Connections Between the General Theories of Ordered Vector Spaces and C^*-Algebras

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Preface

Two areas of functional analysis which have been the subject of extensive study are operator theory and ordered vector spaces. While these have developed into flourishing independent areas, each with their own experts, it is the author's impression that the theories have a bit more in common than the literature might suggest. The goal of this thesis is therefore to exhibit connections between the general theories of C^*-algebras and ordered vector spaces.

Coming into this project, I was already familiar with the theory of C^*-algebras, but I had little experience with ordered vector spaces. Therefore most of the theory is motivated from the C^*-algebra point of view. The thesis is built up in two parts. In the first part, we take ideas and concepts from the theory of C^*-algebras to ordered vector spaces, leading to concepts such as *order ideals*, *order semisimplicity*, and *order unitisations*. Conversely, in the second part we ask questions about the order structure of C^*-algebras, with an emphasis on the lattice-like structure of a C^*-algebra. Detailed outlines of each of these parts are given below.

This thesis is written with a reader in mind having roughly the same background as I had coming into this project. As such, we assume familiarity with graduate level functional analysis and operator theory, as for instance provided by the Dutch MasterMath courses *Functional Analysis* and *Operator Algebras*.

Concretely, the first part relies heavily on notions such as topological vector spaces, locally convex spaces, the Hahn–Banach theorems, weak and weak-$*$ topologies, and the Krein–Milman theorem.

Most of the second part only requires familiarity with the basic theory of bounded linear operators on a Hilbert space, and the notion of a C^*-algebra. Only in Chapter 8 do we use slightly more advanced concepts from the theory of C^*-algebras, such as irreducible representations, the strong operator topology, and Kaplansky's density theorem.

Note that prior knowledge of partially ordered vector spaces is *not* assumed. The reason for this is purely circumstantial: no courses on this subject are taught at the author's university (or elsewhere in the Netherlands) at the time of writing. We cannot aim to give a full overview of the theory; the interested reader is encouraged to consult [AT07].

Overview of the first part

The first part of this thesis aims to build up part of the theory of ordered vector spaces in a way that most closely resembles the theory of C^*-algebras. We introduce concepts of *order ideals*, *order semisimple spaces*, and *order unitisations*, each of these being analogous to its algebraic counterpart for Banach algebras. Apart from that, a technical obstacle has to be overcome: the classical theory of ordered vector spaces deals exclusively with real vector spaces, whereas the theory of Banach algebras is by far more successful in the complex setting. To overcome this, we develop a theory of complex ordered vector spaces parallel to the real theory. However, we end up proving most of the results for real ordered vector spaces only, and merely listing modifications to be made in the complex case at the end of the corresponding chapter.

[AT07] has been our main reference for the theory of ordered vector spaces. It provides a comprehensive overview of the general theory, and indeed goes much further than we do. However, the authors are motivated by questions from economics and econometrics, and the book is therefore written for a relatively broad audience. As a result, some (if not most) of the connections with other areas of mathematics are lost, or at least not pointed out very clearly. This is unfortunate, as it makes learning the ideas and techniques unnecessarily difficult for students and mathematicians already familiar with functional analysis and C^*-algebras. The first part of this thesis aims to complement the treatment in [AT07]. Roughly speaking, the algebraic approach is developed here, and the geometric intuition and more advanced results can be obtained from [AT07].

Chapter 1 contains an introduction to the basic theory of ordered vector spaces. We treat concepts such as real and complex ordered vector spaces, positive linear maps, generating cones, the Archimedean property, full sets, and order units. There is nothing new here, except that the theory of complex ordered vector spaces is non-standard. We mostly follow notation and terminology from [AT07].

