Stopping Rules for Algebraic Iterative Reconstruction Methods in Computed Tomography

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HD TOMO
HIGH - DEFINITION TOMOGRAPHY

DTU Compute
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Just a Pitch for Our Work

Read the paper in arxiv.org/abs/2106.10053

Stopping Rules for Algebraic Iterative Reconstruction Methods in Computed Tomography

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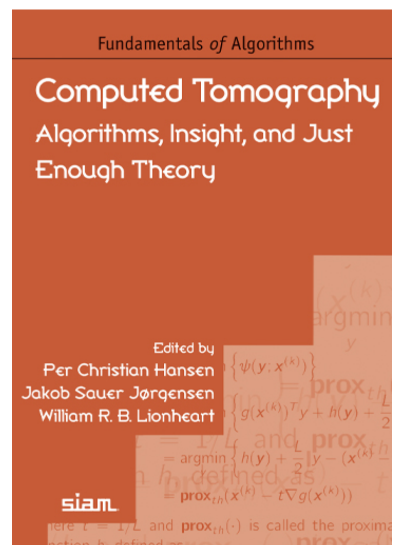
***Abstract*—Algebraic models for the reconstruction problem in X-ray computed tomography (CT) provide a flexible framework that applies to many measurement geometries. For large-scale problems we need to use iterative solvers, and we need stopping rules for these methods that terminate the iterations when**

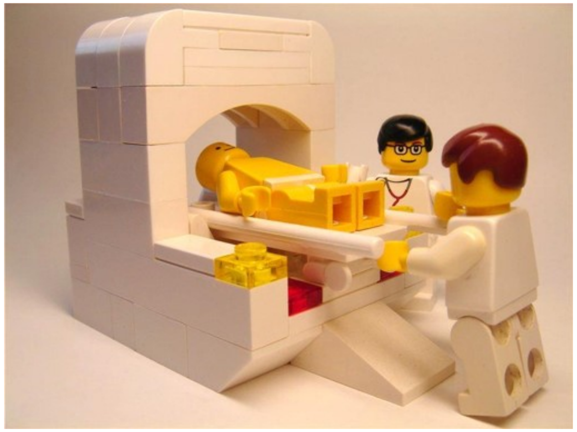
are common, depending on the measurement setup. The matrix A^T represents the so-called back projector which maps the data back onto the solution domain [21]; it plays a central role in filtered back projection and similar methods.

More stuff in our SIAM book



Computed Tomography & Iterative Methods

Used in medical imaging, industrial inspection, materials science, ...



Algebraic Iterative Reconstruction (AIR)

The reconstruction problem is formulated as a discretized problem $A\,x = b$ that is solved by an iterative method, such as:

Landweber: $x_{k+1} = x_k + \omega\, A^T (b - A\,x_k)$, $\quad \omega$ = relaxation parameter,

SIRT: $x_{k+1} = x_k + \omega\, D\, A^T M\, (b - A\,x_k)$, $\quad D$ and M diagonal matrices.

- Can use fast implementation of forward & back projections A and A^T.
- Well suited for noisy data and/or underdetermined problems.
- Easy to incorporate non-negativity or box constraints.
- Rely on semi-convergence and a stopping rule.

