APPENDIX

Directional Statistics with the Spherical Normal Distribution — Supplementary Material —

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The isotropic spherical normal distribution has density

$$\mathsf{SN}(\mathbf{x} \mid \boldsymbol{\mu}, \lambda) = \frac{1}{\mathcal{Z}_1(\lambda)} \exp\left(-\frac{\lambda}{2}\mathrm{Log}_{\boldsymbol{\mu}}(\mathbf{x})^{\intercal}\mathrm{Log}_{\boldsymbol{\mu}}(\mathbf{x})\right), \qquad \|\boldsymbol{\mu}\| = 1, \quad \lambda > 0. \tag{1}$$

In the main paper, we claim that the normalization constant is

$$\begin{aligned}\mathcal{Z}_1^{(\text{even})}(\lambda) = {}& \frac{A_{D-2}}{2^{D-2}}\binom{D-2}{D/2-1}\sqrt{\frac{\pi}{2\lambda}}\mathrm{erf}\left(\pi\sqrt{\frac{\lambda}{2}}\right) + \frac{A_{D-2}}{2}\sqrt{\frac{2\pi}{\lambda}}\frac{(-1)^{D/2-1}}{2^{D-3}} \\ & \cdot \sum_{k=0}^{D/2-2}(-1)^k\binom{D-2}{k}\exp\left(-\frac{(D-2-2k)^2}{2\lambda}\right)\mathrm{Re}\left[\mathrm{erf}\left(\frac{\pi\lambda-(D-2-2k)i}{\sqrt{2\lambda}}\right)\right]\end{aligned} \tag{2a}$$

$$\begin{aligned}\mathcal{Z}_1^{(\text{odd})}(\lambda) = {}& A_{D-2}\frac{(-1)^{(D-3)/2}}{2^{D-3}}\sqrt{\frac{\pi}{2\lambda}}\sum_{k=0}^{(D-3)/2}(-1)^k\binom{D-2}{k}\exp\left(-\frac{(D-2-2k)^2}{2\lambda}\right) \\ & \cdot\left\{\mathrm{Im}\left[\mathrm{erf}\left(\frac{(D-2-2k)i}{\sqrt{2\lambda}}\right)\right] + \mathrm{Im}\left[\mathrm{erf}\left(\frac{\pi\lambda-(D-2-2k)i}{\sqrt{2\lambda}}\right)\right]\right\}\end{aligned} \tag{2b}$$

when D is even and odd. Here erf is the imaginary error function, while $\mathrm{Re}[\cdot]$ and $\mathrm{Im}[\cdot]$ takes the real and imaginary parts of a complex number, respectively. In this section, we provide the derivation of this constant.

By definition, we have

$$\mathcal{Z}_1(\lambda) = \int_{\mathcal{S}^{D-1}} \exp\left(-\frac{\lambda}{2}\mathrm{Log}_{\boldsymbol{\mu}}(\mathbf{x})^{\intercal}\mathrm{Log}_{\boldsymbol{\mu}}(\mathbf{x})\right)\mathrm{d}\mathbf{x}. \tag{3}$$

We express this integral in the tangent space of the sphere at $\boldsymbol{\mu}$, i.e. we perform the substitution

$$\mathbf{v} = \mathrm{Log}_{\boldsymbol{\mu}}(\mathbf{x}). \tag{4}$$

The appropriate Jacobian is

$$\det(\mathbf{J}) = \left(\frac{\sin(\|\mathbf{v}\|)}{\|\mathbf{v}\|}\right)^{D-2} \tag{5}$$

and the integral becomes

$$\mathcal{Z}_1(\lambda) = \int_{\|\mathbf{v}\|<\pi} \exp\left(-\frac{\lambda\|\mathbf{v}\|^2}{2}\right)\left(\frac{\sin(\|\mathbf{v}\|)}{\|\mathbf{v}\|}\right)^{D-2}\mathrm{d}\mathbf{v}. \tag{6}$$

