Ray Transforms

For a scalar field f we define the x-ray transform as the integral over a lien through $\mathbf{x}$ in the direction of a unit vector ξ

$$Xf(\mathbf{x}, \xi) = \int_{-\infty}^{\infty} f(\mathbf{x} + s\xi)\,\mathrm{d}s$$

For a rank-2 symmetric tensor field there are some obvious choices. We can resolve the tensor field in the direction of ξ: we will write this as $\xi \cdot f\xi$ by considering f as a matrix. This is called the longitudinal ray transform

$$If(\mathbf{x}, \xi) = \int_{-\infty}^{\infty} \xi \cdot f(\mathbf{x} + s\xi)\xi\,\mathrm{d}s$$

The Transverse Transform

Another option is to resolve in the direction perpendicular to the ray. The matrix $P_\xi = \mathbf{I} - \xi\xi^T$ projects vectors in the plane normal to ξ. For the tensor field f the projection is $P_\xi f P_\xi$ and the integral of this gives the transverse ray transform

$$Jf(\mathbf{x}, \xi) = \int_{-\infty}^{\infty} P_\xi f(\mathbf{x} + s\xi) P_\xi \,\mathrm{d}s$$

Note that this integral results in a matrix for each ray

The Truncated Transverse Transform

In photoelastic tomography we will meet a more complicated beast. Let
$Q_\xi(f) = P_\xi f P_\xi - \frac{1}{2}\mathrm{Trace}(P_\xi f P_\xi) P_\xi$
which projects normal to ξ but also sets the trace to zero
For example if $\xi^T = (0, 0, 1)$ then

$$Q_\xi(f) = \frac{1}{2}\begin{pmatrix} f_{11} - f_{22} & 2f_{22} & 0 \\ 2f_{22} & f_{22} - f_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The Truncated Transverse Transforms (TTRT) is

$$Kf(\mathbf{x}, \xi) = \int_{-\infty}^{\infty} Q_\xi(f(\mathbf{x} + s\xi))\,\mathrm{d}s$$

a trace free matrix for each ray.

Calculus for symmetric tensors

The divergence of a symmetric tensor field f is a vector $g = \delta f$ where

$$g_i = \sum_{j=1}^{3} \frac{\partial f_{ij}}{\partial x_j}$$

For example if f is the stress tensor δf is the force per unit volume. The symmetric gradient of a vector field $dg = \nabla g + (\nabla g)^T$. For example if g is the deformation vector field dg is twice the infinitesimal strain. By analogy with vector calculus we say a rank-2 symmetric tensor field f is *potential* if $f = dg$ for some vector field g, and f is *solenoidal* if $\delta f = 0$. This is important for us as $If = 0$ if and only if f is potential. The TTRT K also has a null space consisting of scalar multiples of the identity (that is isotropic tensor fields)

Inverting ray transforms

Consider first the scalar x-ray transform (in three-dimensional space). There is an inversion formula of the filtered back projection type

$$f = \frac{1}{\pi}(-\Delta)^{1/2} BXf$$

where B is back projection over all rays through each point and $\Delta = \nabla^2$ is the Laplacian.

This is not usually very useful as one does not typically measure a complete (four dimensional) space of rays. For example in x-ray tomography using parallel beams one measures only the rays normal to some direction (the rotation axis). The importance of the inversion formula is to understand the unstable problem inversion on a limited but four-dimensional set of rays. For example tilting a rotation axis in two directions, but only rotating about that axis through some range less than $\pi/2$.

There are filtered back projection type formulas for J, K (up to an isotropic tensor field), and for I (up to a potential tensor field). The backprojection operator is the same, but the filter involves other differential operators.

TRT Slice by slice inversion

For the scalar case with data corresponding to using parallel projection and a single rotation axis, the problem reduces to the two dimensional case (the Radon transform). One can then perform the reconstruction using filtered backprojection on the data for each 'slice', that each plane through the object normal to the axis.

For the TRT the same trick works. For a unit vector $\mathbf{n}$, the rotation axis, and for $\xi \cdot \mathbf{n} = 0$

$$\mathbf{n}^T Jf(\mathbf{x}, \xi)\mathbf{n} = X(\mathbf{n}^T f \mathbf{n})(\mathbf{x}, \xi)$$

so we can use slice by slice scalar Radon transform inversion for $\mathbf{n}^T f \mathbf{n}$. Repeating this for six directions $\mathbf{n}$ with independent outer products we can solve for f.

Which six directions determine a tensor?

Suppose you have a symmetric matrix A and you want to find a_{ij} from a knowledge of $b_i = x_i^T A x_i$ for six different x_i. Which directions x_i will do?

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This the same as saying the directions x_i lie on a projective conic.

