost of buying or selling power on the short-term market, must be paid.

The cost of producing the power over a period of time, is  $\phi_k$ . This economic cost,  $\phi_k$ , consists of the cost of operating a power generator,  $c_k$ , and the penalties,  $\rho_k$ , related to the use of slack variables,  $s_k$

$$\phi_k = \sum_{j=0}^{N-1} c'_k u_k + \sum_{j=0}^{N-1} \rho'_k s_k \quad (4)$$

## III. OPERATION

Operating a power system means making real-time decisions as planning the power production in response to the customers demand. In this section we introduce the Economic MPC strategy to operate a power system as in Figure 1. Such optimization based controller balances power supply and demand in the most economic way. The control strategy includes a Kalman filter and a Linear Economic MPC regulator.

### A. Kalman Filter and Certainty Equivalence

Predictions are necessary to compute the optimal trajectory in our control problem. For this purpose a Kalman filter [18]–[21] estimates the state  $\hat{x}_{k+1+j|k}$  of a system like (1) [17]. Using the certainty equivalence principle, all previous variables are replaced by their conditional mean values [17].

### B. Linear Economic Model Predictive Control (MPC)

A power system consists of independent and dynamically decoupled power units, as described in Section II. Consider a power system where a set,  $\mathcal{P} = \{1, \dots, P\}$ , of power units jointly create the total portfolio power production  $\hat{z}_{k+j+1|k}$  subject to the following connecting constraints

$$\hat{z}_{k+j+1|k} = \sum_{i=1}^P \tilde{C}_i \hat{x}_{i,k+j+1|k} \quad (5a)$$

$$\hat{z}_{k+j+1|k} + \tilde{s}_{k+j+1|k} \geq \hat{r}_{min,k+j+1|k} \quad (5b)$$

$$\hat{z}_{k+j+1|k} - \tilde{s}_{k+j+1|k} \leq \hat{r}_{max,k+j+1|k} \quad (5c)$$

$$\tilde{s}_{k+j+1|k} \geq 0 \quad (5d)$$

As described in (3), as well the power system production should be within an interval, extended by adding the slack variable  $\tilde{s}$ .

The power units in a power system have linear dynamics, costs functions are linear, constraints are linear; as a result, the Economic MPC is formulated as a linear program. Given the mean value of the forecasts in Section II and in Appendix VIII, the predicted state  $\hat{x}_{k+1+j|k}$  from the Kalman filter and

the previous input  $u_{k-1}$ , the Linear Economic MPC for a power system of  $P$  power units, is defined as

$$\min \phi_k = \sum_{i=1}^P \phi_{i,k} + \sum_{j=0}^{N-1} \hat{\rho}'_{k+j+1|k} \tilde{s}_{k+j+1|k} \quad (6)$$

subject to the local constraints  $\forall i \in \mathcal{P}$  and  $\forall j \in \mathcal{N}$

$$\hat{x}_{i,k+j+1|k} = A_i \hat{x}_{i,k+j|k} + B_i u_{i,k+j|k} + E_i \hat{d}_{k+j|k} \quad (7a)$$

$$\hat{z}_{i,k+j+1|k} = C_{z,i} \hat{x}_{i,k+j+1|k} \quad (7b)$$

$$u_{\min,i} \leq u_{i,k+j|k} \leq u_{\max,i} \quad (7c)$$

$$\Delta u_{\min,i} \leq \Delta u_{i,k+j|k} \leq \Delta u_{\max,i} \quad (7d)$$

$$\hat{z}_{i,k+j+1|k} + s_{i,k+j+1|k} \geq \hat{r}_{\min,i,k+j+1|k} \quad (7e)$$

$$\hat{z}_{i,k+j+1|k} - s_{i,k+j+1|k} \leq \hat{r}_{\max,i,k+j+1|k} \quad (7f)$$

$$s_{i,k+j+1|k} \geq 0 \quad (7g)$$

and subject to the connecting constraints  $\forall j \in \mathcal{N}$  in (5).

The linear program (5)-(7) has a block-angular structure that may be used for its efficient solution using the Dantzig-Wolfe decomposition algorithm.