Semi-Convergence

Notation

$b = A\,\bar{x} + e$,

$\bar{x}$ = exact solution,

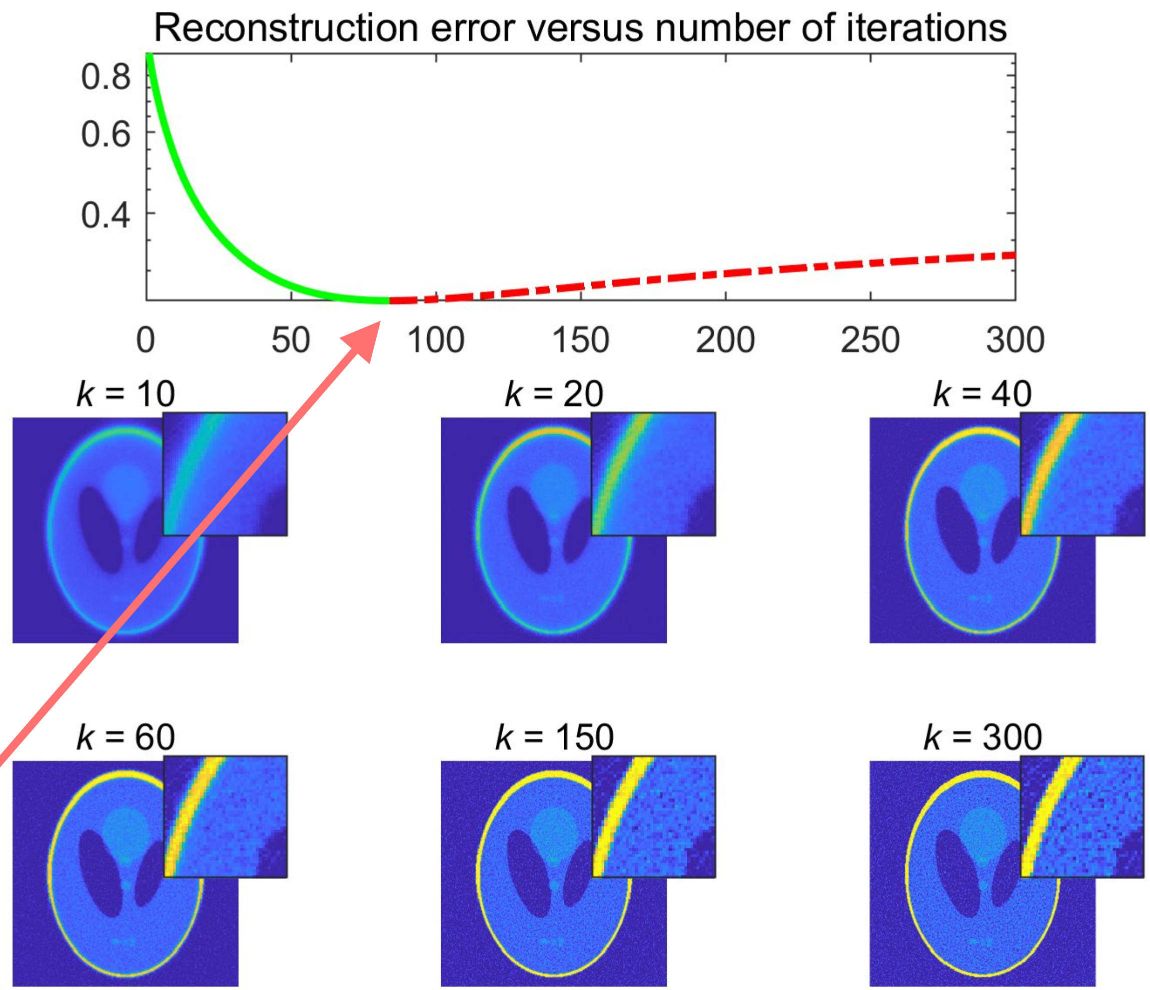
e = noise.

Initially

the reconstruction error $\|x_k - \bar{x}\|_2$ decreases.

Later

the error increases and $x_k \to$ noisy solution.



Need to stop at the transition point!

Stopping Rules I *see the paper for details*

Set the stage

- For each AIR method there is a matrix $A_k^{\#}$ such that $x_k = A_k^{\#} b$.
- Define trace $t_k = \mathrm{trace}(A\, A_k^{\#})$ and residual norm $\rho_k = \|A\, x_k - b\|_2$.
- Assume that the noise is Gaussian: $e \sim \mathcal{N}(0, \eta^2 I)$.

The trace t_k can be estimated by Monte Carlo techniques that involve an additional random right-hand side and thus double the amount of work.