Pascal's theorem

Euclidean lines through the origin are projective points, Euclidean planes through the origin are projective lines (think of great circles on the sphere)

Theorem

Six distinct points x_i in the projective plane lie on a conic if and only if for any partition of the points in to three pairs x_{i_k}, x_{j_k}, $k = 1, 2, 3$ the lines through pairs meet at points that themselves lie on a projective line

Pascal's theorem

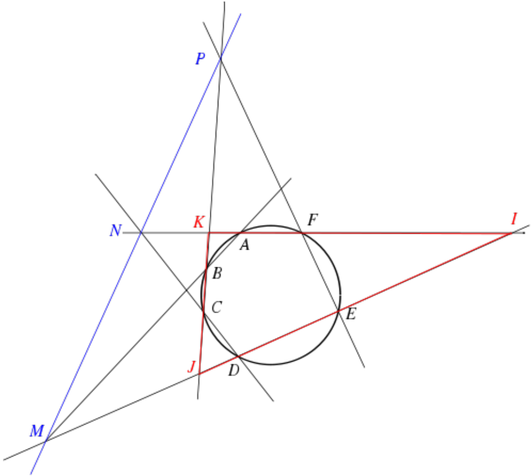


Figure: The classical Pascal theorem is explained using a hexagon joining the six points ABCDEF. The opposite sides of the hexagon are extended and meet at co-linear points PMN

TTRT Three axis inversion formula

L and Sharafutdinov derive an inversion formula for TTRT for the anisotropic part of a general tensor field.
Off diagonal component $K^1_{\mathbf{n}}(x,\xi) = (\xi \times \mathbf{n}) \cdot Kf(x,\xi)\mathbf{n}$
Diagonal component $K^2_{\mathbf{n}}(x,\xi) = \mathbf{n} \cdot Kf(x,\xi)\mathbf{n}$
Let $B_{\mathbf{n}}$ be the slice-by-slice backprojection normal to $\mathbf{n}$ and let y be the Fourier transform variable then we set

$$\lambda_{\mathbf{n}}(y) = \frac{i}{2}|P_{\mathbf{n}}y| \left(\widehat{B_{\mathbf{n}}\frac{\partial K^1_{\mathbf{n}} f}{\partial p}} \right)$$

$$\mu_{\mathbf{n}}(y) = |P_{\mathbf{n}}y|^3 (\widehat{B_{\mathbf{n}} K^2_{\mathbf{n}} f})$$

The system in Fourier space

Taking $\mathbf{n}$ to be the unit basis vectors we obtain the system

$$
\begin{aligned}
y_2\hat{f}_{12} + y_3\hat{f}_{13} &= \lambda_{\mathbf{e}_1}\\
y_1\hat{f}_{12} + y_3\hat{f}_{23} &= \lambda_{\mathbf{e}_2}\\
y_1\hat{f}_{13} + y_2\hat{f}_{23} &= \lambda_{\mathbf{e}_3}\\
(2y_2^2 + y_3^2)\hat{f}_{22} + 2y_2y_3\hat{f}_{23} + (y_2^2 + 2y_3^2)\hat{f}_{33} &= -\mu_{\mathbf{e}_1}\\
(2y_1^2 + y_3^2)\hat{f}_{11} + 2y_1y_3\hat{f}_{13} + (y_1^2 + 2y_3^2)\hat{f}_{33} &= -\mu_{\mathbf{e}_2}\\
(2y_1^2 + y_2^2)\hat{f}_{11} + 2y_1y_2\hat{f}_{12} + (y_1^2 + 2y_2^2)\hat{f}_{22} &= -\mu_{\mathbf{e}_3}\\
f_{11} + f_{22} + f_{33} &= 0
\end{aligned}
$$

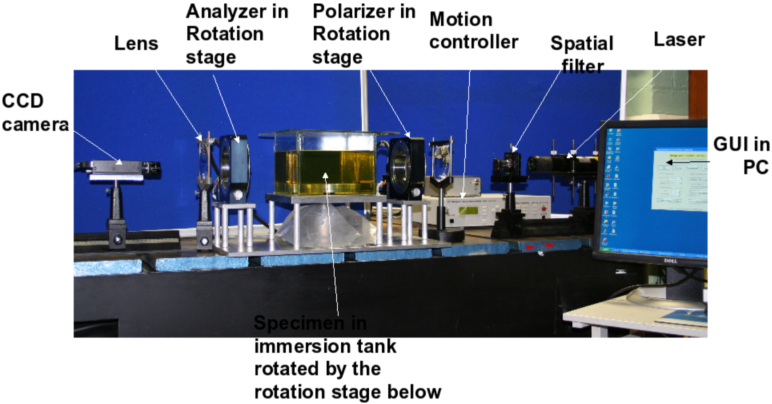
Which can be solved off the coordinate planes and works well numerically.

