What You Can See in Limited Data Tomography

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X-ray Tomography

X-ray Tomography (CT or CAT) is the mathematics, science, and engineering used to find internal information about an object using X-ray images.

Our Goals:

1. Learn what limited data tomography is.
2. Determine what features of the body will be easy to reconstruct from limited CT data, and which will be difficult.
3. Understand, geometrically, how this depends on the data.

Some History: The first CAT Scanner

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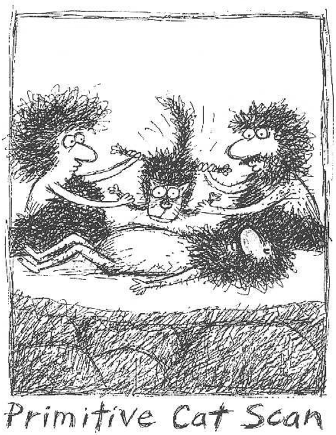

Modern GE scanner

Cost: DKK 12,000,000

GE Reconstruction

The Mathematical Model of X-ray CT

f a function in the plane representing the density of an object
L a line in the plane over which the photons travel.
The X-ray (Radon) Transform:

$$\boxed{\text{Tomographic Data} \sim \mathcal{R}f(L) = \int_{x \in L} f(x)dx}$$

–The 'amount' of material on the line the X-rays traverse.

The goal: Recover a picture of the body (values of $f(x)$), from X-ray CT data over a finite number of lines.

With *complete data* (lines throughout the object in fairly evenly spaced directions), good reconstruction methods exist, such as Filtered Backprojection (FBP) [Natterer, Natterer-Wübbling, Hansen-Jørgensen-Lionheart].

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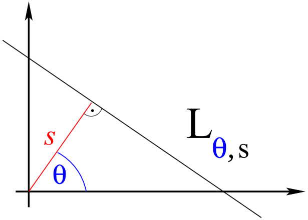
Parallel Beam Scanning Geometry

The angle: $\theta \in [0^\circ, 360^\circ]$ $\quad \overline{\theta} = (\cos(\theta), \sin(\theta))$

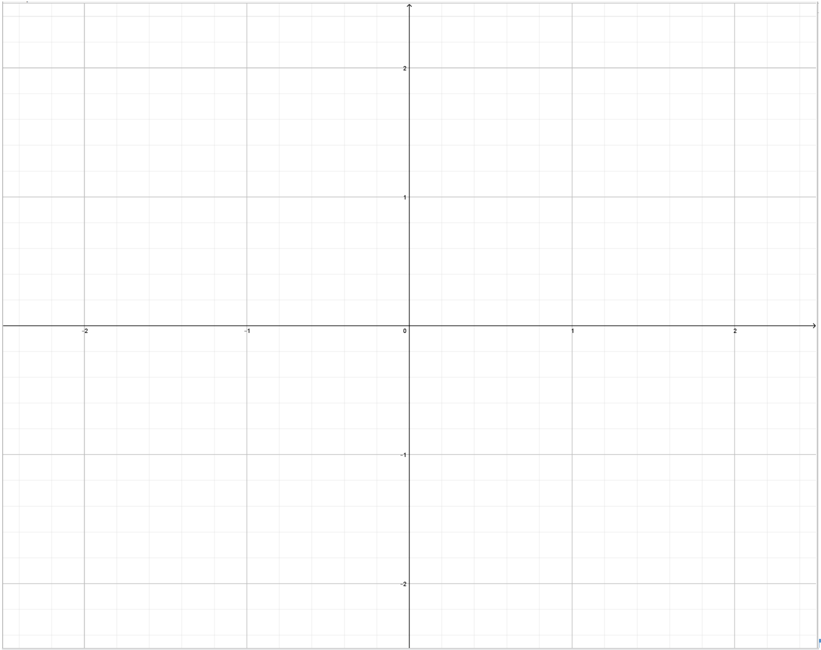
The lines over which X-rays travel: $L_{\theta,s}$ is the line perpendicular to θ and s units from the origin (in the opposite direction of θ if $s < 0$)

($\sim$fan beam but simpler mathematically)



Note $L_{\theta+180^\circ,-s} = L_{\theta,s}$ *(pic!)*

- Each line can be parameterized by a unique $(\theta, s) \in [0^\circ, 180^\circ] \times [-1, 1]$
  or redundantly by two $(\theta, s) \in [0^\circ, 360^\circ[\times [-1, 1]$.
- So sometimes we will have $\theta \in [0^\circ, 180^\circ]$ and sometimes in $[0^\circ, 360^\circ]$.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X-ray Tomographic Data

The object: f is the density function of an object in the plane–inside the unit disk (radius 1 centered at $(0,0)$).

Tomographic data: $\mathcal{R}f(\theta, s) = \int_{x \in L_{\theta,s}} f(x)dx$ is calculated using X-rays traveling along the line $L_{\theta,s}$ (by the Lambert-Beer law).