FTNL (fit to noise level):

$$\text{stop as soon as } \rho_k^2 \le \eta^2\,(m - t_k), \qquad m = \mathtt{size}(\mathtt{A},2)$$

This is the classical "discrepancy principle" if we neglect t_k.

Stopping Rules II

Noise-free data

Two methods that seek to minimize the prediction error $\|(A\,\bar{x}) - A\,x_k\|_2$.

UPRE (unbiased predictive risk estimation):

$$\text{minimize } \rho_k^2 + 2\,\eta^2(t_k - m)$$

GCV (generalized cross valiation):

$$\text{minimize } \rho_k^2 \,/\, (m - t_k)^2$$

Identify when all relevant information is extracted from the noisy data b, i.e., when the residual $A\,x_k - b$ resembles the white noise.

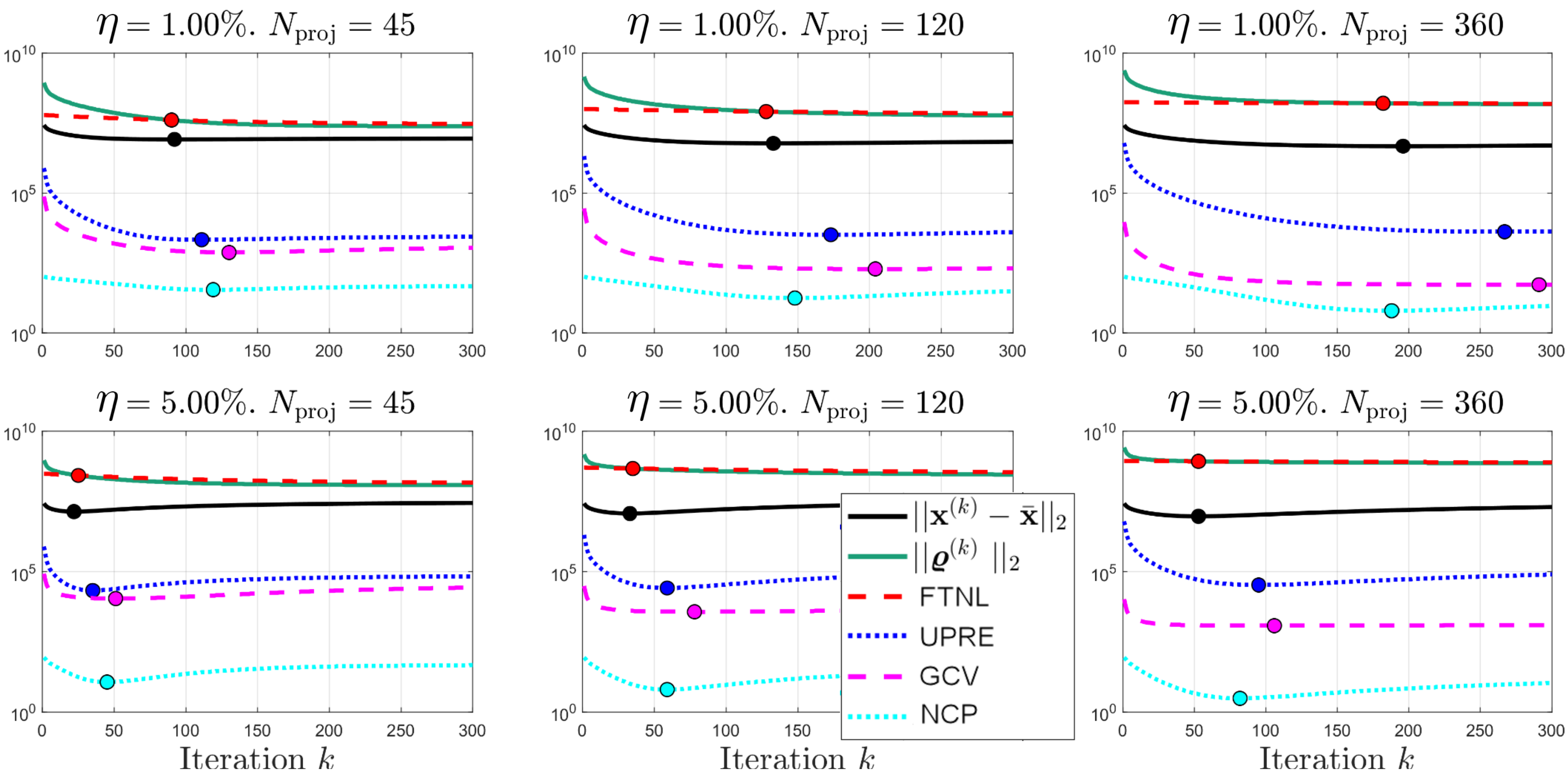
NCP (normalized cumulative periodogram):