#### IV. DANTZIG-WOLFE DECOMPOSITION

The Dantzig-Wolfe decomposition algorithm is an efficient computational solution of linear programs having a block-angular structure, as (5)-(7), [7], [8]. This decomposition technique is a distributed optimization method for solution of the centralized Economic MPC controller. In this section we describe the Dantzig-Wolfe decomposition and we illustrate the related algorithm.

##### A. Block-angular structure

Our control problem (8) obtained from (5)-(7) is, for  $i \in \mathcal{M}$ ,

$$\min_{\{q_{i,k}\}_{i=1}^M} \varphi = \sum_{i=1}^M e'_i q_{i,k} \quad (8a)$$

$$s.t. \quad \begin{bmatrix} F_1 & F_2 & \dots & F_M \\ G_1 & & & \\ & G_2 & & \\ & & \ddots & \\ & & & G_M \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_M \end{bmatrix} \geq \begin{bmatrix} g \\ h_1 \\ h_2 \\ \vdots \\ h_M \end{bmatrix} \quad (8b)$$

The Dantzig-Wolfe breaks the linear (8) into  $M$  independent subproblems and a Master Problem (MP), that coordinates these subproblems.

From here on we assume that the feasible region of each subproblems is closed and bounded.

##### B. Master Problem

The *convex combination* theorem is the basis for the Dantzig-Wolfe decomposition [22].

[Convex Combination] Consider  $\mathcal{Q} = \{q \mid Gq \geq h\}$  be nonempty, bounded and closed set, i.e. a polytope.  $v^j$  denotes the extreme point of  $\mathcal{Q}$  with  $j \in \{1, 2, \dots, V\}$ .

Then any point  $q$  in the polytope  $\mathcal{Q}$  can be written as a convex combination of its extreme points

$$q = \sum_{j=1}^V \lambda_j v^j \quad (9a)$$

$$s.t. \lambda_j \geq 0, \quad j = 1, 2, \dots, V \quad (9b)$$

$$\sum_{j=1}^V \lambda_j = 1 \quad (9c)$$

See [22]. Substituting (9) into (8) yields to the following linear program

$$\min_{\lambda} \varphi = \sum_{i=1}^M \sum_{j=1}^{V_i} f_{ij} \lambda_{ij} \quad (10a)$$

$$s.t. \quad \sum_{i=1}^M \sum_{j=1}^{V_i} p_{ij} \lambda_{ij} \geq g \quad (10b)$$

$$\sum_{j=1}^{V_i} \lambda_{ij} = 1, \quad i = 1, 2, \dots, M \quad (10c)$$

$$\lambda_{ij} \geq 0, \quad i = 1, 2, \dots, M; j = 1, 2, \dots, V_i \quad (10d)$$

where the coefficients are

$$f_{ij} = e'_i v_i^j, \quad p_{ij} = F_i v_i^j \quad (11)$$

The linear program (10), known as *Master Problem (MP)*, is equivalent to the block-angular linear problem (8). It is worth noting that (10) has fewer constraints than the original problem (8).

However the MP considers the extreme points of each subproblem, thus the number of variables is larger than in the original problem (8). The Dantzig-Wolfe decomposition algorithm overcomes this problem by including a reduced number of extreme points, and adding new vertices when needed. As a result, the *Reduced Master Problem (RMP)* is defined as

$$\min_{\lambda} \varphi = \sum_{i=1}^M \sum_{j=1}^l f_{ij} \lambda_{ij} \quad (12a)$$

$$s.t. \quad \sum_{i=1}^M \sum_{j=1}^l p_{ij} \lambda_{ij} \geq g \quad (12b)$$

$$\sum_{j=1}^l \lambda_{ij} = 1, \quad i = 1, 2, \dots, M \quad (12c)$$

$$\lambda_{ij} \geq 0, \quad i = 1, 2, \dots, M; j = 1, 2, \dots, l \quad (12d)$$

where  $l \leq V_i$  for all  $i \in \{1, 2, \dots, M\}$ .

Solving the RMP provides the Lagrangian multipliers  $\pi$  associated with the inequality constraint (12b), the Lagrangian multipliers  $\rho$ , associated with equalities (12c), and the Lagrange multipliers  $\kappa$  for the positivity constraints (12d). These are playing a key role in the Dantzig-Wolfe algorithm as they represent the information sent from the Master Problem to each subproblem.