Chapter 2 deals with the question of representing an ordered vector space as a space of functions. This is done by defining the *order radical* of an ordered space, which leads to a notion of *order semisimple* spaces. An important tool is wedge duality and the bipolar theorem, introduced in Section 2.5. The main result of this section is Theorem 2.28, showing that a topological ordered vector space V whose (topological) dual V^* separates points is topologically order semisimple if and only if the weak closure of V^+ is a cone (as opposed to a wedge). Finally, towards the end of the chapter we briefly consider the question of representing an ordered vector space as a space of *continuous* functions on a compact Hausdorff space.

In Chapter 3, we attempt to find ways of adjoining an order unit to an ordered vector space, analogously to the algebraic unitisation of a (Banach) algebra without unit. After briefly considering a failed construction for general spaces, we study an Archimedean order unitisation for normed, topologically order semisimple spaces in Section 3.2. The remainder of the chapter is devoted to studying the properties of this order unitisation $\tilde{V}$. One of the main results, proved in Section 3.5, is that any continuous positive linear map $V \to W$ to an Archimedean space with an order unit can be extended to a continuous positive linear map $\tilde{V} \to W$.

Finally, Chapter 4 contains an assortment of interesting examples and counterexamples regarding the theory from the first three chapters. Some of the longer examples have been moved here so as not to interrupt the general flow of ideas in the main text. Additionally, in Section 4.5 we briefly compare the algebraic and order structure of $C_0(\Omega)$ spaces (Ω locally compact Hausdorff).

The general approach from this first part of the thesis is considered new, even if large parts of the theory are not particularly ground-breaking. Some of the results are also considered to be original; in particular, concepts such as (topological) order semisimplicity and order unitisations do not seem to be covered in the literature.

Overview of the second part

The main objective of the second part is to study the lattice-like structure of C^*-algebras. It is well-known that every *commutative* C^*-algebra is isometrically $*$-isomorphic to some $C_0(\Omega)$ space (Ω locally compact Hausdorff), so in particular it is lattice-ordered. This leads one to ask whether non-commutative C^*-algebras are also lattice-ordered.

In this setting, we prove two classical theorems. The first is Kadison's anti-lattice theorem, which states that two self-adjoint operators on a Hilbert space have a supremum (of infimum) if and only if they were comparable to begin with (in that case the larger of the two clearly is the supremum). The second main result is Sherman's theorem, stating that a lattice-ordered C^*-algebra must necessarily be commutative.

In Chapter 5 we show how to use the Gelfand representation to define a *quasi-supremum* (and a *quasi-infimum*) for every pair of self-adjoint operators in a C^*-algebra. We show that it is an upper (resp. lower) bound for a and b, and we show that $*$-homomorphisms preserve the quasi-lattice operations. Furthermore, it is shown that the quasi-supremum of two trace-class operators has a special property: it is the unique upper bound of minimal trace.

Chapter 6 aims to prove Kadison's anti-lattice theorem. Unlike standard proofs in the literature, the proof does not rely so much on techniques from the theory of operator algebras, and instead stays much closer to the geometry of the underlying Hilbert space. Furthermore, our proof does not depend on the full-blown spectral theorem, and as such constitutes an "elementary" proof.

The study of the anti-lattice theorem is continued in Chapter 7, where we provide a geometric interpretation of a large class of minimal upper bounds. We show that the anti-lattice theorem follows from a simple statement about decompositions of the Hilbert space $\mathcal{H}$ into a "positive" and a "negative" part for $a - b$. The main results in this chapter are the correspondence between (certain) minimal upper bounds and subspace decompositions (Theorem 7.17), and a strengthening of the anti-lattice theorem showing that the set of minimal upper bounds for a pair of incomparable self-adjoint operators is in fact unbounded in norm (Theorem 7.23).

In Chapter 8 we prove Sherman's theorem as a corollary to the anti-lattice theorem. This is done via representation theory of C^*-algebras.

Finally, in Appendix A we treat the theory of complementary subspaces of a Hilbert space. The results play an important role in Chapter 7, but are not needed in any of the other chapters. The main result of this appendix is Corollary A.10, stating that the sum $V + W$ of two closed subspaces $V, W \subseteq \mathcal{H}$ is closed if and only if $V^\perp + W^\perp$ is closed.