stop when the residual's power spectrum is flat

No need for t_k or η^2. Needs just one FFT for each iteration.

Numerical Example

We *simulate* nano-CT reconstruction of a piece of chalk – a porous material – from the North Sea Basin (i.e., we know the ground-truth image).

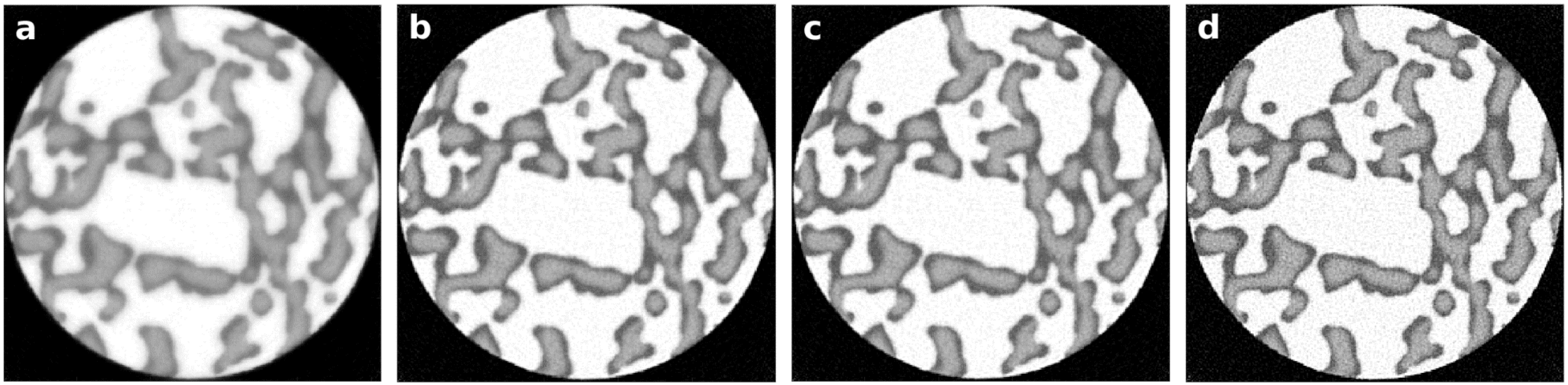


NCP is consistently doing well, for all noise levels and all amounts of data.

Numerical Example – Reconstructions

Reconstructions ofter different number of iterations:

$k = 100$ | $k_{\min} = 157$ | $k_{\mathrm{NCP}} = 189$ | $k = 1000$



Too few iterations, lack of details.

Optimal number of iterations.

Stopped by NCP.

Too many iterations, noisy reconstruction.

Conclusions

- We surveyed state-of-the-art stopping rules for AIR methods in CT.
- They aim to terminate the iterations at the point of semi-convergence.
- They are easy to use and integrate in existing software.
- We illustrated the use of the methods for a realistic CT problem, related to the study of multiphase flow in chalk.
- The NCP stopping rule works well for this problem, and it does not depend on knowledge of the noise level.