01622 Advanced Dynamical Systems: Applications in Science and Engineering

Week 5: Feedback control

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Feedback control

Feedback control – Linear systems

Linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

Control law

$$u(t) = -Lx(t) \tag{2}$$

Closed-loop system

$$\dot{x}(t) = Ax(t) - BLx(t) = \underbrace{(A - BL)}_{\bar{A}} x(t) \tag{3}$$

Question: Is the closed-loop system stable?

Question: Are the real parts of the eigenvalues of $\bar{A}$ negative?

Feedback control – Nonlinear systems

Nonlinear system

$$\dot{x}(t) = f(x(t), u(t), d(t), p) \tag{4}$$

Control law

$$u(t) = \mu(x(t)) \tag{5}$$

Closed-loop system

$$\dot{x}(t) = f(x(t), \mu(x(t)), d(t), p) = F(x(t), d(t), p) \tag{6}$$

Typical objectives

- In steady state, $x(t) = \bar{x}$, for some given setpoint $\bar{x}$
- The steady state should be reached quickly
- The steady state should be stable
- The steady state should be robust against uncertainty in f, d, and p
- The manipulated inputs should be used "sparingly"
- Optimize the economy

Closed-loop stability

Closed-loop system

$$\dot{x}(t) = f(x(t), \mu(x(t)), d(t), p) = F(x(t), d(t), p) \tag{7}$$

Steady state

$$0 = f(x_s, \mu(x_s), d_s, p) = F(x_s, d_s, p), \qquad u_s = \mu(x_s) \tag{8}$$

Jacobian matrix

$$A = \frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial \mu}{\partial x} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix} \tag{9}$$

State and output feedback

State feedback: All states are known to the controller

$$u(t) = \mu(x(t)) \tag{10}$$

Output equation (variables of interest/variables that are measurable)

$$y(t) = g(x(t), p) \tag{11}$$

Output feedback: Only the outputs are known

$$u(t) = \mu(y(t)) \tag{12}$$

Rule of thumb: You can only control as many states/outputs as you have manipulated inputs

Proportional-integral-derivative (PID) control

Proportional controller

$$u(t) = K_p(x(t) - \bar{x}) \tag{13}$$

Proportional-integral controller

$$u(t) = K_p(x(t) - \bar{x}) + K_i \int_0^t x(s) - \bar{x} \, \mathrm{d}s \tag{14}$$

Proportional-integral-derivative controller ($\bar{x}$ constant)

$$u(t) = K_p(x(t) - \bar{x}) + K_i \int_0^t x(s) - \bar{x} \, \mathrm{d}s + K_d \dot{x}(t) \tag{15}$$

Proportional-integral-derivative control with nominal input

$$u(t) = \bar{u} + K_p(x(t) - \bar{x}) + K_i \int_0^t x(s) - \bar{x} \, \mathrm{d}s + K_d \dot{x}(t) \tag{16}$$

Closed-loop simulation of PID-controlled system

Integral

$$w(t) = \int_{t_0}^{t} x(s) - \bar{x}\, \mathrm{d}s \tag{17}$$

Formulate integral as initial value problem

$$\dot{w}(t) = x(s) - \bar{x}, \qquad w(t_0) = 0 \tag{18}$$

Closed-loop system

$$\dot{x}(t) = f(x(t), u(t), d(t), p), \tag{19a}$$
$$\dot{w}(t) = x(s) - \bar{x}, \tag{19b}$$
$$u(t) = K_p(x(t) - \bar{x}) + K_i w(t) + K_d \dot{x}(t) \tag{19c}$$

This is a set of *implicit* differential equations

Use, e.g., Matlab's ode15i to simulate (19)

Discrete-time PID control

Integral (right rectangle rule)

$$w_k = w_{k-1} + (x_k - \bar{x})\Delta t, \qquad\qquad w_0 = 0 \tag{20}$$

Derivative term

$$K_d \frac{x_k - x_{k-1}}{\Delta t} \tag{21}$$

Discrete-time PID controller ($\bar{x}$ constant)

$$u_k = \bar{u} + K_p(x_k - \bar{x}) + K_i w_k + K_d \frac{x_k - x_{k-1}}{\Delta t} \tag{22}$$