The general approach taken here is new: the main theorems are presented as part of a single theory, and Sherman's theorem is deduced from the anti-lattice theorem via representation theory. Furthermore, both the elementary proof (Chapter 6) and the subsequent geometric interpretation (Chapter 7) of Kadison's anti-lattice theorem are believed to be new.

A similarly coherent study of various commutativity theorems is given in [Top65], but the method of proof is completely different. (In particular, the paper does not deal with Kadison's anti-lattice theorem at all.)

Notation and terminology

The set $\mathbb{N}$ is assumed to contain 0. The set of all (strictly) positive integers will be denoted by $\mathbb{N}^+$ instead.

All vector spaces are over the ground field $\mathbb{F}$, which is either $\mathbb{R}$ or $\mathbb{C}$. An *algebra* over the field $\mathbb{F}$ is an $\mathbb{F}$-vector space A together with an associative bilinear multiplication map $A \times A \to A$, $(a,b) \mapsto ab$. (In other words, algebras are assumed to be associative, but not necessarily unital or commutative.) Regarding Banach and C^*-algebras, we follow terminology from [Mur90].

Inner products (and more generally sesquilinear maps) are assumed to be linear in the first coordinate and conjugate-linear in the second.

Regarding function spaces (e.g. $C(\Omega)$, $C_0(\Omega)$, $C_b(\Omega)$, $C_c(\Omega)$, $\mathscr{L}^p(\mu)$, $L^p(\mu)$) and sequence spaces (e.g. ℓ^p, c, c_0, c_{00}), we follow notation from [Con07].

We shall have no philosophical objections to the axiom of choice (indeed, large parts of linear algebra and functional analysis collapse without it), so in particular its use is often not mentioned explicitly. We leave it to the interested expert to figure out which theorems do and do not require choice.

Small numbers in superscript refer to *end* notes (as opposed to footnotes). The end notes for each chapter are collected at the end of the chapter.

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I furthermore find myself indebted to all those great mathematicians (past and present), those metaphorical giants, whose work allowed me to stand on their (metaphorical) shoulders. In this respect, special mention goes out to Richard V. Kadison, whose early work was a great inspiration for this thesis. (The first part builds on [Kad51a], while the anti-lattice theorem from [Kad51b] plays a major role in the second part of this thesis.)

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Part I

Ideals, radicals and unitisations of ordered vector spaces

1 Ordered vector spaces

This chapter is a brief introduction to the basic concepts from the theory of ordered vector spaces. We mostly use terminology from [AT07], with the notable exception that we build a theory of complex ordered vector spaces parallel to the real theory.

The interested reader is encouraged to consult [AT07] for further reading, as it paints a clear geometric picture of cones in real vector spaces (contrasting – and complementing – our more algebraic approach).

1.1 Cones in real vector spaces

Definition 1.1. Let V be a real vector space. A non-empty subset $W \subseteq V$ is said to be a *wedge* if it satisfies the following properties:

1. If $x, y \in W$, then $x + y \in W$;
2. If $x \in W$ and $\alpha \in \mathbb{R}_{\geq 0}$, then $\alpha x \in W$.

Furthermore, a wedge $W \subseteq V$ is said to be a *cone* if it also satisfies the following additional property:

3. $W \cap -W = \{0\}$.

A word of warning: in other areas of mathematics (such as convex geometry), what we call *wedges* and *cones* are sometimes called *cones* and *pointed cones*, respectively. In keeping with [AT07], we shall make no use of this alternate terminology.

Note that wedges and cones are convex sets. Furthermore, the intersection of a non-empty family of wedges in V is again a wedge in V. Since every subset $S \subseteq V$ is contained in a wedge (namely in V), it follows that there is a smallest wedge W_S containing S, called the *wedge generated by* S. Concretely, W_S is the non-negative linear span of S.

