Hybrid Logic

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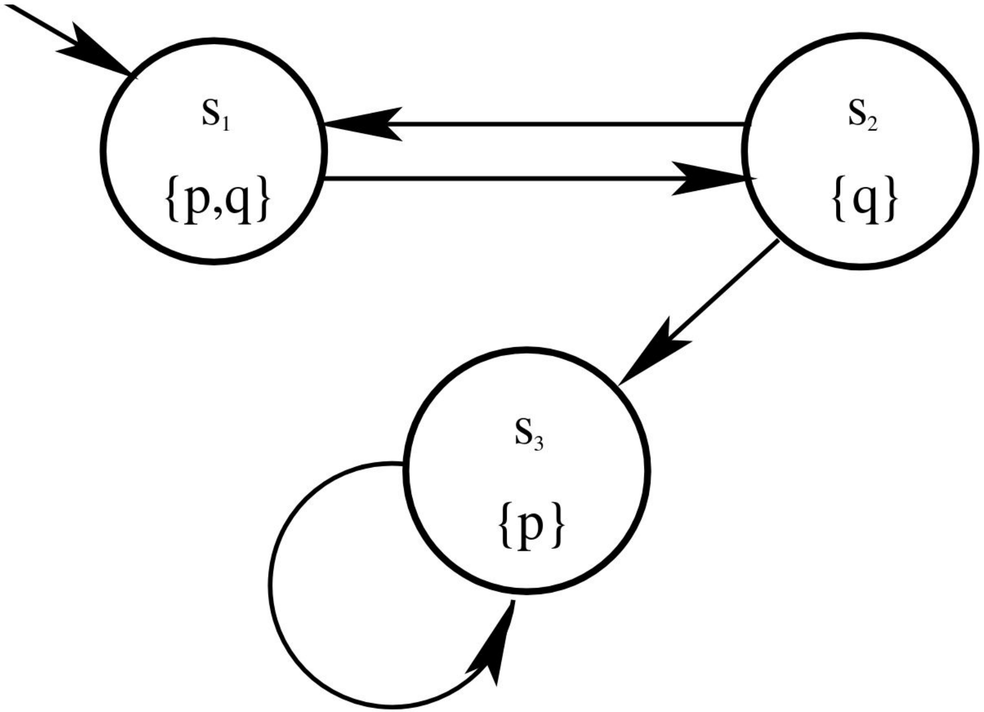
Modal Logic

Let's talk about relational structures
- People
- Program states
- Possible worlds
- …

Internal perspective

◊, □ modalities talk about relations

How do we talk about points?



Hybrid Logic

Introduce ***nominals***:

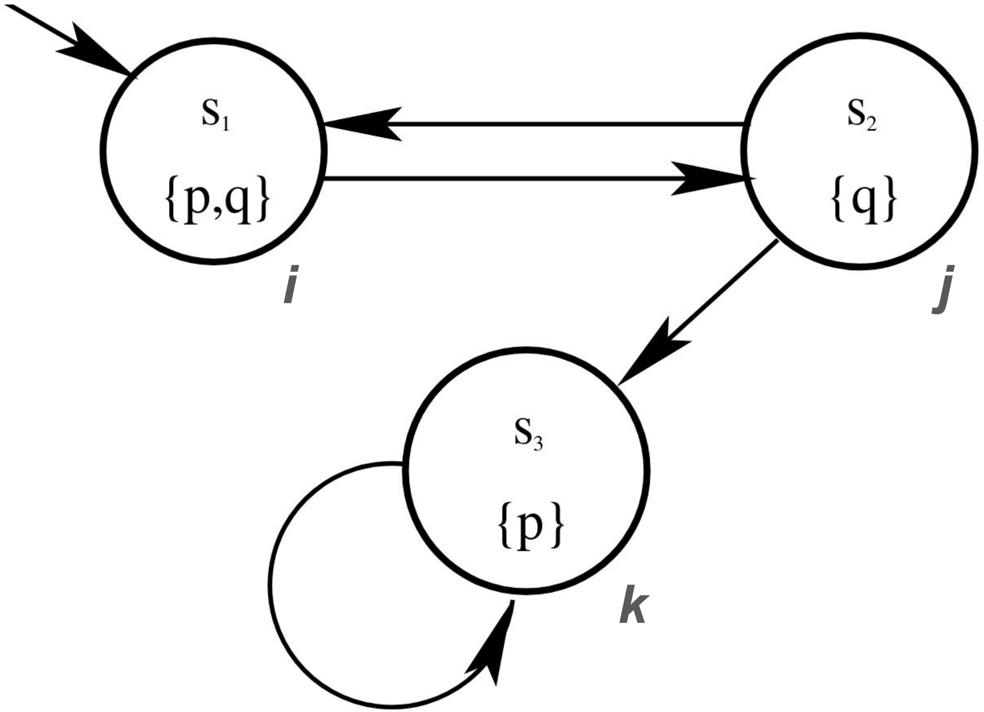
- True at one place
- *name* worlds

Induce *satisfaction operator*:

- $@_k\ \varphi$ "jumps" to k
- $\neg @_i \neg\varphi$ iff $@_i\ \varphi$

Added expressivity:

- Irreflexive frame: $i \rightarrow \Box \neg i$
- Asymmetric frame: $i \rightarrow \Box \neg \Diamond i$



Syntax and Semantics

$$\phi, \psi ::= p \mid i \mid \neg\phi \mid \phi \vee \psi \mid \Diamond\phi \mid @_i\phi$$

Kripke model ((*W*, *R*), *V*) and assignment *g*.

W: underlying set, *R*: binary relation, *V*: unary relation, *g*: map from nominals to worlds.

$$\begin{array}{lll}
\mathfrak{M}, g, w \models p & \text{iff} & w \in V(p) \\
\mathfrak{M}, g, w \models i & \text{iff} & g(i) = w \\
\mathfrak{M}, g, w \models \neg\phi & \text{iff} & \mathfrak{M}, g, w \not\models \phi \\
\mathfrak{M}, g, w \models \phi \vee \psi & \text{iff} & \mathfrak{M}, g, w \models \phi \text{ or } \mathfrak{M}, g, w \models \psi \\
\mathfrak{M}, g, w \models \Diamond\phi & \text{iff} & \text{for some } w',\ wRw' \text{ and } \mathfrak{M}, g, w' \models \phi \\
\mathfrak{M}, g, w \models @_i\phi & \text{iff} & \mathfrak{M}, g, g(i) \models \phi
\end{array}$$

Seligman-Style Tableau

Truth relative to a world

$@_i$-formulas!

Very global

$$
\begin{array}{c}
@_i \phi_1 \\
@_i \phi_2 \\
\vdots \\
\\
@_j \psi_1 \\
\vdots
\end{array}
$$

Named blocks
(+ explicit context switch)

Local perspective!

$$
\begin{array}{c}
i \\
\phi_1 \\
\phi_2 \\
\vdots \\
\hline
j \\
\psi_1 \\
\vdots
\end{array}
$$

$$\frac{\begin{array}{c} a \\ \phi \vee \psi \end{array}}{\begin{array}{c} a \\ / \quad \backslash \\ \phi \qquad \psi \end{array}} \quad (\vee)$$

$$\frac{\begin{array}{c} a \\ \neg(\phi \vee \psi) \end{array}}{\begin{array}{c} a \\ | \\ \neg\phi \\ \neg\psi \end{array}} \quad (\neg\vee)$$

$$\frac{\begin{array}{c} a \\ \neg\neg\phi \end{array}}{\begin{array}{c} a \\ | \\ \phi \end{array}} \quad (\neg\neg)$$

$$\frac{\begin{array}{c} a \\ \Diamond\phi \end{array}}{\begin{array}{c} a \\ | \\ \Diamond i \\ @_i\phi \end{array}} \quad (\Diamond)^1$$

$$\frac{\begin{array}{cc} a & a \\ \neg\Diamond\phi & \Diamond i \end{array}}{\begin{array}{c} a \\ | \\ \neg @_i\phi \end{array}} \quad (\neg\Diamond)$$

$$\frac{|}{i} \quad \mathsf{GoTo}^2$$

$$\frac{\begin{array}{cc} b & b \\ a & \phi \end{array}}{\begin{array}{c} a \\ | \\ \phi \end{array}} \quad \mathsf{Nom}$$

$$\frac{\begin{array}{cc} b & b \\ \phi & \neg\phi \end{array}}{\begin{array}{c} a \\ | \\ \times \end{array}} \quad \text{Closing}$$

$$\frac{\begin{array}{c} b \\ @_a\phi \end{array}}{\begin{array}{c} a \\ | \\ \phi \end{array}} \quad (@)$$

$$\frac{\begin{array}{c} b \\ \neg @_a\phi \end{array}}{\begin{array}{c} a \\ | \\ \neg\phi \end{array}} \quad (\neg @)$$

Aside: Isabelle

Formalize!

```
| Neg:
    ‹(¬ ¬ p) at a in (ps, a) # branch ⟹
    new p a ((ps, a) # branch) ⟹
    A, Suc n ⊢ (p # ps, a) # branch ⟹
    A, n ⊢ (ps, a) # branch›
```

Soundness and completeness is checked mechanically

Verify the definitions, trust the results

Example Tableau

Start from refuted formula
(at arbitrary nominal)