Complete Tomographic Data: *X-ray data are given over all lines going through the body (e.g., $L_{\theta,s}$ for $(\theta, s) \in [0^\circ, 180^\circ] \times [-1, 1]$). In practice: a finite number of evenly distributed lines.*

Limited Data Tomography: *When data over some lines are missing.*
The ***Data Domain*** *is the set of lines $L_{\theta,s}$ over which data are taken–equivalently, the set of (θ, s) parameterizing those lines.*

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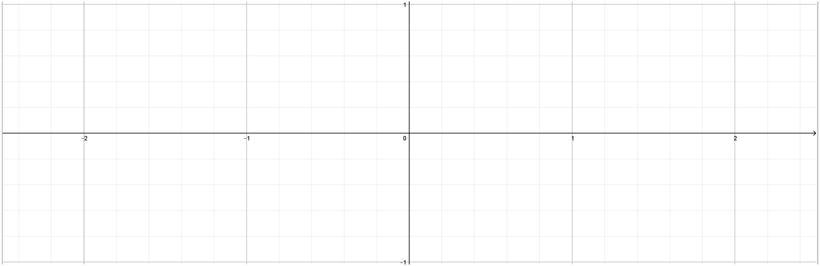
The missing data: data **not** on lines $L_{\theta,s}$ for $(\theta, s) \in S$.



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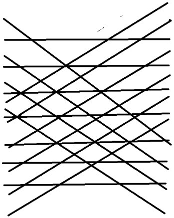
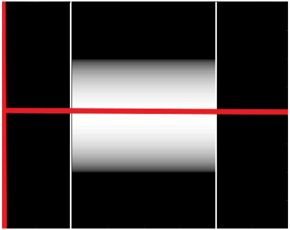
Example: The set of (θ, s) over which data are taken, the *data domain*, includes only horizontal-ish lines–$L_{\theta,s}$ with $(\theta, s) \in S = [45^\circ, 135^\circ] \times [-1, 1]$





Horizontal-*ish* lines are in the data domain, (shown in (θ, s) space on right).

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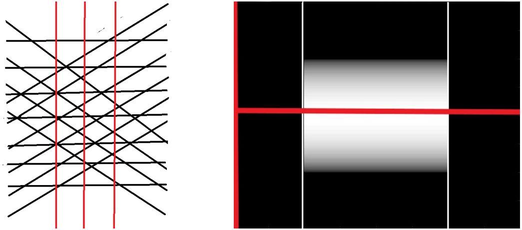
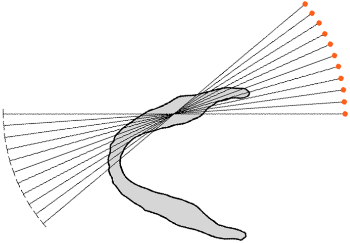


Vertical-*ish* lines are missing (data domain shown in (θ, s) space on right).

Limited Angle CT in Dental Imaging

Dental Scanner–head goes in “Π”

Jaw showing X-ray projection angles

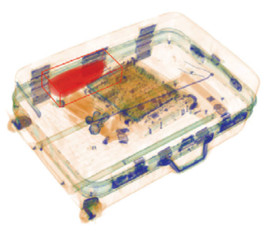


http://www.siltanen-research.net

Limited Angle CT in Luggage Testing

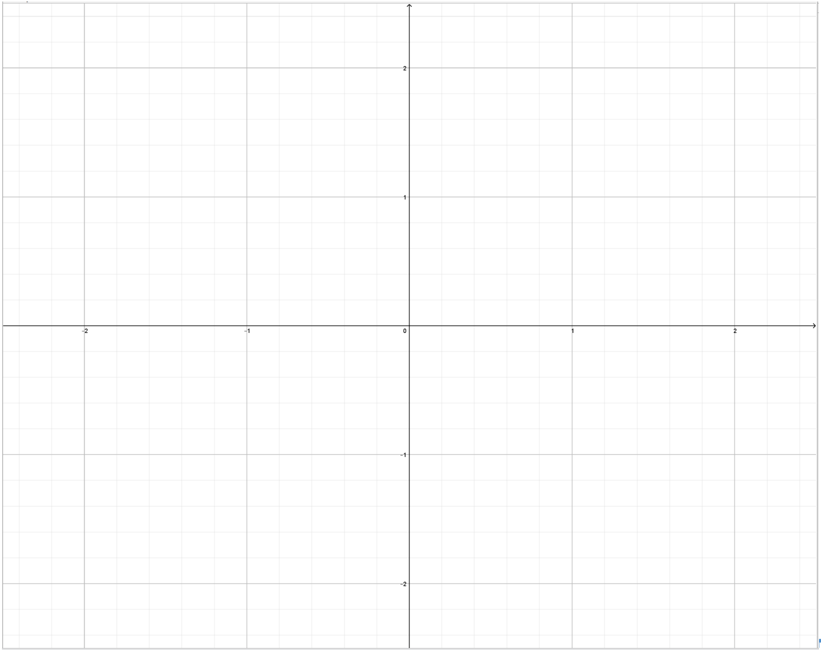
Luggage Scanner

Sample Luggage scan



Scanner moves above and below suitcase *(pic!)*

Analogic COBRA carry-on luggage scanner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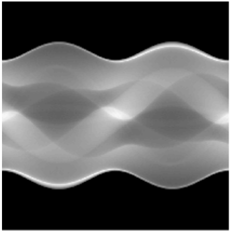
FBP and limited data