Application to photoelastic tomography

In photoelastic tomography f is the anisotropic part of the permittivity tensor assumed to be linearly related to the deviatoric (=anisotropic) stress.
The change in the polarization state of collimated monochromatic light is measured through the sample.
Provided the anisotropic part of the permittivity is small the measurements problem reduces to the TTRT. The ‘truncation’ is due the overall phase change in the light not being measured.

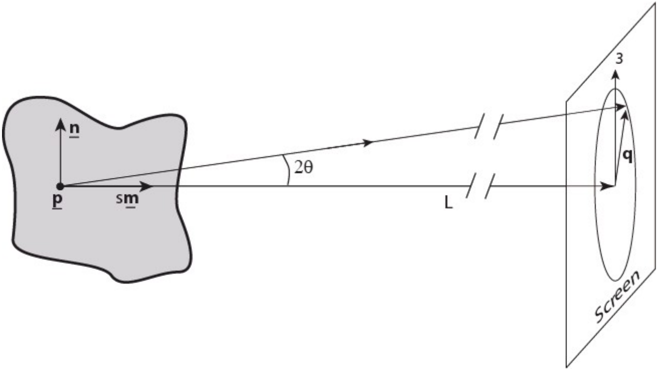
Photo of apparatus

Apparatus in Tomlinson's lab in Sheffield



X-ray diffraction strain tomography

Polycrytstaline materials such as metals illuminated with a beam of monochromatic x-rays produce Bragg diffraction rings on a distant screen. These are circular if the orientation of the crystals is uniformly distributed (ie isotropic on a large scale)



The effect of deformation

We now consider the effect of the deformation on the separation d of the planes normal to k in one crystal. Let $d'k' = DFdk$, where $DF = \mathbf{I} + \nabla U$ is the derivative of F, be the deformed separation vector we have

$$\begin{aligned} d'^2 &= d^2(1 + k^T(\nabla U^T + \nabla U)k + O(|\nabla U|^2)) && (1) \\ d' &= d\sqrt{1 + k^T 2\epsilon k + O(|\nabla U|^2)} && (2) \\ &= d(1 + k^T \epsilon k) + O(|\nabla U|^2) && (3) \end{aligned}$$

where $\epsilon = (\nabla U^T + \nabla U)/2$ is the infinitesimal strain tensor. Hence we have the relative change in d

$$\frac{d' - d}{d} = k^T \epsilon k + O(|\nabla U|^2). \quad (4)$$

By contrast $k' \cdot k = 1 + k^T \nabla U^T k + O(|\nabla U|^2)$ so to the leading order term the direction is unchanged. We see that the circles in the diffraction pattern of the unstrained case, Debye-Scherrer rings, are distorted by the strain.

Small angles

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k_i approx parallel to the screen.
This gives rise to a the deformed Debye-Scherrer ring being simply an ellipse defined by the points $y \in \xi^{\perp}$ such that

$$|y| = \frac{Ln\lambda}{(1 + k^T \epsilon k)} + O(\theta^2) \tag{5}$$

or with the small strain approximation.

$$|y| = Ln\lambda(1 - k^T \epsilon k) + O(\theta^2, |\nabla U|^2) \tag{6}$$

and using the small θ approximation in k we have approximatly

$$y^T(\mathbf{I} - P_\xi \epsilon)y = R_{i,n}^2 \tag{7}$$

where $R_{i,n} = nL\lambda/d_i$ is the radius of the unstrained Debye-Scherrer ring for the crystallographic plane k_i and order n.

Distributions of ellipses

This describes the situation for uniform strain. The diffraction pattern observed will be the superposition of (slightly blurred) ellipses, and any given point on the detector plane will include x-rays diffracted from points along the line with different strains that meet the Bragg condition.

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This describes the situation for uniform strain. The diffraction pattern observed will be the superposition of (slightly blurred) ellipses, and any given point on the detector plane will include x-rays diffracted from points along the line with different strains that meet the Bragg condition. An ellipse centred at the origin in the plane can in general be represented as a symmetric 2×2 matrix A with the points on the ellipse satisfying $y^T A y - 1 = 0$. In our case $A(s)$ is the matrix

$$\frac{d^2}{n\lambda^2 L^2} P_\xi \left(\mathbf{I} - \varepsilon(x + s\xi) \right) \tag{8}$$

in a suitable basis on $\xi^\perp$

Density of ellipses

In general there is some density of ellipses g with

$$\mathcal{I}'(\mathbf{y}) = \int_{\{A:\mathrm{y}^T A\mathrm{y}=1\}} g(A)\, da_{11} da_{22} da_{12}. \tag{9}$$

