Complexity Results in Epistemic Planning

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Automated planning (or, simply, planning):

- Given is a planning task: initial state + goal formula + finite set of actions.
- Aim is to compute a solution: sequence of actions that leads from the initial state to a state satisfying the goal formula.

**Example.**

**Goal**: On(A,B) \(\land\) On(B,C).

In automated planning, such a graph is called a state space (induced by a planning task).
From classical to epistemic planning

In many scenarios, classical (deterministic, fully observable, single-agent, static) planning is not enough. Restack $\begin{array}{c} C \\ B \\ A \end{array}$ as $\begin{array}{c} A \\ B \\ C \end{array}$:
**Epistemic planning**

We will here use *epistemic planning* to refer to planning based on Dynamic Epistemic Logic (DEL).

**Essentially** our framework of epistemic planning is obtained by replacing the propositional logic underlying classical planning by *Dynamic Epistemic Logic* (DEL).

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Epistemic planning is a framework for multi-agent planning that allows *(arbitrary levels of) higher-order reasoning*. Higher-order reasoning here means reasoning about the beliefs of yourself and other agents (and nesting of such).
DEL by example: A private announcement

- **Action models**: Only *propositional* preconditions and no *postconditions*. Means: Purely epistemic planning, no change of ontic facts.
- **Action model above**: Private announcement of $p$ to agent $a$.
- **Actual world/event**: Colored black.
- **Product update**: As in [Baltag et al., 1998].
- **In resulting model**: Agent $a$ knows $p$ ($\square_a p$ holds), but agent $b$ didn’t learn anything.
**Planning interpretation of DEL**

- **Epistemic states**: Pointed, finite epistemic models.
- **Epistemic actions**: Pointed, finite action models.
- **Result of applying an action in a state**: Product update of state with action.
Definition (Action models and epistemic actions)

An action model is $A = (E, Q, pre)$ where:

- $E$ is a finite set of events.
- $Q : Ag \rightarrow 2^{E \times E}$ assigns an epistemic (accessibility) relation to each agent.
- $pre : E \rightarrow L_{Prop}$ assigns a precondition of the propositional language to each event.

We write $Q_a$ for $Q(a)$. For $e \in E$, the pair $(A, e)$ is called an epistemic action whose actual event is $e$.

Note: Only propositional preconditions and no postconditions.

Applicability: An epistemic action $\alpha = (A, e)$ is said to be applicable in an epistemic state $s$ if $s \models pre(e)$. 
Planning tasks and plan existence problem

Definition (Planning tasks)
A planning task is $T = (s_0, \mathcal{L}, \varphi_g)$, where
- $s_0$ is the initial state: a finite epistemic state.
- $\mathcal{L}$ is the action library: a finite set of epistemic actions.
- $\varphi_g$ is the goal formula: a formula of epistemic logic.

Definition (Plans)
A plan (or solution) for a planning task $T = (s_0, \mathcal{L}, \varphi_g)$ is a sequence of epistemic actions $\alpha_1, \ldots, \alpha_n \in \mathcal{L}$ such that $s_0 \models \langle \alpha_1 \rangle \cdots \langle \alpha_n \rangle \varphi_g$
(where, by definition, $s \models \langle \alpha \rangle \varphi$ iff $\alpha$ is applicable in $s$ and $s \otimes \alpha \models \varphi$).

Definition (Plan existence problem)
Let $X$ denote a class of planning tasks. The plan existence problem for $X$ is the following decision problem “Given an epistemic planning task $T \in X$, does it have a solution?”
Example

Consider the planning task \( \{s_0, \{\alpha_1, \alpha_2, \alpha_3\}, \varphi_g\} \) with

\[
\varphi_g = \Box a p \land \Box b p \land \neg \Box a \Box b p \land \neg \Box b \Box a p
\]

\[ s_0 = \]

\[
\alpha_1 = \]

\[
\alpha_1 = \]

\[
\alpha_3 = \]

\( \alpha_1 \): privately announcing \( p \) to \( a \); \( \alpha_2 \): privately announcing \( p \) to \( b \); \( \alpha_3 \): publicly announcing \( p \) to both agents.

A solution (plan) is \( \alpha_1, \alpha_2 \), since \( s_0 \models \langle \alpha_1 \rangle \langle \alpha_2 \rangle \varphi_g \). Another solution is \( \alpha_2, \alpha_1 \). Also \( \alpha_1, \alpha_2, \alpha_1 \) and \( \alpha_1, \alpha_1, \alpha_2 \) are solutions, etc.
## Summary of complexity results for plan existence

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The green results will be covered in this talk. From Bolander, Jensen, Schwarzentruber: Complexity Results in Epistemic Planning (under submission).
Why study very expressively restricted fragments?