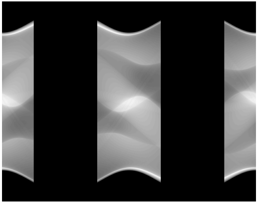
FBP for complete data: $f = \frac{1}{4\pi}\mathcal{R}^{\#}\Lambda\left(\mathcal{R}f\right)$

- $\mathcal{R}^{\#}$ is the backprojection operator, Λ is the filter, $\mathcal{R}f$ the data.
- $\mathcal{R}^{\#}$ (also Λ!) needs value of $\mathcal{R}f$ over *all* (θ, s)!
- For limited data, the data domain, S, is a proper subset of $[0^\circ, 360^\circ] \times [-1, 1]$–some data are missing.

Question: How should we modify FBP if we want to use FBP on limited data?



(a) Full data sinogram



(b) A limited angle sinogram

Limited data FBP with data domain S

Moral

To use FBP on limited data with data domain S, we set the data equal to zero off of S!

The Characteristic function of S: $\chi_S(\theta, s) = \begin{cases} 1 & (\theta, s) \in S \\ 0 & (\theta, s) \notin S \end{cases}.$

- for $(\theta, s) \in S$, then $\chi_S(\theta, s)\mathcal{R}f(\theta, s) =$???
- for $(\theta, s) \notin S$, then $\chi_S(\theta, s)\mathcal{R}f(\theta, s) =$???

We do FBP on “completed” data $\chi_S \mathcal{R}f$

$$\boxed{f \sim \frac{1}{4\pi}\mathcal{R}^{\#}\Lambda\left(\chi_S \mathcal{R}f\right)}$$

FBP with general data domain S.

$$\chi_S(\theta, s) = \begin{cases} 1 & (\theta, s) \in S \\ 0 & (\theta, s) \notin S \end{cases}.$$

$$f(x) \sim \frac{1}{4\pi}\mathcal{R}^{\#}\Lambda\left(\chi_S \mathcal{R} f\right) = \frac{1}{4\pi}\int_{0^\circ}^{360^\circ} \Lambda\left(\chi_S \mathcal{R} f\right)\left(\theta, x \cdot \overline{\theta}\right) d\theta.$$

Example: Limited Angle FBP: The data domain for limited angle CT is: $S = [a^\circ, b^\circ] \times [-1, 1]$.
So the limited angle FBP becomes ($b - a < 180^\circ$ and Λ doesn't depend on θ):

$$f \sim \frac{1}{2\pi}\int_{a^\circ}^{b^\circ} \Lambda\left(\mathcal{R} f\right)\left(\theta, x \cdot \overline{\theta}\right) d\theta.$$

- By integrating from a to b, we reconstruct using only data in the data domain, $S = [a^\circ, b^\circ] \times [-1, 1]$.

We do standard FBP on data that is set to zero off of S.

Features, Boundaries and CT

Our Goal: learn what object boundaries can be reconstructed from limited data.

In many tomography problems, the shapes/boundaries of features in the test object are diagnostically important:

- Cracks in industrial objects,
- Illegal stuff in carry-on luggage,
- Cavities in teeth,
- Blockages in blood vessels,
- Uneven boundaries in some tumors.

- So, knowing the boundaries of structures in the test object is important.
- We don't always need to know the exact density values of the object.
- Algorithms such as limited data FBP can be useful!

Which features of the body are sharpest in this X-ray image?

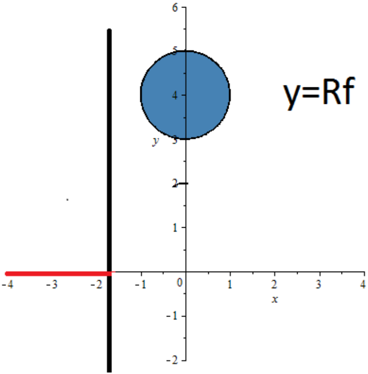
My Answer: The the edges/boundaries of the bones!

Which X-ray beams show edges (boundaries) *(pic!)***?**

Answer: The beams tangent to the edges (boundaries) of the bones!

Now see why mathematically.

CT data of a disk of radius 1 over vertical lines



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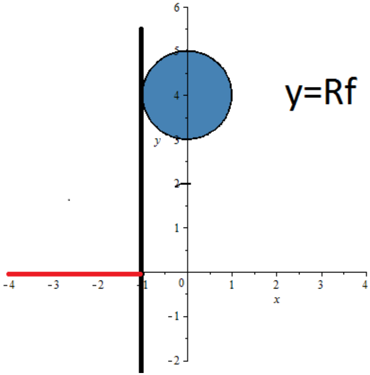

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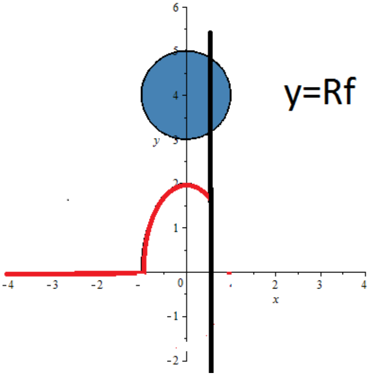