Derive conclusions

Search for contradictions

[Potential] *restricts* **GoTo**

0.	a		
1.	$\neg(\neg @_i \phi \vee @_i \phi)$		[0]
2.	$\neg\neg @_i \phi$	$(\neg\vee)$ 1	[1]
3.	$\neg @_i \phi$	$(\neg\vee)$ 1	[1]
4.	$@_i \phi$	$(\neg\neg)$ 2	[2]
5.	i	GoTo	[1]
6.	$\neg\phi$	$(\neg @)$ 3	[2]
7.	ϕ	$(@)$ 4	[3]
	$\times$		

Restrictions

Don't apply rules ad infinitum

Simple to formalize

S1 The output of a non-GoTo rule must include a formula new to the current block type.

S2 The ($\Diamond$) rule can only be applied to input $\Diamond\phi$ on an a-block if $\Diamond\phi$ is not already witnessed at a by formulas $\Diamond i$ and $@_i\phi$ for some witnessing nominal i.

S3 We associate *potential*, a natural number n, with each line in the tableau. GoTo must decrement the number, the other rules increment it and we may start from any amount.

S4 We parameterize the proof system by a fixed set of nominals A and impose the following:

a. The nominal introduced by the ($\Diamond$) rule is not in A.

b. For any nominal i, Nom only applies to a formula $\phi = i$ or $\phi = \Diamond i$ when $i \in A$.

Completeness Landscape

Analytic completeness for a **restricted** internalized tableau system
Bolander, Blackburn (2007)

Completeness for a **restricted** Seligman-style tableau system [via translation]
Blackburn, Bolander, Braüner, Jørgensen (2017)

Synthetic completeness for **unrestricted** Seligman-style tableau systems
Jørgensen, Blackburn, Bolander, Braüner (2016)

Synthetic completeness for **restricted** Seligman-style tableau systems
F. (under review but formalized)

Completeness Approach

Start from existing synthetic approach

Tweak Hintikka sets to restrictions

Adapt model from internalized calculus

Isabelle guides the way

$$\frac{\begin{array}{cc} i & j \\ \Diamond j & k \end{array}}{\begin{array}{c} i \\ \Diamond k \end{array}}$$

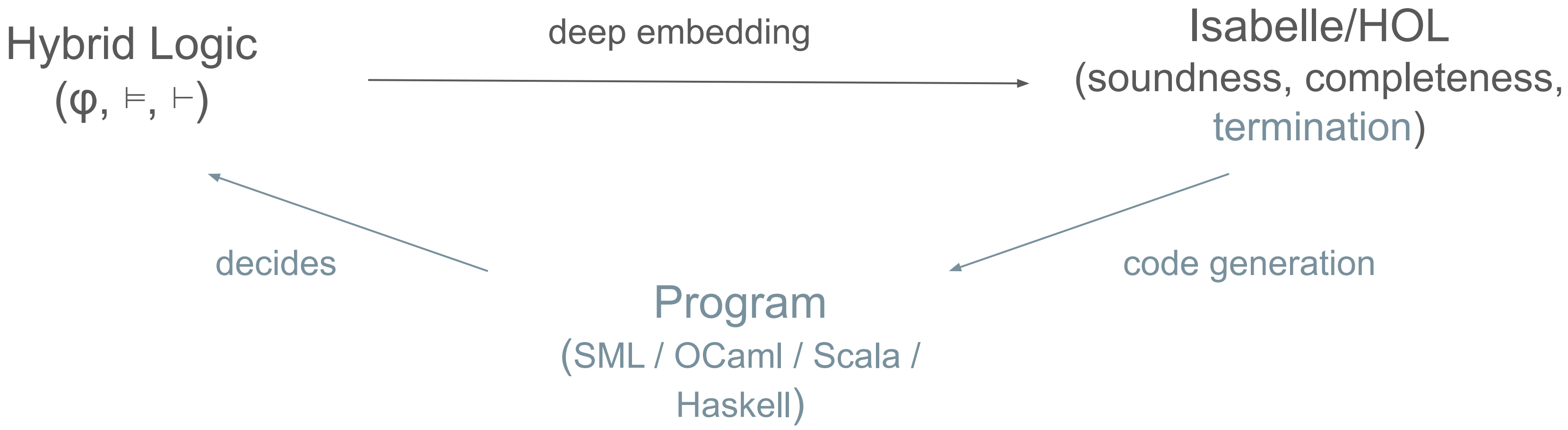
Aside: Worlds are sets, no need for **Bridge**

Status

Tableau system ST^A for basic hybrid logic

Informal termination proof [via translation]

4900+ lines in the Archive of Formal Proofs (and 2000+ WIP)



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