Motivation for studying complexity of very restrictive fragments of epistemic planning:

- Where does the complexity come from?
- Constructing search heuristics for planning engines (relaxed problems).
- Subclasses of more general fragments might be translatable into simpler fragments.
**Definition (n-ary product)**

Let \( \alpha = (A, e) \) be an epistemic action where \( A = (E, Q, pre) \). We denote by \( A^n = (E^n, Q^n, pre^n) \) the n-ary product of \( A \).

1. \( E^n = \{(e_1, \ldots, e_n) \mid e_i \in E \text{ for all } i = 1, \ldots, n\} \),
2. \( Q^a_n = \{((e_1, \ldots, e_n), (f_1, \ldots, f_n)) \mid e_i Q_a f_i \text{ for all } i = 1, \ldots, n\} \)
3. \( pre^n((e_1, \ldots, e_n)) = \bigwedge_{i=1,\ldots,n} pre(e_i) \).

The n-ary product of \( \alpha \) is defined as \( \alpha^n = (A^n, e^n) \), where \( e^n \) denotes \((e, e, \ldots, e)\).

\[
\begin{pmatrix}
  \begin{array}{c}
  a \\
  e_1 : p
  \end{array}
  & \begin{array}{c}
  a, b \\
  e_2 : \top
  \end{array}
\end{pmatrix}^2
= \begin{pmatrix}
  \begin{array}{c}
  a \\
  e_1 : p
  \end{array}
  & \begin{array}{c}
  a, b \\
  e_2 : \top
  \end{array}
\end{pmatrix}
\]

**Lemma**

For any epistemic action \( \alpha \) and any \( \varphi \in L_E \) we have that \( \langle \alpha \rangle^n \varphi \equiv \langle \alpha^n \rangle \varphi \) (that is, \( \langle \alpha \rangle \langle \alpha \rangle \cdots \langle \alpha \rangle \varphi \) is modally equivalent to \( \langle \alpha^n \rangle \varphi \)).
**Definition (Bisimilarity)**

Two epistemic actions $\alpha = ((E, Q, pre), e)$ and $\alpha' = ((E', Q', pre'), e')$ are called *bisimilar*, written $\alpha \leftrightarrow \alpha'$, if there exists a (bisimulation) relation $Z \subseteq E \times E'$ containing $(e, e')$ and satisfying for every $a \in Ag$:

- **[atom]** If $(f, f') \in Z$ then $pre(f) \equiv pre'(f')$.
- **[forth]** If $(f, f') \in Z$ and $fQ_ag$ then $\exists g' \in E': f' Q'_ag'$ and $(g, g') \in Z$.
- **[back]** Other direction.
\textbf{$n$-bisimilarity on epistemic actions}

\textbf{Definition ($n$-bisimilarity)}

Let $\alpha = ((E, Q, \text{pre}), e)$ and $\alpha' = ((E', Q', \text{pre}'), e')$ be epistemic actions. They are 0-bisimilar, written $\alpha \leftrightarrow_0 \alpha'$, if $\text{pre}(e) \equiv \text{pre}'(e')$. For $n > 0$, they are $n$-bisimilar, written $\alpha \leftrightarrow_n \alpha'$, if for every $a \in Ag$:

\begin{itemize}
  \item \textbf{[atom]} $\text{pre}(e) \equiv \text{pre}'(e')$.
  \item \textbf{[forth]} If $eQ_a f$ then $\exists f' \in E'$ : $e'Q_a f'$ and $(A, f) \leftrightarrow_{n-1} (A', f')$.
  \item \textbf{[back]} Other direction.
\end{itemize}

\textbf{Equivalently}: $\alpha \leftrightarrow_n \alpha'$ if for any path of length $m \leq n$ in $\alpha$:

\begin{itemize}
  \item there exists a path in $\alpha'$:
\end{itemize}

\begin{itemize}
  \item with $\text{pre}(f) \equiv \text{pre}'(f')$; and vice versa.
\end{itemize}
Stabilisation

$md(\varphi)$ denotes modal depth of $\varphi$.

Lemma ([Sadzik, 2006], slightly reformulated)

Let $\alpha$, $\alpha'$ be two epistemic actions and $\varphi$ any formula.

1. If $\alpha \leftrightarrow \alpha'$, then $\langle \alpha \rangle \varphi \equiv \langle \alpha' \rangle \varphi$.
2. If $md(\varphi) \leq n$ and $\alpha \leftrightarrow_n \alpha'$, then $\langle \alpha \rangle \varphi \equiv \langle \alpha' \rangle \varphi$.