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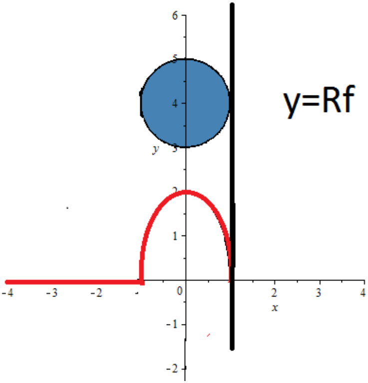

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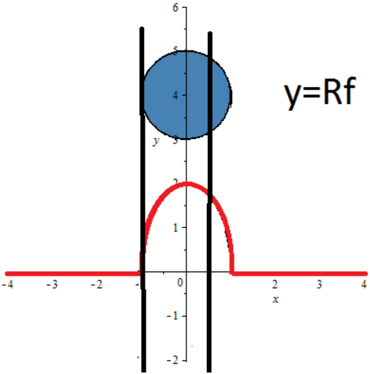

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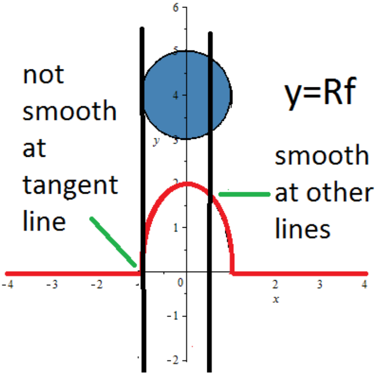

- The CT data has a “corner” (graph not smooth) at any line tangent to the boundary of the disk.

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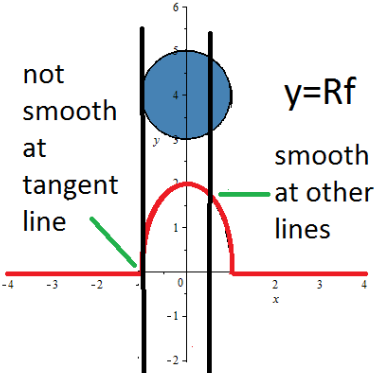

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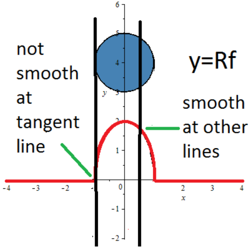
- So *that* boundary will be easy to see in the data.

In limited data CT, data over some lines are missing.



- If data is *not smooth* ⟶ easy to see the feature that caused it in this data.
- If data is *smooth* ⟶ features on that line are "washed out" in this data.

If a boundary in the object is tangent to a line in the data domain, then it will be easy to see in the data (like the chest X-ray!).

∴ easy to reconstruct from limited data.

If a boundary in the object is *not* tangent to *any* line in the data domain, then it will be had to see in the data.

∴ hard to reconstruct from limited data.

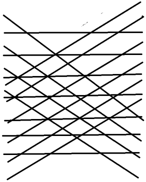
Moral

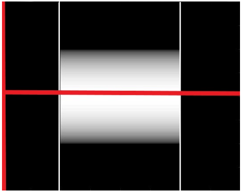
- *Boundary tangent to some line in the data domain ⟶ boundary easy to reconstruct.* *It is a visible boundary.*
- *Boundary tangent to no line in the data domain ⟶ boundary hard to reconstruct.* *It is an invisible boundary.*

Example

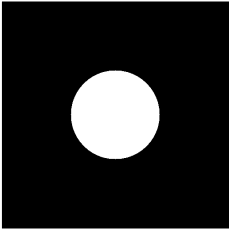
Limited angle CT data of a disk over lines $L_{\theta,s}$ with $(\theta, s) \in [45^\circ, 135^\circ] \times [-1, 1]$

The data





Data Domain



[Frikel, Q 2013] *Left:* disk,

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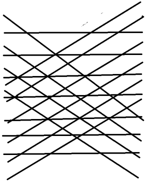
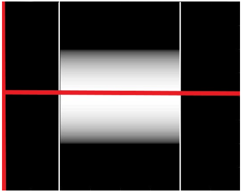
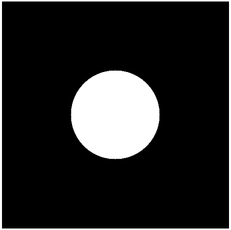
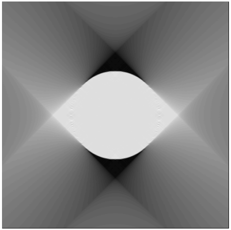
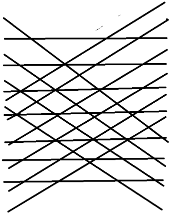
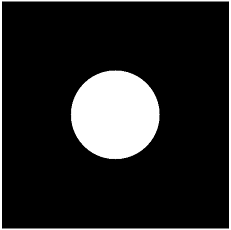
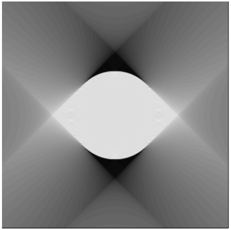
The data   **Data Domain**

[Frikel, Q 2013] *Left:* disk, *Right:* Limited angle FBP reconstruction

The data domain: all lines with $L_{\theta,s}$ with $(\theta, s) \in [45^\circ, 135^\circ] \times [-1, 1]$.

