Regularization Techniques for Tomography Problems

Chapter 12.3

Yiqiu Dong

DTU Compute
Technical University of Denmark

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MAP estimate

The **MAP estimation** problem is

$$\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{b} - \boldsymbol{A}\,\boldsymbol{x}\|_2^2 + \alpha\, J(\boldsymbol{x}).$$

- The term $\frac{1}{2}\|\boldsymbol{b} - \boldsymbol{A}\,\boldsymbol{x}\|_2^2$ is called the *data-fidelity* term.
- The term $J(\boldsymbol{x})$ is called the *regularization* term.
- $\alpha > 0$ is the regularization parameter.
- *Tikhonov regularization*: $J(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}\|_2^2$.

Tikhonov regularization in general form

A general version of the Tikhonov problem is

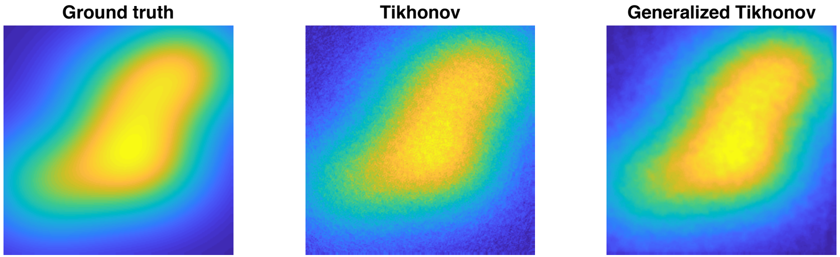
$$\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{b}-\boldsymbol{A}\boldsymbol{x}\|_2^2+\alpha\frac{1}{2}\|\boldsymbol{D}\boldsymbol{x}\|_2^2.$$

- $\boldsymbol{D}\in\mathbb{R}^{p\times n}$ is a discrete approximation of a derivative operator.
- If $\boldsymbol{D}=\boldsymbol{I}$ then we obtain the standard Tikhonov problem.
- If $\boldsymbol{D}$ approximates the first order derivative together with Neumann (symmetric) boundary condition, we have

$$\boldsymbol{D}_{M\times N}=\begin{pmatrix}\boldsymbol{I}_N\otimes\boldsymbol{D}_M\\ \boldsymbol{D}_N\otimes\boldsymbol{I}_M\end{pmatrix}\quad\text{and}\quad\boldsymbol{D}_P=\begin{pmatrix}-1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1\\ & & & 0\end{pmatrix},$$

with $\boldsymbol{D}_{M\times N}\in\mathbb{R}^{2n\times n}$, $\boldsymbol{D}_P\in\mathbb{R}^{P\times P}$ and $n=M\times N$.

Example: Comparison with Tikhonov



- The ground truth is smooth, so $\|\boldsymbol{D}\boldsymbol{x}\|_2^2$ is small, but $\|\boldsymbol{x}\|_2^2$ is not necessarily small.
- We set $\boldsymbol{D} = \boldsymbol{D}_{M \times N}$.

The solution of Tikhonov regularization in general form

Reformulate as a linear least squares problem

$$\min_{\boldsymbol{x}} \frac{1}{2} \left\| \begin{pmatrix} \boldsymbol{A} \\ \sqrt{\alpha}\boldsymbol{D} \end{pmatrix} \boldsymbol{x} - \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{pmatrix} \right\|_2^2 .$$

The normal equation is

$$(\boldsymbol{A}^T\boldsymbol{A} + \alpha\boldsymbol{D}^T\boldsymbol{D})\,\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b},$$

- The general-form Tikhonov solution $\boldsymbol{x}_{\mathrm{GTik}}$ is unique when

$$\mathrm{Null}(\boldsymbol{A}) \cap \mathrm{Null}(\boldsymbol{D}) = \{\boldsymbol{0}\} \ .$$

Exercises

12.4 **Tikhonov Solutions in General Form**

12.5 **Finite Different Approximation of the Gradient**

12.6 **Importance of the Choice of Regularization Terms**