Hybrid Logic

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Modal Logic

Let’s talk about relational structures
- People
- Program states
- Possible worlds
- …

Internal perspective

◊, □ modalities talk about relations

How do we talk about points?
Hybrid Logic

Introduce **nominals**:
- True at one place
- *name* worlds

Induce **satisfaction operator**:
- $\@_k \varphi$ “jumps” to $k$
- $\neg @_i \neg \varphi$ iff $@_i \varphi$

Added expressivity:
- Irreflexive frame: $i \rightarrow \square \neg i$
- Asymmetric frame: $i \rightarrow \square \neg \diamond i$
Syntax and Semantics

\[ \phi, \psi ::= p \mid i \mid \neg \phi \mid \phi \lor \psi \mid \Diamond \phi \mid \@ i \phi \]

Kripke model \( ((W, R), V) \) and assignment \( g \).

\( W \): underlying set, \( R \): binary relation, \( V \): unary relation, \( g \): map from nominals to worlds.

\[ M, g, w \models p \quad \text{iff} \quad w \in V(p) \]
\[ M, g, w \models i \quad \text{iff} \quad g(i) = w \]
\[ M, g, w \models \neg \phi \quad \text{iff} \quad M, g, w \not\models \phi \]
\[ M, g, w \models \phi \lor \psi \quad \text{iff} \quad M, g, w \models \phi \text{ or } M, g, w \models \psi \]
\[ M, g, w \models \Diamond \phi \quad \text{iff} \quad \text{for some } w', wRw' \text{ and } M, g, w' \models \phi \]
\[ M, g, w \models \@ i \phi \quad \text{iff} \quad M, g, g(i) \models \phi \]
Seligman-Style Tableau

Truth relative to a world

@\_i\_\_\_\_\_-formulas!

Very global

\begin{align*}
@i \phi_1 \\
@i \phi_2 \\
\vdots \\
@j \psi_1 \\
\vdots
\end{align*}

Named blocks
(+ explicit context switch)

Local perspective!
Aside: Isabelle

Formalize!

| Neg: 

\((\neg \neg p) \text{ at } a \text{ in } (ps, a) \ # \text{ branch } \Rightarrow \)
new \ p \ a ((ps, a) \ # \text{ branch}) \Rightarrow 

A, Suc \ n \vdash (p \ # \ ps, a) \ # \text{ branch } \Rightarrow 

A, n \vdash (ps, a) \ # \text{ branch} \rangle 

Soundness and completeness is checked mechanically

*Verify the definitions, trust the results*
Example Tableau

Start from refuted formula (at arbitrary nominal)

Derive conclusions

Search for contradictions

[Potential] restricts GoTo
Restrictions

Don’t apply rules ad infinitum

*Simple to formalize*

**S1** The output of a non-\texttt{GoTo} rule must include a formula new to the current block type.

**S2** The $(\Diamond)$ rule can only be applied to input $\Diamond \phi$ on an $a$-block if $\Diamond \phi$ is not already witnessed at $a$ by formulas $\Diamond i$ and $@_i \phi$ for some witnessing nominal $i$.

**S3** We associate *potential*, a natural number $n$, with each line in the tableau. \texttt{GoTo} must decrement the number, the other rules increment it and we may start from any amount.

**S4** We parameterize the proof system by a fixed set of nominals $A$ and impose the following:

\begin{itemize}
  \item[a.] The nominal introduced by the $(\Diamond)$ rule is not in $A$.
  \item[b.] For any nominal $i$, \texttt{Nom} only applies to a formula $\phi = i$ or $\phi = \Diamond i$ when $i \in A$.
\end{itemize}
Completeness Landscape

Analytic completeness for a restricted internalized tableau system
Bolander, Blackburn (2007)

Completeness for a restricted Seligman-style tableau system [via translation]
Blackburn, Bolander, Braüner, Jørgensen (2017)

Synthetic completeness for unrestricted Seligman-style tableau systems
Jørgensen, Blackburn, Bolander, Braüner (2016)

Synthetic completeness for restricted Seligman-style tableau systems
F. (under review but formalized)
Completeness Approach

Start from existing synthetic approach
Tweak Hintikka sets to restrictions
Adapt model from internalized calculus

*Isabelle guides the way*

Aside: Worlds are sets, no need for *Bridge*
Status

Tableau system $ST^A$ for basic hybrid logic

Informal termination proof [via translation]

4900+ lines in the Archive of Formal Proofs (and 2000+ WIP)

Hybrid Logic ($\varphi$, $\models$, $\vdash$)

Isabelle/HOL (soundness, completeness, termination)

Program (SML / OCaml / Scala / Haskell)

decides

deep embedding

code generation
References