Which object boundaries are visible in the reconstruction?
Answer: what do you think?

We claimed that, if a line in the data domain is tangent to a boundary, that boundary will be easy to see in the reconstruction from that data.

Is that true in this picture? *Answer: what do you think?*

Moral

What would you formulate as a moral looking at the pictures??

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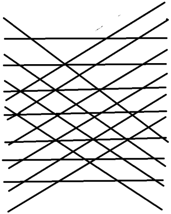

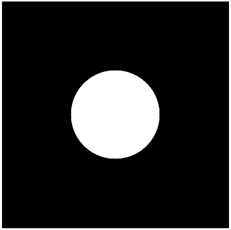

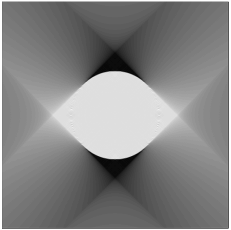

Which boundaries of the disk are not visible in the reconstruction?

Answer: what do you think?

We claimed that, if no line in the data domain is tangent to a boundary, that boundary will be hard to see in the reconstruction.

Is that true in this picture? *Answer: what do you think?*

Moral

A boundary will be difficult to see in the reconstruction from limited data if. . . what do you think?

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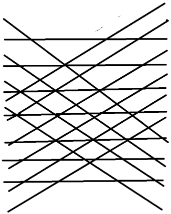
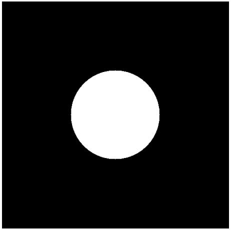
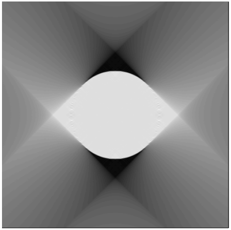






How do the streaks relate to the object?

Answer: ????

What are the value of θ for the streak lines?

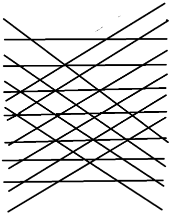
Answer: ????

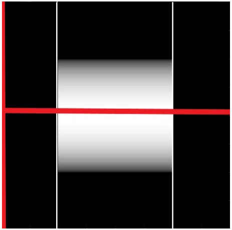
How do the artifact lines relate to the data domain?

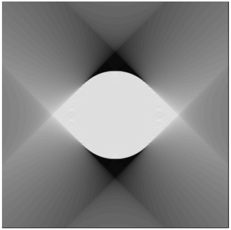
Moral

How would you summarise this?

The data domain: all lines with $L_{\theta,s}$ with
$(\theta, s) \in [45^\circ, 135^\circ] \times [-1, 1]$.







How do the streaks relate to the object?

Answer: ????

What are the value of θ for the streak lines?

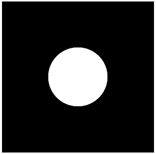
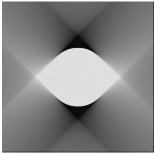
Answer: ????

How do the artifact lines relate to the data domain?

Moral

How would you summarise this?

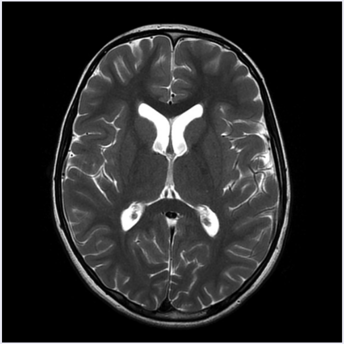
Moral for Limited Data Tomography

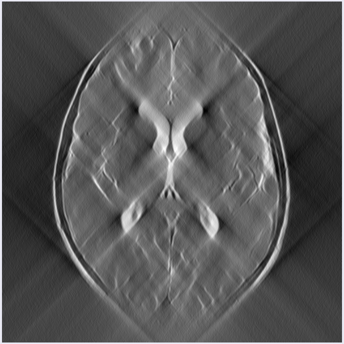
 

- A boundary of the object is (should be) *visible* in the reconstruction if:
  <u>You provide the answer.</u>
- A boundary of the object is (should be) *invisible* (not seen) in the reconstruction if:
  <u>You provide the answer.</u>
- A streak artifact can occur in the reconstruction on a line if:
  <u>You provide the answer.</u>

This is valid for general limited data problems by deep mathematics + a precise concept of singularity–microlocal analysis [3].