For simplicity let us consider now $\mathbf{y} = y_1\mathbf{e}_1 = (y_1, 0)^T$ so that (9) becomes

$$\mathcal{I}'(y_1\mathbf{e}_1) = \int_{\{A:a_{11}=1/y_1^2\}} g(A)\, da_{11} da_{22} da_{12} = \int g(\begin{pmatrix} 1/y_1^2 & a_{12} \\ a_{12} & a_{22} \end{pmatrix})\, da_{22} da_{12} \tag{10}$$

We now define the moment of $\mathcal{I}'$ in the direction of any y as

$$\mathcal{M}(\mathbf{y}) = \int_{r=0}^{\infty} r\mathcal{I}'(r^{-1/2}\mathbf{y})\, dr. \tag{11}$$

as the substitution $y_1 = r^{-1/2}$ in (10) gives $a_{11} = r$.

$$\mathcal{M}(\mathbf{e}_1) = \int_{r=0}^{\infty} r \int g\left(\begin{pmatrix} r & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\right) da_{22} da_{12}\, dr = \int \int \int a_{11} g(A)\, da_{22} da_{12}\, da_{11} \tag{12}$$

noting that as A is positive definite $a_{11} > 0$. Applying a rotation of coordinates we see that for a general unit vector $\mathbf{y}$

$$\mathcal{M}(\mathbf{y}) = \int \mathbf{y}^T \mathbf{A} \mathbf{y} g(\mathbf{A})\, dA \tag{13}$$

and so

$$\int a_{ii}\, ds = \mathcal{M}(\mathbf{e}_i)$$

for three axes we can get the line integral of of $\mathbf{A}(s)$.

Transverse Ray Transform for strain tomography

We see now that the moments of the diffraction pattern for this ray in three directions along the ray give the TRT $J\varepsilon$. For the reconstruction formula for the axial component of ε (eg ε_{33} rotation about the third axis) we need only the moment in that direction eg $\mathcal{M}(\mathbf{e}_3)$ to perform a slice-by slice reconstruction of that strain component.

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However this is a lot of measurement and but does not use all the Bragg ring data!

An inversion formula formula for TRT with all rays

For x-ray CT there is an inversion formula $f = (-\Delta)^{1/2} BXf$ where the backprojection B is over all lines through each point.
For the TRT we have

$$f = \frac{4}{\pi}\left(4\mathbf{I} - \mathbf{I}\mathrm{Tr} - 4\Delta^{-1} d\delta + \Delta^{-1} d^2 \mathrm{Tr} + \Delta^{-1}\mathbf{I}\delta^2\right)(-\Delta)^{1/2} BJf \quad (14)$$

Note that first we apply the inverse x-ray transform to each element, then we apply derivatives and a filter.

A linear associative algebra

Let $T = (1/3)\mathbf{I}\mathrm{Tr}, S = \Delta^{-1}d\delta, P = \Delta^{-1}d^2\mathrm{Tr}, Q = \Delta^{-1}\mathbf{I}\delta^2$ and $R = \Delta^{-2}d^2\delta^2$ then we have the multiplication table

	T	S	P	Q	R
T	T	$Q/3$	T	Q	$Q/3$
S	$P/3$	$(R+S)/2$	P	R	R
P	P	R	P	$3R$	R
Q	T	Q	$3T$	Q	Q
R	$P/3$	R	P	R	R

Hence the associative algebra generated by these elements and the identity is closed, and as the elements are linearly independent the algebra is six dimensional. T, P, Q and R are idempotent.
The inversion of J was performed by calculating an inverse in this algebra.

Bragg edge tomography does not work!

Neutrons also undergo Bragg diffraction from crystals. The idea is to use polychromatic neutrons and a distinct dip in the neutrons transmitted occurs when the Bragg angle $2\theta = \pi$ and the neutrons are backscattered.

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$$\int_{-\infty}^{\infty} \xi^T \varepsilon(x + t\xi)\xi \mathrm{d}t = I\varepsilon(x, \xi). \tag{15}$$

Here $\varepsilon = (1/2)dU$. For simplicity take $\xi = e_3$ and assume D is convex

$$\int_{-\infty}^{\infty} e_3^T \varepsilon(x + te_3)e_3 \mathrm{d}t = \int_{-\infty}^{\infty} \varepsilon_3(x + te_3)\mathrm{d}t \tag{16}$$

$$= \int_{-\infty}^{\infty} \partial U_3/\partial x_3 \mathrm{d}x_3 \tag{17}$$

$$= U_3(x_3^+) - U_3(x_3^-) \tag{18}$$

so it measures change in thickness

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