## Additional exercise – The Role of the Null Space

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**Exercise: the role of the null space of *A***

This exercise demonstrates that a non-trivial null space of A (i.e., a null space whose dimension is greater than zero) has an influence on the image that is able to reconstruct.

1. Generate a noise-free test problem by means of the function **paralleltomo**. Use the parameters **N = 64**, **theta = 3:3:90**, and **p = 35**. This corresponds to measurements where we record projections for angles 3, 6, 9, …, 90 degrees, and with 35 parallel rays for each angle.

2. Use **svd** or **rank** – or both – to compute the rank of the matrix A. You should see that the matrix has more rows than columns, and that it is rank deficient; hence it has a non-trivial null space.

3. The null space of A is spanned by the last $n-k$ right singular vectors, i.e., the last $n-k$ columns of the SVD-matrix $V$. If we partition $V$ as

$$V = [\, V_k , V_0 \,] , \qquad V_k = [\, v_1, v_2, \dots, v_k \,] , \qquad V_0 = [\, v_{k+1}, \dots, v_n \,] ,$$

then we can partition the exact solution $x$ into two orthogonal components – one in the null space and one orthogonal to it:

$$x = x_0 + x_0^{\perp} , \qquad x_0 \in \mathrm{null}(A) = \mathrm{range}(V_0) , \qquad x_0^{\perp} \in \mathrm{null}(A)^{\perp} = \mathrm{range}(A^T) = \mathrm{range}(V_k) .$$

In MATLAB these two components are computed as

$x_0^{\perp}$ = **V(:,1:k)*(V(:,1:k)'*x)** and $x_0 = x - x_0^{\perp}$ = **V(:,k+1:n)*(V(:,k+1:n)'*x)** .

Plot $x$, $x_0^{\perp}$ and $x_0$ as images, and discuss what you see. Try to relate the “artifacts” of these images to the angles of the rays that penetrate the domain.

Note that since all the algebraic iterative reconstruction methods work with multiplications with $A$ and $A^T$, we can only hope to compute approximations to the component $x_0^{\perp} \in \mathrm{range}(A^T)$. Any component in the null space of $A$ is unrecoverable!

4. Assuming that you want to keep using a spacing of 3 degrees between the angles of the projections; can you achieve a matrix $A$ with full rank if you are using enough projections?

5. Assuming that you want to keep the angles of the projections between 0 and 90 degrees; can you obtain a full-rank matrix if you choose enough projections in this range? Repeat the question for projection angles between 0 and 45 degrees?