Optimal control

Optimal control problem

$$\min_{u} \quad \phi(x,u) = \int_{t_0}^{t_f} \Phi(x(t), u(t), d(t), p)\, \mathrm{d}t, \tag{23a}$$

subject to

$$x(t_0) = x_0, \tag{23b}$$
$$\dot{x}(t) = f(x(t), u(t), d(t), p), \qquad t \in [t_0, t_f], \tag{23c}$$
$$x_{\min} \le x(t) \le x_{\max}, \qquad t \in [t_0, t_f], \tag{23d}$$
$$u_{\min} \le u(t) \le u_{\max}, \qquad t \in [t_0, t_f] \tag{23e}$$

Zero-order-hold parametrization

$$u(t) = u_k, \qquad t \in [t_k, t_{k+1}[, \tag{24a}$$
$$d(t) = d_k, \qquad t \in [t_k, t_{k+1}[\tag{24b}$$

Optimal control – Linear quadratic regulator

Optimal control problem

$$\min_{u} \quad \phi(x,u) = \int_{t_0}^{\infty} x^T(t)Qx(t) + u^T(t)Ru(t)\,\mathrm{d}t, \tag{25a}$$

subject to

$$x(t_0) = x_0, \tag{25b}$$

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad t \in [t_0, t_f] \tag{25c}$$

Riccati equation

$$A^T S + SA - SBR^{-1}B^T S + Q = 0 \tag{26}$$

Optimal controller [1, Section 5.5], [2, Thm. 14.3 and 14.4]

$$u(t) = -Kx(t), \qquad K = R^{-1}B^T S \tag{27}$$

Use, e.g., Matlab's icare or lqr

Deviation variables

Linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{28}$$

Desired steady state

$$0 = A\bar{x} + B\bar{u}, \qquad \bar{x} = -A^{-1}B\bar{u} \qquad \text{if } A \text{ is invertible} \tag{29}$$

Deviation variables

$$X(t) = x(t) - \bar{x}, \qquad U(t) = u(t) - \bar{u} \tag{30}$$

Linear system for deviation variables

$$\begin{aligned} \dot{X}(t) &= \dot{x}(t) - \dot{\bar{x}} = Ax(t) + Bu(t) \\ &= A(X(t) + \bar{x}) + B(U(t) + \bar{u}) = AX(t) + BU(t) + \underbrace{A\bar{x} + B\bar{u}}_{=0} \\ &= AX(t) + BU(t) \end{aligned} \tag{31}$$

Control law

$$U(t) = -KX(t), \quad u(t) = \bar{u} + U(t) = \bar{u} - K(x(t) - \bar{x}) \tag{32}$$

Closed-loop simulation

Closed-loop simulation

Nonlinear system

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t), d(t), p), && t \in [t_0, t_f], && \text{(33a)} \\
y_k &= g(x(t_k), p), && k = 0, \ldots, N && \text{(33b)}
\end{aligned}
$$

Zero-order-hold parametrization

$$
\begin{aligned}
u(t) &= u_k, && t \in [t_k, t_{k+1}[, && \text{(34a)} \\
d(t) &= d_k, && t \in [t_k, t_{k+1}[&& \text{(34b)}
\end{aligned}
$$

Control law

$$
u_k = \mu(y_k) \qquad \text{(35)}
$$

Closed-loop simulation

Initial value problems ($\{d_k\}_{k=0}^{N-1}$ are given)

$$x_k(t_k) = \begin{cases} x_0, & k = 0, \\ x_{k-1}(t_k), & k = 1, \dots, N-1, \end{cases} \tag{36a}$$

$$y_k = g(x_k(t_k), p), \tag{36b}$$

$$u_k = \mu(y_k), \tag{36c}$$

$$\dot{x}_k(t) = f(x_k(t), u_k, d_k, p), \quad t \in [t_k, t_{k+1}], \quad k = 0, \dots, N-1 \tag{36d}$$

Closed-loop simulation

1. Determine the k'th initial state from (36a)
2. Compute the k'th measurement, y_k, from (36b)
3. Compute the k'th manipulated input, u_k, from (36c)
4. Solve the initial value problem (36d)

Nuclear reactor model

Nuclear reactor model 8 – Model 5 revisited (again)