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We then write this in hyper-spherical coordinates[1]

$$
\begin{aligned}
\mathcal{Z}_1(\lambda) = \int_{r=0}^{\pi}\int_{\phi_1=0}^{\pi}\int_{\phi_2=0}^{\pi}\cdots\int_{\phi_{D-3}=0}^{\pi}\int_{\phi_{D-2}=0}^{2\pi} \exp\left(-\frac{\lambda r^2}{2}\right)\left(\frac{\sin(r)}{r}\right)^{D-2} r^{D-2} \\
\sin(\phi_1)^{D-3}\sin(\phi_2)^{D-4}\ldots\sin(\phi_{D-3})\mathrm{d}\phi_{D-2}\mathrm{d}\phi_{D-3}\ldots\mathrm{d}\phi_2\mathrm{d}\phi_1\mathrm{d}r.
\end{aligned} \tag{7}
$$

Since the radius r and the angles ϕ are never mixed in the integrand, we can split this into the product of two integrals

$$
\begin{aligned}
\mathcal{Z}_1(\lambda) = \int_{r=0}^{\pi} \exp\left(-\frac{\lambda r^2}{2}\right)\sin^{D-2}(r)\mathrm{d}r \qquad (8)\\
\int_{\phi_1=0}^{\pi}\int_{\phi_2=0}^{\pi}\cdots\int_{\phi_{D-3}=0}^{\pi}\int_{\phi_{D-2}=0}^{2\pi}\sin(\phi_1)^{D-3}\sin(\phi_2)^{D-4}\cdots\sin(\phi_{D-3})\mathrm{d}\phi_{D-2}\mathrm{d}\phi_{D-3}\cdots\mathrm{d}\phi_2\mathrm{d}\phi_1.
\end{aligned}
$$

The second term is merely the surface area of the $D-2$ dimensional unit sphere

$$
A_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma\left((D-1)/2\right)}, \tag{9}
$$

where Γ is the usual Gamma function. The integral (9) then reduces to

$$
\mathcal{Z}_1(\lambda) = A_{D-2}\int_{r=0}^{\pi}\exp\left(-\frac{\lambda r^2}{2}\right)\sin^{D-2}(r)\mathrm{d}r. \tag{10}
$$

To evaluate this expression we need the trigonometric power formulas [2]

$$
\sin^{2n}(x) = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\cos(2(n-k)x), \tag{11}
$$

$$
\sin^{2n+1}(x) = \frac{(-1)^n}{4^n}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}\sin((2n+1-2k)x). \tag{12}
$$

We are now ready evaluate the normalization constant. *First we consider the case were $D-2$ is even.*

$$
\mathcal{Z}_1^{(\text{even})}(\lambda) = A_{D-2}\int_{r=0}^{\pi}\exp\left(-\frac{\lambda r^2}{2}\right)\sin^{D-2}(r)\mathrm{d}r \tag{13}
$$

$$
\begin{aligned}
&= \frac{A_{D-2}}{2^{D-2}}\binom{D-2}{D/2-1}\int_{r=0}^{\pi}\exp\left(-\frac{\lambda r^2}{2}\right)\mathrm{d}r \\
&+ A_{D-2}\frac{(-1)^{D/2-1}}{2^{D-3}}\sum_{k=0}^{D/2-2}(-1)^k\binom{D-2}{k}\int_{r=0}^{\pi}\exp\left(-\frac{\lambda r^2}{2}\right)\cos((D-2-2k)r)\mathrm{d}r.
\end{aligned} \tag{14}
$$

