Micro Project 2A: AIR Methods for Incomplete-Data Tomography

Per Christian Hansen, DTU, Denmark – pcha@dtu.dk
Jakob Sauer Jørgensen, DTU, Denmark – jakj@dtu.dk

1 The Problem: Reduction of Reconstruction Artifacts

There are many examples of incomplete-data tomography problems, some of which we have met in the course so far, for example

- **Limited-angle tomography:** Projections are missing for a range of angles.
- **Few/sparse-view tomography:** Projections are available for the full angular range, but only a reduced number (with large angular increments).
- **The interior/region-of-interest problem:** Projections are truncated on both sides, i.e., the detector is not wide enough to cover the width of the object scanned.
- **The exterior problem:** An area within the object completely blocks/absorps any X-ray that intersects the area.

All incomplete-data tomography problems are subject to *reconstruction artifacts* that can look different depending on the problem. Note that, unlike the other problems, the interior problem typically has *inconsistent data*, since projections at any two different angles contain contributions from passing through different parts of the object outside the interior region.

The algebraic approach allows us to set up a reconstruction model that involves precisely the given data – without the need to create "artificial data" such as by padding as required in FBP. Software packages with implementations of algebraic iterative reconstruction (AIR) methods provide a number of different methods that solve such large-scale reconstruction problems. To fully specify an AIR method the user must choose a method, choose any constraints if applicable, set any method parameters, and choose an available stopping rule (the only stopping criterion currently available in AIR methods in CIL is a maximum number of iterations).

The goal of the micro-project is to investigate how well AIR methods can help suppress reconstruction artifacts in incomplete-data tomography problems.

2 The Micro Project

This micro project is quite open-ended with the overall goal to investigate the use of AIR methods for one or more incomplete-data tomography problems. It is interesting to study the potential *improvement* of reconstructions computed by AIR methods, compared to those produced by FBP, and (if time permits) to study if any of the proposed stopping rules work well for this problem.

It is up to you to choose which incomplete-data problem(s), iterative methods, stopping rules, and test problems (synthetic and/or real data) to use, as well as how to evaluate their performance. Here are some suggestions and ideas for what to explore:

- Start with experiments using a small artificial test problems where you know the ground truth, such that you can compare the reconstruction with the exact image and determine the optimal number of iterations.
- Various phantoms can be generated with AIR Tools II by means of `phantomgallery`. In CIL use the `TomoPhantom` plugin.
- Choose one or a few of the AIR methods available in AIR Tools II and CIL.
- Once you have familiarized yourself with the behavior of the AIR methods and the stopping rules, perform experiments with CIL on larger test problems.
- Consider whether employing constraints on pixel values may be useful.
- Explore how the exterior problem can be set up in CIL – maybe multiple `ProjectionOperator`s can be used?

3 Practical information and assessment

You will be working together in groups of 3–4 students. At the end of Friday, from around 2pm, each group will present their work to the lecturers and other groups in 10–15 min oral presentations, and each group member is expected to contribute, and to attend presentations by all groups. There is no written report – assessment is solely based on the oral presentation. In the presentation please explain what you have chosen to investigate, which theory/tools you have used, show your results (plots, reconstructions, etc.) and state your conclusions.

Appendix

Let the discretized model for the full problem be denoted $\boldsymbol{A}_{\mathrm{full}}\,\boldsymbol{x} = \boldsymbol{b}_{\mathrm{full}}$. Without loss of generality we can assume that those data in the sinogram that are unavailable are permuted to be bottom, and that we permute the rows of $\boldsymbol{A}_{\mathrm{full}}$ in the same order, to obtain

$$\boldsymbol{A}_{\mathrm{full}} = \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{A}_{\mathrm{un}} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b}_{\mathrm{full}} = \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{b}_{\mathrm{un}} \end{pmatrix}$$

where $\boldsymbol{b}_{\mathrm{un}}$ is unknown. This leads to two simultaneous systems

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b} \qquad \text{and} \qquad \boldsymbol{A}_{\mathrm{un}}\,\boldsymbol{x} = \boldsymbol{b}_{\mathrm{un}}$$

of which we can obviously only use the first one. This shows how to set up the discretized model for an incomplete-data problem, as some already did in the first micro project.

Notice that the right singular vectors of $\boldsymbol{A}_{\mathrm{full}}$ are different from those of $\boldsymbol{A}$, thus showing that the solutions will be different. If you do an SVD analysis, will it make a difference if you consider the matrix $\boldsymbol{A}$ or a matrix with $\boldsymbol{A}_{\mathrm{un}}$ replaced by zeros? The answer is that it does not matter, because

$$\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{U}\,\boldsymbol{\Sigma}\,\boldsymbol{V}^T \\ \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{U} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Sigma} \\ \boldsymbol{0} \end{pmatrix} \boldsymbol{V}^T ,$$

showing that the right singular vectors are identical.