Exercise (In Class)





Brain phantom [radiopedia.org], FBP reconstruction [Frikel, Q 2013]

(a) *Which features of the brain are visible in the reconstruction?*

(b) *Which are invisible?*

(c) *Are there added streak artifacts?*

(d) *Use this information to determine the data domain for this reconstruction.*

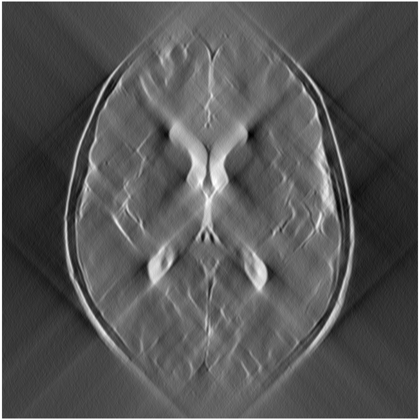
Artifact Reduction

$$\boxed{f(x) \sim \frac{1}{4\pi}\mathcal{R}^{\#}\Lambda\left(\chi_S \mathcal{R} f\right), \qquad \chi_S(\theta, s) = \begin{cases} 1 & (\theta, s) \in S \\ 0 & (\theta, s) \notin S \end{cases}.}$$

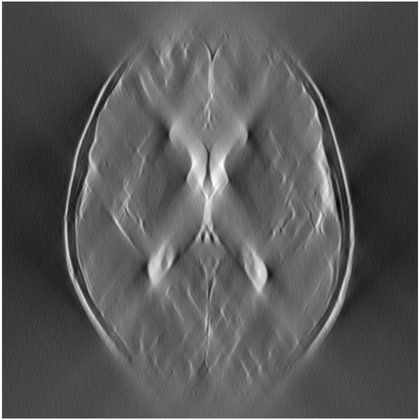
- By multiplying by χ_S, we restrict to data in the given data domain, S.
- *The streaks occur along lines at the ends of the data domain.*
- *The cause of streaks:* the sharp cutoff in χ_S at the ends of the data domain, S.
- *The solution:* Make a smooth, gradual cutoff instead of the sharp cutoff in χ_S.
- Replace χ_S by a smooth function $\psi(\theta, s)$ that is 1 on most of S (e.g., for *limited angle* equal to 1 on most of $[a, b]$ and smoothly goes to 0 at a, b).

Artifact reduced limited angle FBP: $\boxed{f \sim \frac{1}{4\pi}\mathcal{R}^{\#}\Lambda\left(\psi \mathcal{R} f\right)}$

Sample Reconstruction [Frikel Q 2013]



Reconstruction



Artifact reduced reconstruction

Region of Interest (ROI) or Interior Tomography

ROI Tomography: tomography using only lines that pass through a small part of the object to reconstruct that part of the object. This is often because the object is too large or we are interested in only imaging that small part of the object. *(pic!)*

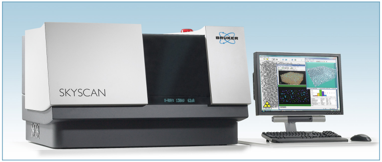


Figure: Skyscan Micro-CT Scanner

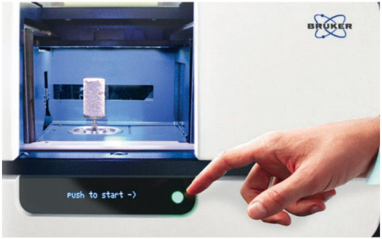


Figure: Object in scanner

ROI CT is used for nondestructive evaluation of parts of small objects.

The data domain for ROI CT: $S = [0^\circ, 360^\circ] \times [-r, r]$ where $r < 1$ is the radius of the ROI.

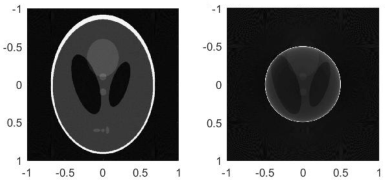


Figure: The Shepp Logan Phantom + ROI

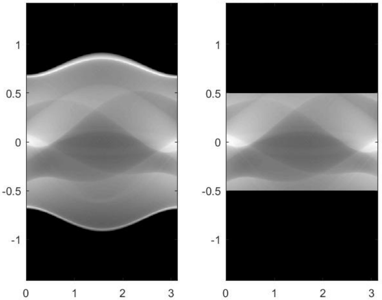


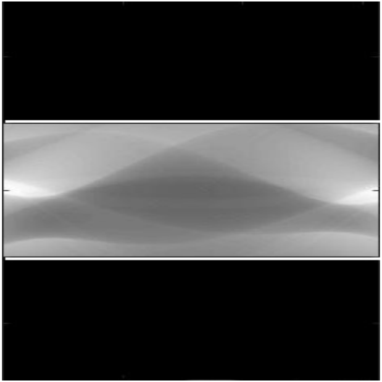
Figure: Complete Data Sinogram and ROI Sinogram

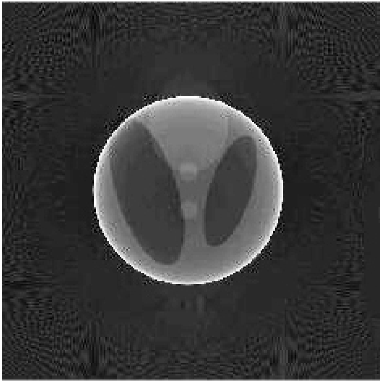
Exercise

Let's say you have a ROI data domain of an object.