To evaluate this we need two simple integrals, which we evaluate using Maple,

$$
\int_{r=0}^{\pi}\exp\left(-\frac{\lambda r^2}{2}\right)\mathrm{d}r = \frac{\sqrt{\pi}}{\sqrt{2\lambda}}\mathrm{erf}\left(\frac{\pi\sqrt{\lambda}}{\sqrt{2}}\right) \tag{15}
$$

$$
\int_{r=0}^{\pi}\exp\left(-\frac{\lambda r^2}{2}\right)\cos(a\cdot r)\mathrm{d}r = \frac{\sqrt{2\pi}}{4\sqrt{\lambda}}\exp\left(-\frac{a^2}{2\lambda}\right)\left(\mathrm{erf}\left(\frac{\pi\lambda - ai}{\sqrt{2\lambda}}\right) + \mathrm{erf}\left(\frac{\pi\lambda + ai}{\sqrt{2\lambda}}\right)\right), \tag{16}
$$

where i denotes the complex unit. Inserting these expressions into Eq. 14 and simplifying expressions gives the desired result (2a). Note that in the simple special case $D = 2$, the spherical normal is a distribution over the unit circle. Here, the normalization constant reduce to

$$
\mathcal{Z}_1^{(D=2)}(\lambda) = \frac{\sqrt{2\pi}}{\sqrt{\lambda}}\mathrm{erf}\left(\frac{\pi\sqrt{\lambda}}{\sqrt{2}}\right). \tag{17}
$$

[1]Using the convention of https://en.wikipedia.org/wiki/N-sphere#Spherical_coordinates

We now consider the case where $D-2$ is odd. Akin to the previous derivation, we insert Eq. 12 into Eq. 10 and get

$$\mathcal{Z}_1^{(\mathrm{odd})}(\lambda) = A_{D-2} \int_{r=0}^{\pi} \exp\left(-\frac{\lambda r^2}{2}\right) \sin^{D-2}(r) \mathrm{d}r \tag{18}$$

$$= A_{D-2} \frac{(-1)^{(D-3)/2}}{2^{D-3}} \sum_{k=0}^{(D-3)/2} (-1)^k \binom{D-2}{k} \int_{r=0}^{\pi} \exp\left(-\frac{\lambda r^2}{2}\right) \sin((D-2-2k)r) \mathrm{d}r. \tag{19}$$

To evaluate this, we need to evaluate the integral

$$\begin{aligned} &\int_{r=0}^{\pi} \exp\left(-\frac{\lambda r^2}{2}\right) \sin(a \cdot r) \mathrm{d}r \\ &\quad = -\frac{i\sqrt{2\pi}}{4\sqrt{\lambda}} \exp\left(-\frac{a^2}{2\lambda}\right) \left(2\mathrm{erf}\left(\frac{ia}{\sqrt{2\lambda}}\right) + \mathrm{erf}\left(\frac{\pi\lambda - ia}{\sqrt{2\lambda}}\right) - \mathrm{erf}\left(\frac{\pi\lambda + ia}{\sqrt{2\lambda}}\right)\right), \end{aligned} \tag{20}$$

which we have evaluated using Maple. Inserting this expression into Eq. 19 and simplifying gives the result in Eq. 2b. In the important special-case $D = 3$, the normalization constant reduce to

$$\mathcal{Z}_1^{(D=3)}(\lambda) = \frac{-i\pi^{3/2}}{\sqrt{2\lambda}} \exp\left(-\frac{1}{2\lambda}\right) \left\{2\,\mathrm{erf}\left(\frac{i}{\sqrt{2\lambda}}\right) + \mathrm{erf}\left(\frac{\pi\lambda - i}{\sqrt{2\lambda}}\right) - \mathrm{erf}\left(\frac{\pi\lambda + i}{\sqrt{2\lambda}}\right)\right\}. \tag{21}$$

This concludes the derivation.