There is no such thing as the *cone generated by* S. One should consider the wedge generated by S and check whether or not it is a cone.[1]

Definition 1.2. An *pre-ordered vector space* is a real vector space V equipped with a pre-order $\leq$ satisfying the following properties:

1. If $x \leq y$, then $x + z \leq y + z$ for all $z \in V$;
2. If $x \leq y$, then $\alpha x \leq \alpha y$ for all $\alpha \in \mathbb{R}_{\geq 0}$.

If $\leq$ is a partial order, then V is called an *ordered vector space*.

If $(V, \leq)$ is a pre-ordered vector space, then $V^+ := \{x \in V : 0 \leq x\}$ is a wedge. Conversely, if $S \subseteq V$ is a wedge, then we can define a pre-order $\leq$ on V by letting $x \leq y$ if and only if $y - x \in S$. These constructions define a bijective correspondence between vector space pre-orders and wedges on V. Furthermore, under this correspondence, the pre-order $\leq$ is a partial order if and only if V^+ is a cone.

In light of the correspondence between vector space orders and positive cones, we will usually think of an ordered vector space $(V, \leq)$ as the vector space V equipped with the positive cone V^+.

1.2 Cones in complex vector spaces

Most of the literature on ordered vector spaces only deals with cones in real vector spaces. Of course, a complex vector space can be seen as a real vector space via restriction of scalars, so we already have a concept of cones in complex vector spaces. However, in most situations it is undesirable to dispose of the complex structure altogether. Instead, we shall restrict our attention to spaces carrying a *complex conjugation*, as defined below.

Definition 1.3. Let V be a complex vector space. A *complex conjugation* on V is a conjugate-linear involution $f : V \to V$. In other words, for all $x, y \in V$ and $\lambda, \mu \in \mathbb{C}$ we have $f(f(x)) = x$ and $f(\lambda x + \mu y) = \overline{\lambda} f(x) + \overline{\mu} f(y)$.

A complex conjugation f is usually written $^- : V \to V$, and the conjugate of an element $x \in V$ is written as $\overline{x}$.

We say that an element $x \in V$ is *real* or *self-conjugate* if $\overline{x} = x$ holds. The subset consisting of all real elements of V is denoted $\mathrm{Re}(V)$. It is a real subspace of V, but not a complex one. Every $x \in V$ can be uniquely written as $x = y + iz$ with $y, z \in \mathrm{Re}(V)$; these y and z are given by $y = \mathrm{Re}(x) := \frac{1}{2}(x + \overline{x})$ and $z = \mathrm{Im}(x) := \frac{1}{2i}(x - \overline{x})$.

A subset $S \subseteq V$ is called *real* if $S \subseteq \mathrm{Re}(V)$ holds, and *self-conjugate* if $\overline{S} = S$ holds. (Note that these two terms are no longer synonymous here.)

Example 1.4. If V is a space of complex functions (e.g. $\mathbb{C}^n$, $\ell^\infty(S)$, $C(\Omega)$, $\mathscr{L}^p(\Omega, \mathscr{A}, \mu)$, etcetera), then the pointwise conjugation map $^- : f \mapsto \overline{f}$ defines a complex conjugation on V. ⊠

Example 1.5. If V is a $*$-algebra, then $*$ is a complex conjugation. Note that not every complex conjugation on an algebra turns it into a $*$-algebra; for this it is also required that $(ab)^* = b^* a^*$ holds for all $a, b \in V$. ⊠

Example 1.6. The algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices carries two competing complex conjugations. On the one hand, there is the entry-wise conjugation $^- : M_n(\mathbb{C}) \to M_n(\mathbb{C})$. It is a ring homomorphism ($\overline{ab} = \overline{a}\,\overline{b}$), and the real part with respect to this conjugation is $M_n(\mathbb{R})$. On the other hand, there is the C^*-conjugation, taking a to its conjugate transpose a^*. This is an antihomomorphism ($(ab)^* = b^* a^*$), and the real part with respect to this conjugation is the (real) space of complex self-adjoint $n \times n$ matrices. ⊠

Definition 1.7. A *complex ordered vector space* is a complex vector space V equipped with a complex conjugation $^- : V \to V$ and a cone $V^+ \subseteq \mathrm{Re}(V)$. While V^+ also defines a cone in the larger space V, we restrict the notation $a \leq b$ to the case where a and b are real. (After all, in the one-dimensional case we wouldn't say that $i + \frac{1}{2}$ is larger than $i - \frac{1}{2}$, even though their difference is a positive real number.)