1. *According to the theory, what object boundaries would be easy to reconstruct from the ROI data inside the ROI?*
2. *According to the theory, what types of object boundaries would be difficult to reconstruct from the ROI data inside the ROI?*
3. *According to the theory, what object boundaries would be easy to reconstruct from the ROI data outside the ROI?*
4. *According to the theory, what types of object boundaries would be difficult to reconstruct from the ROI data outside the ROI?*
5. *Did you observe this in the ROI reconstruction exercise using* `iRadon`*?*

Artifact Curves



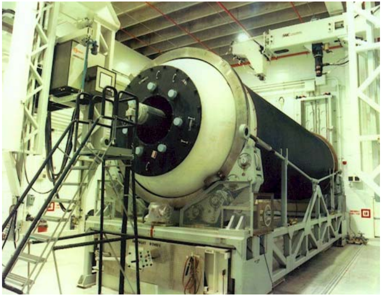


Moral

Artifacts can occur on curves generated from lines in the boundary of the data domain. We have a formula for them in [3]!

Exterior Tomography

Exterior Tomography: only rays through an outer annulus of object are measured, not the rays through its center.



Exterior CT is used for nondestructive evaluation (NDE) of rockets because industrial X-ray CT scanners can't penetrate the thick central part of the rocket, but they can penetrate the outside annulus. Often scientists are interested in cracks, etc., in the rocket shell, anyway.

Exterior Tomography

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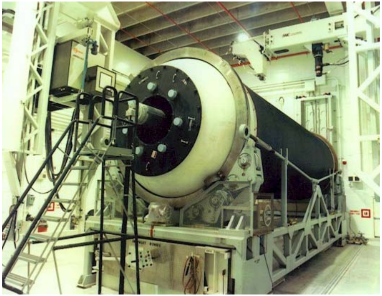
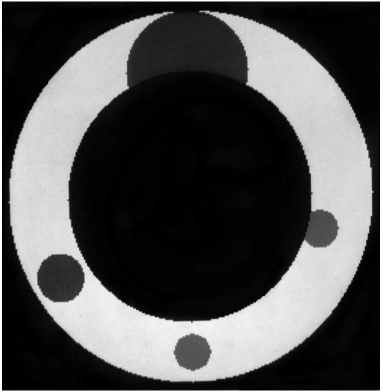
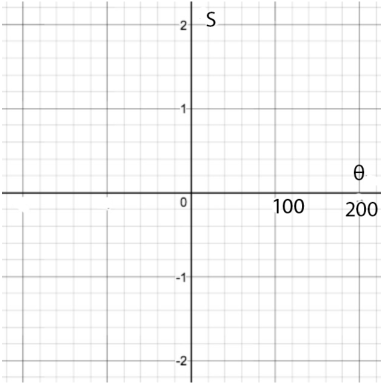


Exterior CT is used for nondestructive evaluation (NDE) of rockets because industrial X-ray CT scanners can't penetrate the thick central part of the rocket, but they can penetrate the outside annulus. Often scientists are interested in cracks, etc., in the rocket shell, anyway.

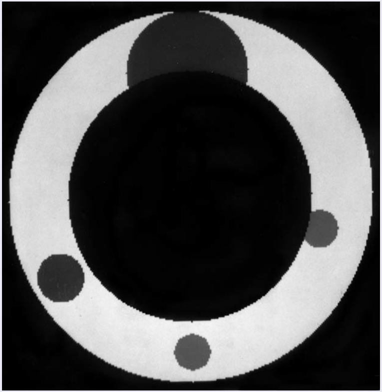
The data domain for exterior CT: If the central disk has radius $r < 1$, then $S = [0^\circ, 180^\circ] \times ([-1, -r] \cup [r, 1])$.





Exercise

Use what we've learned to answer the following questions about an exterior reconstruction of this phantom [Q1988]



(a) *What boundaries should be easy to see in an exterior reconstruction of the phantom?*

(b) *What boundaries should be difficult to see in an exterior reconstruction of the phantom?*

(c) ⚠ *Could there be artifact curves? (HINT: think about artifacts in ROI CT.)*

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Exercise

Defects in rocket shells are generally along the circumference direction of the shell.

1. ***Would exterior CT be a good modality for such defects?***
2. ***According to the theory, what types of defects would be easy to reconstruct from exterior CT?***
3. ***According to the theory, what types of defects would be difficult to reconstruct from exterior CT?***
4. ***Do you think there could be added artifacts in reconstructions from exterior data? Why or why not?***

Summary I

- **Visible boundary:** boundary with *some* line in the data domain tangent to it.
- **Invisible boundary:** boundary with *no* line in the data domain tangent to it.
- **Added Artifacts:** streaks on lines at the boundary of the data domain that are tangent to the object.
  - More subtle added artifacts can occur on lines or curves generated by lines at the boundary of the data domain.
- **Artifact reduction:** smooth the mask at the ends of the data domain. (ROI/Exterior simple extensions work well (limited angle????).)
- We make this characterization mathematically precise using the Fourier transform and microlocal analysis. See:

[3] Leise Borg, Jürgen Frikel, Jakob Sauer Jørgensen, and ETQ, Analyzing reconstruction artifacts from arbitrary incomplete X-ray CT data, SIAM J. Imaging Sci., 11(4)(2018), 2786–2814.