A. *Quality of the approximation*

We approximate the inverse normalization constant $1/\mathcal{Z}_1$ with a straight line in order to derive an expression for the anisotropic normalization constant. Figure 1 (center) show the inverse normalization constant for varying values of λ. The right panel of the figure show the difference between the inverse normalization and a fitted straight line. From this we draw two conclusions: 1) the inverse normalization is indeed *not* a straight line; 2) a straight line is, however, a good approximation. While using one globally fitted straight line gives a fairly accurate estimate of the normalization constant, we find that accuracy can be slightly improved by fitting the line locally. We make this local fit through $1/\mathcal{Z}_1(\lambda_2)$ and $1/\mathcal{Z}_1(\lambda_2 + \alpha(\lambda_1 - \lambda_2))$. By extensive numerical optimization we have found that $\alpha^{-1} = 0.46\,\lambda_2 + 1.55$ minimizes the worst-case approximation error of the integral.

At times it may be easier to interpret a variance parameter rather than a concentration parameter. The variance of the spherical normal distribution is defined as [1]

$$\mathsf{Var}[\mathbf{x}] = \int_{\mathcal{S}^{D-1}} \arccos^2(\mathbf{x}^\intercal \boldsymbol{\mu})\ \mathsf{SN}(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Lambda}) \mathrm{d}\mathbf{x}. \tag{22}$$

When the distribution is isotropic, this expression can be evaluated for $\mathcal{S}^2$ similarly to the proof of proposition 1 to give

$$\begin{aligned} \mathsf{Var}[\mathbf{x}] = -\frac{\pi}{\lambda^{5/2}} \Bigg(&- \frac{i\sqrt{\pi}(\lambda-1)}{\sqrt{2}} \exp(-1/2\lambda)\,\mathrm{erf}\left(\frac{i - \pi\,\lambda}{\sqrt{2\lambda}}\right) \\ &- \frac{i\sqrt{\pi}(\lambda-1)}{\sqrt{2}} \exp(-1/2\lambda)\,\mathrm{erf}\left(\frac{i + \pi\,\lambda}{\sqrt{2\lambda}}\right) \\ &+ i(\lambda-1)\sqrt{2\pi}\,\exp(-1/2\lambda)\mathrm{erf}\left(\frac{i}{\sqrt{2\lambda}}\right) \\ &- 2\sqrt{\lambda}\exp(-\pi^2\lambda/2) - 2\sqrt{\lambda}\Bigg). \end{aligned} \tag{23}$$

The left panel of Fig. 1 show how the variance change as a function of λ. Notice that the curve is roughly shaped as $1/\lambda$.

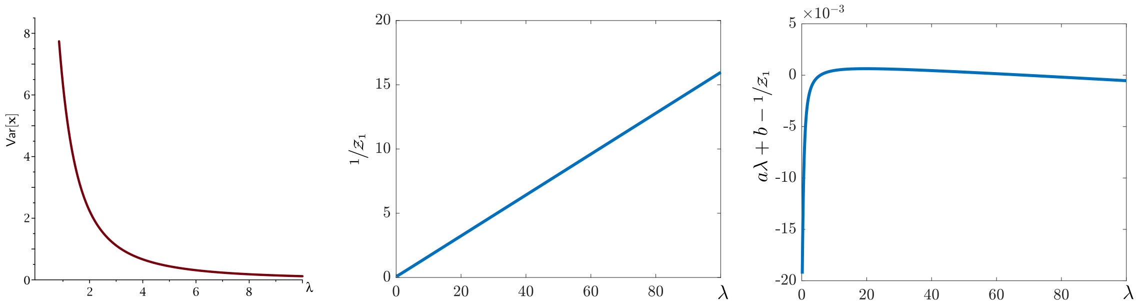

Fig. 1. *Left:* the variance as a function of the concentration. *Center:* the inverse normalization constant for the isotropic distribution. *Right:* The deviation between the inverse normalization constant and a single linear approximation.

References

[1] X. Pennec. Intrinsic Statistics on Riemannian Manifolds: Basic Tools for Geometric Measurements. *Journal of Mathematical Imaging and Vision (JMIV)*, 25(1):127–154, 2006.

[2] E. W. Weisstein. Trigonometric power formulas. *From MathWorld – A Wolfram Web Resource.* *http://mathworld.wolfram.com/TrigonometricPowerFormulas.html*.