Since most (if not all) order theory takes place in $\mathrm{Re}(V)$, much of the theory of real ordered vector spaces can be extended to the complex setting with minor adjustments. We formulate a few results simultaneously for real and complex ordered vector spaces; in this case the notation $\mathrm{Re}(V)$ is understood to mean all of V if the ground field is $\mathbb{R}$.

As mentioned, there is no universally accepted notion of complex ordered vector spaces in the literature. If we wanted to stay a little closer to terminology from operator theory, we could have called spaces like this $*$-*ordered spaces*, analogously to Banach $*$-algebras, so that the term complex (pre)-ordered space could be used for any complex vector space equipped with a cone (or wedge). However, most interesting examples are of the form defined above, so we will stick to the chosen terminology. Furthermore, in Remark 2.48 we will see a compelling reason *not* to broaden our definition.

Typical examples of complex ordered vector spaces are complex function spaces (Example 1.4) and C^*-algebras. More generally, if X is a unital Banach $*$-algebra, then it can be shown that the *closed* wedge generated by all elements of the form a^*a is a cone if and only if X is $*$-semisimple; see for instance [KV53, remarks on page 51]. This wedge plays a role for instance in abstract harmonic analysis (cf. [Fol15, Section 3.3]).

1.3 Positive linear maps

Definition 1.8. Let V, W be ordered vector spaces over the same ground field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. A linear map $f : V \to W$ is called *positive* if $v \geq 0$ implies $f(v) \geq 0$. Similarly, f is called *bipositive* if one has $f(v) \geq 0$ if and only if $v \geq 0$. Note that a linear map is bipositive if and only if V^+ is the inverse image of W^+. Note furthermore that a bipositive map is automatically injective: for $x \in \ker(f)$ we have $f(x), f(-x) \geq 0$, hence $x, -x \geq 0$, from which it follows that $x = 0$ holds.

Over the ground field $\mathbb{R}$, an *order isomorphism* is an invertible linear map $f : V \to W$ such that both f and f^{-1} are positive. It is clear that the order isomorphisms are precisely the surjective bipositive maps. More generally, a bipositive map can be thought of as being an *order embedding*.

Definition 1.9. Let V be a real or complex ordered vector space, and let V' denote its algebraic dual. Then the *dual wedge* of V is the subset $(V^+)' \subseteq V'$ consisting of all positive linear functionals $V \to \mathbb{F}$. (If the ground field is $\mathbb{C}$, then we understand it to be equipped with its standard complex conjugation and the cone $\mathbb{R}_{\geq 0} \subseteq \mathrm{Re}(\mathbb{C}) = \mathbb{R}$.)

While these definitions suffice for the real case, a little more needs to be said about the complex case.

Definition 1.10. Let V and W be complex vector spaces each equipped with a complex conjugation, and let $L(V, W)$ denote the space of all linear maps $V \to W$. The *induced conjugation* on $L(V, W)$ is the map that sends a linear map $f : V \to W$ to the linear map $\overline{f} : V \to W$ given by $v \mapsto \overline{f(\overline{v})}$. This is easily seen to be a well-defined complex conjugation.

The self-conjugate[2] elements with respect to the induced conjugation are precisely the linear maps which preserve conjugation, or equivalently, those that map real elements to real elements (and imaginary to imaginary).

We say that an *order isomorphism* of complex ordered vector spaces is a *self-conjugate* linear isomorphism $f : V \to W$ such that both f and f^{-1} are positive, or equivalently, a *self-conjugate* surjective bipositive map.