### C. Optimality Conditions

The Lagrangian associated to the Master Problem (10) yields to the following necessary and sufficient optimality conditions, for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, V_i$

$$\nabla_{\lambda_{ij}} \mathcal{L} = f_{ij} - p'_{ij} \pi - \rho_i - \kappa_{ij} = 0 \quad (13a)$$

$$\sum_{i=1}^M \sum_{j=1}^{V_i} p_{ij} \lambda_{ij} - g \geq 0 \quad \perp \quad \pi \geq 0 \quad (13b)$$

$$\sum_{j=1}^{V_i} \lambda_{ij} - 1 = 0 \quad (13c)$$

$$\lambda_{ij} \geq 0 \quad \perp \quad \kappa_{ij} \geq 0 \quad (13d)$$

We notice that the conditions (13a) and (13d) imply

$$\kappa_{ij} = f_{ij} - p'_{ij} \pi - \rho_i = [e_i - F'_i \pi]' v_i^j - \rho_i \geq 0 \quad (14)$$

such that the KKT-conditions for (10) may be stated as for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, V_i$

$$\sum_{i=1}^M \sum_{j=1}^{V_i} p_{ij} \lambda_{ij} - g \geq 0 \quad \perp \quad \pi \geq 0 \quad (15a)$$

$$\sum_{j=1}^{V_i} \lambda_{ij} - 1 = 0 \quad (15b)$$

$$\lambda_{ij} \geq 0 \quad \perp \quad \kappa_{ij} = [e_i - F'_i \pi]' v_i^j - \rho_i \geq 0 \quad (15c)$$

### D. Subproblems

An optimal solution must satisfy the KKT conditions (15). We denote  $\lambda_{ij}^{RMP}$  a solution of RMP, such that a feasible solution to Master Problem (10) is

$$\lambda_{ij} = \lambda_{ij}^{RMP} \quad i = 1, 2, \dots, M; j = 1, 2, \dots, l \quad (16a)$$

$$\lambda_{ij} = 0 \quad i = 1, 2, \dots, M; j = l+1, l+2, \dots, V_i \quad (16b)$$

This solution satisfies (15a) and (15b). To be optimal it also needs to satisfy (15c). These conditions are already satisfied for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, l$ . We need to verify whether they are satisfied for all  $i = 1, 2, \dots, M$  and  $j = l+1, l+2, \dots, V_i$ . This is complicated by the fact that we only know the extreme points,  $v_i^j$  for  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, l$ . Section IV-E describes how to compute efficiently an optimal initial solution for the Dantzig-Wolfe algorithm.

Condition (15c) is satisfied for all  $i = 1, 2, \dots, M$  and  $j = 1, 2, \dots, V_i$  if  $\min_i \psi_i - \rho_i \geq 0$  where

$$\psi_i = \min_{v_i^j} [e_i - F'_i \pi]' v_i^j \quad (17)$$

$v_i^j$  is an extreme point of the polytope  $\mathcal{Q}_i = \{q_i \mid G_i q_i \geq h_i\}$ . Therefore, using the Simplex Algorithm we compute the solution of (17) as a solution of the following linear program

$$\psi_i = \min_{q_i} \varphi = [e_i - F'_i \pi]' q_i \quad (18a)$$

$$s.t \quad G_i q_i \geq h_i \quad (18b)$$

These linear programs are called *subproblems* and can be solved via either parallel or sequential computation; this

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### Algorithm 1 The Dantzig-Wolfe algorithm for block-angular LP

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Compute the coefficients (11) for the RMP using the initial feasible vertex available, as described in Section (V).

$l=1$ . Converged=false

**while** Not Converged **do**

Solve the  $l$ -th RMP (12).

Solve all the subproblems ( $i = 1, 2, \dots, M$ ) (18) considering the  $\pi$  from (12b) and  $\rho_i$  from (12c).

**if**  $\psi_i - \rho_i \geq 0 \quad \forall i \in \{1, 2, \dots, M\}$  **then**

Converged = true

The optimal solution is given by (20).

**else**

Compute the coefficients for the new columns in the RMP

$$f_{i,l+1} = e'_i v_i^{l+1}$$

$$p_{i,l+1} = F_i v_i^{l+1}$$

$$l = l + 1$$

**end if**

**end while**

---

possible parallel computation of the subproblems represents one of the advantages of the Dantzig-Wolfe decomposition algorithm.