Reactivity and thermal reactivity

$$\rho(t) = \rho_{th}(t) + \rho_{ext}(t), \qquad \dot{\rho}_{th}(t) = -\kappa \dot{T}_r \tag{37}$$

Mass balance equations

$$\dot{C}_n(t) = \frac{\rho(t) - \beta}{\Lambda} C_n(t) + \sum_{i=1}^{m} \lambda_i C_i(t), \tag{38a}$$

$$\dot{C}_i(t) = \frac{\beta_i}{\Lambda} C_n(t) - \lambda_i C_i(t), \qquad i = 1, \ldots, m \tag{38b}$$

Energy balance equations (reactor core and heat exchanger)

$$\dot{T}_r(t) = \frac{f(t)}{n_r}(T_{hx}(t) - T_r(t)) + \frac{Q_g(t)}{n_r c_P}, \tag{39a}$$

$$\dot{T}_{hx}(t) = \frac{f(t)}{n_{hx}}(T_r(t) - T_{hx}(t)) - \frac{k_{hx}}{n_{hx} c_P}(T_{hx}(t) - T_c) \tag{39b}$$

Nuclear reactor model 9 – Model 7 revisited

Reactivity and thermal reactivity

$$\rho(t) = \rho_{th}(t) + \rho_{ext}(t), \qquad \dot{\rho}_{th}(t) = -\kappa \dot{T}_r \tag{40}$$

Mass balance equations

$$\dot{C}_n(t) = \frac{\rho(t) - \beta}{\Lambda} C_n(t) + \sum_{i=1}^{m} \lambda_i C_i(t), \tag{41a}$$

$$\dot{C}_i(t) = \frac{\beta_i}{\Lambda} C_n(t) - \lambda_i C_i(t) + (C_{i,in}(t) - C_i(t))D \tag{41b}$$

Inlet concentration and dilution rate

$$C_{i,in}(t) = e^{-\lambda_i \tau} C_i(t - \tau), \quad D = \frac{F}{V}, \quad F = Av, \quad \tau = L/v \tag{42}$$

Energy balance equations

$$\dot{T}_r(t) = \frac{f(t)}{n_r}(T_{hx}(t - \tau/2) - T_r(t)) + \frac{Q_g(t)}{n_r c_P}, \tag{43a}$$

$$\dot{T}_{hx}(t) = \frac{f(t)}{n_{hx}}(T_r(t - \tau/2) - T_{hx}(t)) - \frac{k_{hx}}{n_{hx} c_P}(T_{hx}(t) - T_c) \tag{43b}$$

Questions?

Bibliography I

[1] E. Hendricks, O. Jannerup, and P. H. Sørensen, *Linear systems control: Deterministic and stochastic methods*. Springer, 2008.

[2] C. M. Kellett and P. Braun, *Introduction to nonlinear control: Stability, control design, and estimation*. Princeton University Press, 2023.

[3] D. E. Seborg, T. F. Edgar, D. A. Mellichamp, and F. J. Doyle III, *Process dynamics and control*. Wiley, 3rd ed., 2011.

[4] A. Ilka and N. Murgovski, "Novel results on output-feedback LQR design," *IEEE Transaction on Automatic Control*, vol. 68, no. 9, pp. 5187–5200, 2023.

[5] T. Binder, L. Blank, H. G. Bock, R. Bulirsch, W. Dahmen, M. Diehl, T. Kronseder, W. Marquardt, J. P. Schlöder, and O. von Stryk, "Introduction to model based optimization of chemical processes on moving horizons," in *Online Optimization of Large Scale Systems* (M. Grötschel, S. O. Krumke, and J. Rambau, eds.), Springer, 2001.

Bibliography II

[6] T. K. S. Ritschel and S. Stange, “Numerical optimal control for delay differential equations: A simultaneous approach based on linearization of the delayed state.” arXiv:2410.02687, 2024. Preprint.

[7] T. K. S. Ritschel, “Numerical optimal control for distributed delay differential equations: A simultaneous approach based on linearization of the delayed variables.” arXiv:2410.15083, 2024. Preprint.

[8] T. K. S. Ritschel and J. Wyller, “An algorithm for distributed time delay identification based on a mixed Erlang kernel approximation and the linear chain trick.” arXiv:2405.07328, 2024. Preprint.