Definition (Stabilisation)

Let $\alpha$ be an epistemic action.

1. $\alpha$ is stabilising at stage $i$ if $\alpha^i \leftrightarrow \alpha^{i+k}$ for all $k \geq 0$.
2. $\alpha$ is $n$-stabilising at stage $i$ if $\alpha^i \leftrightarrow_n \alpha^{i+k}$ for all $k \geq 0$.

Lemma

If two epistemic actions are $n$-bisimilar for all $n$, then they are bisimilar.
Example

Recall $\alpha \leftrightarrow \alpha^2$ where $\alpha$ is private announcement of $p$ to agent $a$:

Hence private announcements are stabilising at stage 1. Clearly, so are public announcements.

Epistemic actions (our type) commute: $\langle \alpha_1 \rangle \langle \alpha_2 \rangle \varphi \equiv \langle \alpha_2 \rangle \langle \alpha_1 \rangle \varphi$ [Löwe et al., 2011].

Consequence: If the action library of a planning task only consists of public and private announcements, then we never have to repeat any action more than once.
Stabilisation and plan-existence problem

Lemma

Let $T = (s_0, \{\alpha_1, \ldots, \alpha_m\}, \varphi_g)$ be a planning task and $B \in \mathbb{N}$. Suppose one of the following holds:

1. Every $\alpha_i$ is stabilising at stage $B$, or
2. $md(\varphi_g) = n$ and every $\alpha_i$ is $n$-stabilising at stage $B$.

Then $T$ is solvable iff there exists $k_1, \ldots, k_m \leq B$ s.t.

$s_0 \models \langle \alpha_1 \rangle^{k_1} \cdots \langle \alpha_m \rangle^{k_m} \varphi_g$.

Non-deterministic algorithm for deciding the plan-existence problem when $B$ satisfies 1 or 2 above:

procedure PLANEXISTS$((s_0, \{\alpha_1, \ldots, \alpha_m\}, \varphi_g), B)$

| a) Guess a vector $(k_1, \ldots, k_m) \in \{0, \ldots, B\}^m$. |
| b) Accept when $s_0 \models \langle \alpha_1 \rangle^{k_1} \cdots \langle \alpha_m \rangle^{k_m} \varphi_g$. |
Sadzik’s lemma

Lemma ([Sadzik, 2006])

Let $\alpha = ((E, Q, pre), e)$ be an epistemic action and $n$ a natural number. Then $\alpha$ is $n$-stabilising at stage $|E|^n$.

We will now improve the upper bound on stabilisation...
Stabilisation Lemma

Given event $e$ in action $\alpha$ and $n \in \mathbb{N}$, define:

$$\text{mpaths}_n(e) = \text{the number of distinct maximal paths of length } \leq n \text{ rooted at } e \text{ (ignoring agent labels)}.$$ 

Lemma (Stabilisation Lemma)

Let $\alpha = (A, e_0)$ be an epistemic action and $n$ any natural number. Then $\alpha$ is $n$-stabilising at stage $\text{mpaths}_n(e_0)$.

Proof sketch.

Let $k = \text{mpaths}_n(e_0)$. We then show:

$$(A^{k+1}, (e_0, \ldots, e_0)) \leftrightarrow_n (A^k, (e_0, \ldots, e_0)).$$
Classes of planning tasks

We define the following classes of planning tasks:

- **Singletons**: Every action in the action library is a singleton (i.e. public announcement).
- **Chains**: The underlying graph of every action is a unary tree (a chain). Leafs can be reflexive.
- **Trees**: The underlying graph of every action is a tree. Leafs may be reflexive.
- **Graphs**: Arbitrary actions.

Note that

\[
\text{Singletons} \subseteq \text{Chains} \subseteq \text{Trees} \subseteq \text{Graphs}
\]

and that **Graphs** contain all planning tasks where actions have propositional preconditions (and no postconditions).
Plan existence problem for Singletons and Chains

Theorem
The plan existence problem for both Singletons and Chains is NP-complete.

Proof sketch of NP-hardness for Singletons.
Polynomial-time reduction from SAT. Given propositional formula \( \varphi(p_1, \ldots, p_n) \), we construct planning task \( T = (s_0, \{\alpha_1, \ldots, \alpha_m\}, \varphi(\lozenge a p_1, \ldots, \lozenge a p_m)) \).