### E. Optimal Solution

Let  $(\psi_i, q_i)$  be the optimal value-minimizer pair for the linear problem (18); then if

$$\psi_i - \rho_i \geq 0 \quad \forall i \in \{1, 2, \dots, M\} \quad (19)$$

is satisfied, then the solution computed from the RMP is optimal. Therefore the solution of the original control problem (8) is given by

$$q_i^* = \sum_{j=1}^l v_i^j \lambda_{ij} \quad i \in \{1, 2, \dots, M\} \quad (20)$$

Otherwise, if (19) is not satisfied, then the number of extreme points considered,  $l$ , is not enough and a new vertex  $v_i^{l+1}$  needs to be included.

The Dantzig-Wolfe decomposition can be summarized in Algorithm 1. Let us assume to have an initial feasible vertex of the Master Problem (10); Section V is about finding such initial solution.

## V. INITIALIZATION TECHNIQUE

The Dantzig-Wolfe algorithm requires an initial feasible vertex. In this section we describe how such a vertex can be computed for the Economic MPC considered in this paper.

In general, the computation of an initial feasible vertex requires the solution of an LP that has the same complexity as the original LP. However, in the Economic MPC (5)-(7), the initial feasible vertex can be computed by a significant simpler method. At the beginning, the control sequence  $u_k$  and the power produced  $z_k$  are equal to 0. The connecting

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**Algorithm 2** Initialization technique for a Dantzig-Wolfe algorithm in an Economic MPC for LP

---

Obtain the  $u_{i,k+j|k}$  as a solution of the control problem (5)-(7) via Dantzig-Wolfe algorithm.

Estimate the system outputs,  $\hat{z}_{i,k+j+1|k}$  for the power generators and  $\hat{z}_{k+j+1|k}$  for the whole power system via Kalman filter.

Compute the initial feasible vertexes for the next iteration as:

$$s_{i,k+j+1|k,\min} = \max(0, \hat{r}_{i,k+j+1|k,\min} - \hat{z}_{i,k+j+1|k}) \quad (21a)$$

$$s_{i,k+j+1|k,\max} = \max(0, \hat{z}_{i,k+j+1|k} - \hat{r}_{i,k+j+1|k,\max}) \quad (21b)$$

$$s_{i,k+j+1|k} = \max(s_{i,k+j+1|k,\min}, s_{i,k+j+1|k,\max}) \quad (21c)$$

and for the global constraints

$$\tilde{s}_{k+j+1|k,\min} = \max\{\hat{r}_{k+j+1|k,\min} - \hat{z}_{k+j+1|k}, 0\} \quad (22a)$$

$$\tilde{s}_{k+j+1|k,\max} = \max\{\hat{z}_{k+j+1|k} - \hat{r}_{k+j+1|k,\max}, 0\} \quad (22b)$$

$$\tilde{s}_{k+j+1|k} = \max(\tilde{s}_{k+j+1|k,\min}, \tilde{s}_{k+j+1|k,\max}) \quad (22c)$$


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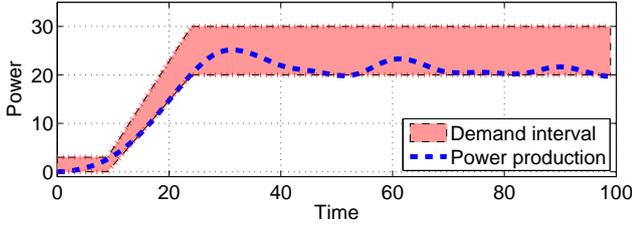


Fig. 2. Closed-loop simulation results. The total power production of the power system is kept within the power demand interval.

constraints (5) and the local constraints (7) compute the initial values of the slack variables as described in Algorithm 2. When the Economic MPC computes the optimal solution via Dantzig-Wolfe algorithm, this value is used in the Kalman filter to estimate  $\hat{x}_{k+1+j|k}$ , in the Economic MPC as previous input, and to obtain the systems outputs  $\hat{z}_{i,k+j+1|k}$ . Such value is then used to initialize the slack variables as described in Algorithm 2.

## VI. RESULTS

In this section, we apply the Dantzig-Wolfe algorithm for an Economic MPC to a system with a large number of power plants. Each power plant is modeled as a third order as described by [23].