Perspective

- The analysis of visible and invisible singularities is intrinsic to limited data CT [2]–invisible singularities are harder for *any algorithm* to reconstruct than visible singularities.
- The artifact characterization we describe [3] applies to **backprojection algorithms** for many limited data tomography problems in X-ray CT, sonar, radar, seismics...
- Backprojection is useful in general and easy to program. However, it isn't perfect for limited data problems.
- **Other effective algorithms:**
  - Use a priori info about the object (general shape, . .) and iterative methods [PCH]
  - Develop and carefully implement inversion formulas or fill in data cleverly (e.g., [Louis, Katsevich]).
  - Use deep learning (e.g., [Bubba et al.]) and good training sets.

THANKS FOR YOUR ATTENTION!

For Further Reading I

References to the work in the talk:

[1] E.T. Quinto *Inverse Problems* **4**(1988), 867-876.

[2] E.T. Quinto, *SIAM J. Math. Anal.* **24**(1993), 1215-1225.

[3] Leise Borg, Jürgen Frikel, Jakob Sauer Jørgensen, and ETQ, Analyzing reconstruction artifacts from arbitrary incomplete X-ray CT data, SIAM J. Imaging Sci., 11(4)(2018), 2786–2814.

[4] *Computed Tomography: Algorithms, Insight and Just Enough Theory*, editors Per Christian Hansen, Jakob Sauer Jørgensen, William R. B. Lionheart, xvii + 337 pages, SIAM, 2021.
Chapter 8: Limited-Data Tomography, pages 125-154 chapter DOI: 10.1137/1.9781611976670.ch8.

For Further Reading II

[5] Characterization and reduction of artifacts in limited angle tomography, joint with Jürgen Frikel, Inverse Problems, 29 (2013) 125007 (21 pages). See also http://iopscience.iop.org/0266-5611/labtalk-article/55769

[6] Artifacts in incomplete data tomography with applications to photoacoustic tomography and sonar, joint with Jürgen Frikel, SIAM J. Appl. Math., 75(2),(2015) 703-725. (23 pages) Preprint on arXiv: http://arxiv.org/abs/1407.3453.

[7] Limited data problems for the generalized Radon transform in $\mathbb{R}^n$, joint with Jürgen Frikel, SIAM J. Math. Anal., 48(4)(2016), 2301-2318, Preprint on arXiv: http://arxiv.org/abs/1510.07151.

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General references:

[8] Frank Natterer, *The Mathematics of Computerized Tomography*, Wiley, New York, 1986 (SIAM 2001).

[9] Frank Natterer, Frank Wuebbling, *Mathematical Methods in Image Reconstruction*, SIAM, 2001.

For Further Reading IV

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- Peter Kuchment, The Radon transform and medical imaging. CBMS-NSF Regional Conference Series in Applied Mathematics, 85. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2014. xvi+240 pp.
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- E.T. Quinto, An Introduction to X-ray tomography and Radon Transforms, Proceedings of Symposia in Applied Mathematics, Vol. 63, 2006, pp. 1-24.

For Further Reading V

Limited Data, Local CT, and Lambda CT

- A. Faridani, E.L. Ritman, and K.T. Smith, *SIAM J. Appl. Math.* **52**(1992), 459–484,
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- A. Katsevich, Cone Beam Local Tomography, SIAM J. Appl. Math. 1999, Improved Cone Beam Local Tomography: Inverse Problems 2006.
- A. Louis and P. Maaß, *IEEE Trans. Medical Imaging*, 12(1993), 764-769.
- T.A. Bubba, et al., Learning the invisible: a hybrid deep learning-shearlet framework for limited angle computed tomography Inverse Problems 35(2019) 064002

For Further Reading VI

- N.A.B. Riis, J. Frøsig, Y. Dong, P.C. Hansen, Limited-data x-ray CT for underwater pipeline inspection. Inverse Problems 34(3)(2018), no. 3, 034002, 16 pp.
- S., Soltani, M.E. Kilmer, P.C. Hansen, A tensor-based dictionary learning approach to tomographic image reconstruction. BIT 56 (2016), no. 4, 1425-1454.

Microlocal references:

Intro + Microlocal: Microlocal Analysis in Tomography, joint with Venkateswaran Krishnan, chapter in Handbook of Mathematical Methods in Imaging, 2e, pp. 847-902, Editor Otmar Scherzer, Springer Verlag, New York, 2015 www.springer.com/978-1-4939-0789-2

Petersen, Bent E., Introduction to the Fourier transform & pseudodifferential operators. Monographs and Studies in Mathematics, 19. Pitman (Advanced Publishing Program), Boston, MA, 1983. xi+356 pp. ISBN: 0-273-08600-6

Strichartz, Robert, A guide to distribution theory and Fourier transforms. Reprint of the 1994 original [CRC, Boca Raton; MR1276724]. World Scientific Publishing Co., Inc., River Edge, NJ, 2003. x+226 pp. ISBN: 981-238-430-8

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- Taylor, Michael E. Pseudodifferential operators. Princeton Mathematical Series, 34. Princeton University Press, Princeton, N.J., 1981. xi+452 pp. ISBN: 0-691-08282-0