Remark 1.11. Note: if we have $V = W$, so that $L(V, W)$ is an algebra, then the induced conjugation does not turn $L(V, W)$ into a $*$-algebra: it is a ring homomorphism ($\overline{ab} = \overline{a}\,\overline{b}$) rather than an anti-homomorphism ($(ab)^* = b^*a^*$). See also Example 1.6.

Remark 1.12. If $f : V \to W$ is a positive linear map between complex ordered vector spaces, then it need not be self-conjugate. (For instance, any linear map $(\mathbb{C}, \{0\}) \to (\mathbb{C}, \mathbb{R}_{\geq 0})$ is positive). In general, a linear map $f : V \to W$ is positive if and only if $\mathrm{Re}(f)$ is positive and $V^+ \subseteq \ker(\mathrm{Im}(f))$ holds.

Note in particular that the dual wedge $(V^+)'$ is not generally contained in $\mathrm{Re}(V')$, so it falls outside our very strict notion of complex (pre-)ordered spaces. However, if $(V^+)'$ is a cone, then it is contained in $\mathrm{Re}(V')$. Indeed, if $f : V \to \mathbb{C}$ is positive, then $\mathrm{Im}(f)$ sends everything in V^+ to zero, so both $\mathrm{Im}(f)$ and $-\mathrm{Im}(f)$ define positive linear functionals. But then we must have $\mathrm{Im}(f) = 0$, since $(V^+)'$ is a cone, so we conclude that $f = \mathrm{Re}(f)$ is self-conjugate. It follows that the induced conjugation and the dual cone turn V' into a complex ordered vector space in this setting.

1.4 Generating cones

Definition 1.13. Let V be a real or complex ordered vector space. The cone V^+ is said to be *generating* if $V = \mathrm{span}(V^+)$ holds. In this case every $v \in \mathrm{Re}(V)$ can be written as $v = a - b$ with $a, b \in V^+$. (Note that this is expression is far from being unique: for any $c \in V^+$ we also have $v = (a + c) - (b + c)$.)

If the ground field is $\mathbb{C}$, then any given $v \in V$ can be written as $v = x + iy$ with $x, y \in \mathrm{Re}(V)$. Consequently, if V^+ is generating, then v can be written as $v = a - b + ic - id$ with $a, b, c, d \in V^+$.

In the literature, sometimes the word *directed* is used to indicate generating cones. This is because a cone V^+ is generating if and only if its corresponding partial order on $\mathrm{Re}(V)$ is directed in the order-theoretic sense (every pair of elements has an upper bound).

1.5 The Archimedean property

Definition 1.14 ([AT07, Definition 1.10 & Lemma 1.11])**.** A real or complex ordered vector space (V, V^+) is said to be *Archimedean*[3] if it satisfies any one (and therefore all) of the following equivalent criteria:

(1) If $x \in \mathrm{Re}(V)$, $y \in V^+$ satisfy $nx \leq y$ for all $n \in \mathbb{N}^+$, then one has $x \leq 0$.

(2) If $x, y \in \mathrm{Re}(V)$ satisfy $nx \leq y$ for all $n \in \mathbb{N}^+$, then one has $x \leq 0$.

In certain cases the Archimedean property is related to topological properties of the cone. To that end, let us say that an *ordered topological vector space* is a space V which is at the same time an ordered vector space and a topological vector space, in such a way that, if the ground field is $\mathbb{C}$, the complex conjugation $\bar{\ } : V \to V$ is continuous. Note that we do not require any kind of compatibility between the positive cone and the topology.

Proposition 1.15 ([AT07, Lemma 2.3])**.** *If W is a closed cone in a topological vector space (V, τ), then τ is Hausdorff and W is Archimedean.*

Proof. We have $\{0\} = W \cap (-W)$, so $\{0\}$ is closed. Therefore τ is Hausdorff.