### A. Closed-Loop simulations

For closed-loop simulations, the scenario includes five power generators and the time horizon is set to  $N = 50$ . The controller developed coordinates these power units, such that the total power production is within the demand interval. This is shown in Figure 2.

The Economic MPC provides the optimal trajectory by solving the control problem via Dantzig-Wolfe decomposition algorithm. Algorithm 1 shows that  $l$  vertexes of the polytope are necessary to find the optimal solution. Therefore, at

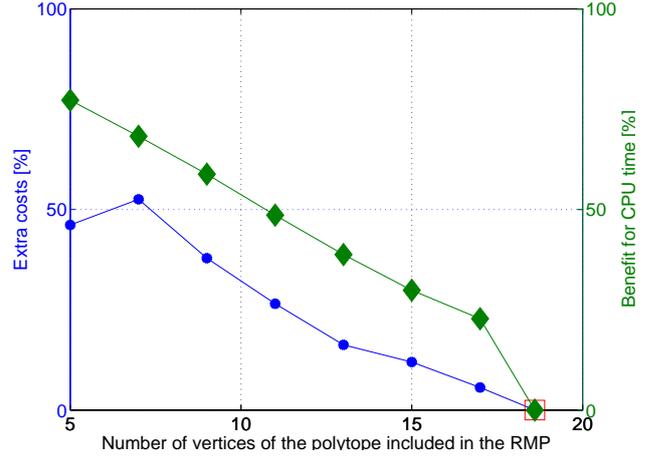


Fig. 3. Extra costs vs. Computational time. The Dantzig-Wolfe algorithm starts by including  $l = 1$  extreme point of the polytope, as described in Algorithm 1. A bound can be set on the maximum number of vertexes,  $l$ , considered in the RMP. Because of this, the computational time is significantly reduced. However such fixed number of extreme points might not be enough to determine the solution in the algorithm. Consequently extra costs must be paid as penalties due to the use of slack variables.

each sampling time,  $l$  extreme points are considered in the RMP. Algorithm 1 shows that including  $l$  vertexes means solving  $l$  RMP, therefore,  $l$  iterations. In this simulation, the controller needs, in average, 18.61  $l$  extreme points to solve the control linear program. An upper bound to  $l$  can be set; as a result for the polytope only a fixed number of vertexes are considered to compute the solution. However, if  $l$  extreme points are not enough then the slack variables assure a feasible solution by paying penalty. A bound on the number of vertexes in the RMP reduce the number of iterations at each sampling time, therefore, the computational time is reduced. Figure 3 reports such relationship. The percentage of extra costs and computational time are computed as

$$\frac{\phi_l - \phi^*}{\phi^*} \cdot 100 \quad \frac{t_l - t^*}{t^*} \cdot 100 \quad (23)$$

where  $\phi_l$  and  $t_l$  denote costs and computational time when a bound of the number of vertexes  $l$  is set.  $\phi^*$  and  $t^*$  denote costs and computational time when the Dantzig-Wolfe algorithm computes the solution with no bounds on the number of vertexes. The blue plot on the left shows the extra costs to pay by setting a bound on the number of vertexes  $l$ . The green graph on the right reports how the computational time is decreased. Without any bound, the controller spends, in average, 18.61  $l$  iterations to compute the solution, as shown at the red square marker in Figure 3. Stopping the Dantzig-Wolfe algorithm, for instance, when  $l$  is equal to 13 leads to reduce the computational time of about 40%, paying about 20% more.

### B. Comparison with other LP solvers

Typical power balancing problems in smart grids involve a very large number of energy components. Consequently, the associated LP of the Economic MPC is a large-scale LP. In this section we compare the solution time of such

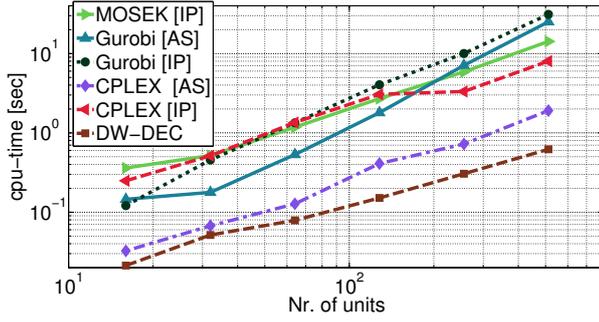


Fig. 4. CPU time for solving the linear Economic MPC controller via Dantzig-Wolfe distributed optimization algorithm, and a centralized Economic MPC solved via several LP solvers. The case-study is a large-scale scenario, with multiple units, i.e. controllable power plants. *DW-DEC* is the decentralized controller developed in this paper, where the LP solver is CPLEX. *[IP]* and *[AS]* denote interior-point and active-set approaches. CPLEX, Gurobi and Mosek are the solvers used to solve the centralized Economic MPC control problem.