We can represent any propositional valuation \( \nu \) by an epistemic state \( s \) satisfying: For all \( p_i \), \( \nu \models p_i \) iff \( s \models \lozenge p_i \).
Recall the plan-existence procedure that decides plan existence when all actions are stabilising at stage $\leq B$ (by previous lemma):

**procedure** PLANEXISTS($(s_0, \{\alpha_1, \ldots, \alpha_m\}, \varphi_g), B$)

a) Guess a vector $(k_1, \ldots, k_m) \in \{0, \ldots, B\}^m$.

b) Accept when $s_0 \models \langle \alpha_1 \rangle^{k_1} \cdots \langle \alpha_m \rangle^{k_m} \varphi_g$.

**Proof sketch of NP-membership of Chains.**

Let $T = (s_0, \{\alpha_1, \ldots, \alpha_m\}, \varphi_g)$ denote a planning task in Chains.

1. $\alpha_i$ has at most one maximal path of length $\leq n$ for all $n$.
2. By Stabilisation Lemma, $\alpha_i$ is $n$-stabilising at stage 1 for any $n$.
3. $\alpha_i$ is stabilising at stage 1 ($\iff_n$ for all $n$ implies $\iff$).
4. The procedure PLANEXISTS$(T, 1)$ is accepting iff $T$ is solvable.
5. PLANEXISTS$(T, 1)$ runs in non-deterministic polynomial time (non-increasing states).
Plan existence problem for Trees

Theorem
The plan existence problem for Trees is PSPACE-complete.
PSPACE-hardness is by a polynomial-time reduction from QSAT (satisfiability of quantified boolean formulas).

Proof sketch of PSPACE-membership of Trees.
Same overall proof strategy as for Chains. Let
\[ T = (s_0, \{\alpha_1, \ldots, \alpha_m\}, \varphi_g) \]
denote a planning task in Trees.

1. Let \( l(\alpha_i) \) denote number of leaves in (underlying graph of) of \( \alpha_i \). Then \( \text{mpaths}_n(e) \leq l(\alpha_i) \) for any \( n \).
2. By Stabilisation Lemma, \( \alpha_i \) is \( n \)-stabilising at stage \( l(\alpha_i) \) for any \( n \).
3. \( \alpha_i \) is stabilising at stage \( l(\alpha_i) \) (\( \Leftrightarrow_n \) for all \( n \) implies \( \Leftrightarrow \)).
4. The procedure \( \text{PLANEXISTS}(T, \max_i l(\alpha_i)) \) is accepting iff \( T \) is solvable.
5. \( \text{PLANEXISTS}(T, \max_i l(\alpha_i)) \) uses polynomial space in the size of \( s_0 \) and
\[ \langle \alpha_1 \rangle^{k_1} \cdots \langle \alpha_m \rangle^{k_m} \varphi_g \] where \( k_i \leq \max_i l(\alpha_i) \)
[Aucher and Schwarzentuber, 2013].
Corollary on plan verification/model checking

**Plan verification problem**: Given a finite epistemic state $s_0$ and a formula of the form $\langle \alpha_1 \rangle \cdots \langle \alpha_j \rangle \varphi_g$, does $s_0 \models \langle \alpha_1 \rangle \cdots \langle \alpha_j \rangle \varphi_g$ hold.

**Theorem**

*The plan verification problem (restricted to propositional action models that are trees) is PSPACE-complete.*

This is a generalisation of a result in van de Pol, van Rooij and Szymanik: How difficult is it to Think that you Think that I think that...? (under submission).

We generalise by: only single-pointed models, no postconditions, only propositional preconditions, only tree structured action models.
Plan existence problem for Graphs

Theorem

*The plan existence problem for Graphs is in EXPSPACE.*

Proof sketch of EXPSPACE-membership of Graphs.

Same overall proof strategy as for Chains and Trees. Let 
\( T = (s_0, \{\alpha_1, \ldots, \alpha_m\}, \varphi_g) \) denote a planning task in Graphs where \( \alpha_i = (E_i, Q_i, pre_i, e_i) \) and let \( k = md(\varphi_g) \).

1. By Sadzik’s Lemma, \( \alpha_i \) is \( k \)-stabilising at stage \( |E_i|^k \).

2. The procedure PLANEXISTS(\( T, \max_i |E_i|^k \)) is accepting iff \( T \) is solvable. This procedure runs in NEXPSpace = EXPSPACE.
### Summary

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Appendix: References I

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DEL planning and some tractable cases.

Exploring the Iterated Update Universe.

Multi-Agent Epistemic Explanatory Diagnosis via Reasoning about Actions.
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