LPs using the DW algorithm to the solution time of state of the art LP solvers. Figure 4 shows the results of this comparison. The Dantzig-Wolfe decomposition technique is significantly faster than all the other solvers and scales in a favorable way. Notice that both active-set methods and interior-point methods are included in our benchmark for CPLEX and Gurobi solvers.

## VII. CONCLUSION

In this paper we have proposed an optimization based controller to operate power systems. The linear Economic MPC of linear systems can be used to balance the power supply and demand in a smart grid.

We have presented a new efficient initialization technique for the Economic MPC problem solved via Dantzig-Wolfe decomposition algorithm.

An early termination has been demonstrated to reduce computational times but with extra costs.

The Economic MPC controller solved via Dantzig-Wolfe decomposition algorithm performs faster than the general state of the art LP solvers.

## VIII. APPENDIX

### A. Forecasts

In Section II we have defined a power system as a stochastic system including stochastic variables. These have been assumed to be available and used by the controller. In this subsection of the Appendix, we describe how these stochastic variables are defined.

We assume that the process noise,  $d_k$ , is predicted by some realization  $I_k^d$  of a stochastic information vector  $\mathcal{I}_k^d$ . As a result, the distribution

$$d_{k+j|k} = (d_{k+j} | \mathcal{I}_k^d = I_k^d) \sim N(\hat{d}_{k+j|k}, R_{dd,k+j|k}) \quad (24)$$

and the mean of these forecasts is

$$\mathcal{D}_k = \{\hat{d}_{k+j|k}\}_{j=0}^{N-1} \quad (25)$$

Similarly, reference bounds are defined as

$$r_{\min,k+j|k} = (r_{\min,k+j} | \mathcal{I}_k^r = I_k^r) \sim \mathbb{F}(\hat{r}_{\min,k+j|k}, R_{(r_{\min}, r_{\min}),k+j|k}) \quad (26a)$$

$$r_{\max,k+j|k} = (r_{\max,k+j} | \mathcal{I}_k^r = I_k^r) \sim \mathbb{F}(\hat{r}_{\max,k+j|k}, R_{(r_{\max}, r_{\max}),k+j|k}) \quad (26b)$$

such that the mean of the forecast,  $\mathcal{R}_k$ , is

$$\mathcal{R}_k = \{\hat{r}_{\min,k+j|k}, \hat{r}_{\max,k+j|k}\}_{j=1}^N \quad (27)$$

Production costs are  $c_k \sim \mathbb{F}(\bar{c}_k, R_{cc,k})$ , and  $\rho_k \sim \mathbb{F}(\bar{\rho}_k, R_{\rho\rho,k})$  as unit costs. The unit price forecasts are the conditional stochastic variables

$$b_{k+j|k} = (b_{k+j} | \mathcal{I}_k^p = I_k^p) \sim \mathbb{F}(\hat{b}_{k+j|k}, R_{bb,k+j|k}) \quad (28a)$$

$$c_{k+j|k} = (c_{k+j} | \mathcal{I}_k^p = I_k^p) \sim \mathbb{F}(\hat{c}_{k+j|k}, R_{cc,k+j|k}) \quad (28b)$$

$$\rho_{k+j|k} = (\rho_{k+j} | \mathcal{I}_k^p = I_k^p) \sim \mathbb{F}(\hat{\rho}_{k+j|k}, R_{\rho\rho,k+j|k}) \quad (28c)$$

and we denote the unit price forecast as

$$\mathcal{F}_k = \{\hat{b}_{k+j+1|k}, \hat{c}_{k+j|k}, \hat{\rho}_{k+j+1|k}\}_{j=0}^{N-1} \quad (29)$$

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