Suppose that $x, y \in \mathrm{Re}(V)$ satisfy $nx \leq y$ for all $n \in \mathbb{N}^+$. Slightly rewriting this yields $\frac{1}{n}y - x \geq 0$ for all $n \in \mathbb{N}^+$. Now let n go to infinity; since W is closed, we find $-x \geq 0$. This shows that W is Archimedean. ∎

In keeping with much of the functional analysis literature, we will henceforth assume that all topological vector spaces are Hausdorff. In this setting we have the following partial converse of Proposition 1.15.

Proposition 1.16 ([AT07, Lemma 2.4]). *If (V,τ) is a (Hausdorff) topological vector space over the reals and $W \subseteq V$ is an Archimedean cone with non-empty interior, then W is closed.*

In general, not every Archimedean cone is closed; see examples 4.1 and 4.10. Things are easier in the finite-dimensional case: there is only one vector space topology on V (cf. [Rud91, Theorem 1.21]), and in this setting a cone is Archimedean if and only if it is closed (cf. [AT07, Corollary 3.4]).

1.6 Full sets

Definition 1.17. Let V be a real or complex ordered vector space. For given $x, z \in \mathrm{Re}(V)$, the *order interval* $[x,z]$ is defined to be the set of all $y \in \mathrm{Re}(V)$ satisfying $x \leq y \leq z$. Equivalently, one has $[x,z] = (x + V^+) \cap (z - V^+)$. From the latter expression it is clear that the order interval $[x,z]$ is convex. (Note that $[x,z]$ is empty unless $x \leq z$ holds.)

Definition 1.18. Let V be a real or complex ordered vector space. A subset $S \subseteq \mathrm{Re}(V)$ is called *full* if $x \leq y \leq z$ and $x, z \in S$ imply $y \in S$. Equivalently, for all $x, z \in S$ one has $[x,z] \subseteq S$.

Clearly V is full, and the intersection of a non-empty collection of full sets is once again full. Consequently, every non-empty set $S \subseteq V$ is contained in a smallest full set $\mathrm{fh}(S)$, the *full hull* of S. It is easy to see that one has

$$\mathrm{fh}(S) \;=\; \bigcup_{x,z\in S} [x,z] \;=\; (S + V^+) \cap (S - V^+).$$

Let us say that a non-empty subset S of a real or complex vector space V is *real balanced* if $s \in S$ and $\lambda \in [-1,1]$ imply $\lambda s \in S$. (In other words, S is a *balanced* subset of V, viewed as a vector space over $\mathbb{R}$.)

Proposition 1.19. *Let V be a real or complex ordered vector space and let $S \subseteq \mathrm{Re}(V)$ be a subset.*

(a) If S is convex, then so is $\mathrm{fh}(S)$.

(b) If S is real balanced, then so is $\mathrm{fh}(S)$.

Proof.

(a) Since S and V^+ are convex, the same is true for $S + V^+$ and $S - V^+$. It follows that $\mathrm{fh}(S)$ is convex, as it is the intersection of two convex sets.

(b) Let $y \in \mathrm{fh}(S)$ and $\lambda \in [-1,1]$ be given. Then we may choose $x, z \in S$ such that $x \leq y \leq z$ holds. Since S is real balanced, we have $\lambda x, \lambda z \in S$. For $\lambda \geq 0$ we find $\lambda x \leq \lambda y \leq \lambda z$, while for $\lambda < 0$ we find $\lambda z \leq \lambda y \leq \lambda x$. Either way, we see that λy lies between two elements of S, so we have $\lambda y \in \mathrm{fh}(S)$. ∎

Note: while order intervals are convex, this is no longer true for full sets in general. Examples of non-convex full sets are given in [AT07, page 6].

1.7 Order units

Definition 1.20. Let V be a real or complex vector space, and let $S \subseteq V$ be a non-empty subset. We say that $s \in S$ is an *internal point* if for each $x \in V$ there exists some $\lambda_0 > 0$ such that $s + \lambda x \in S$ holds for all $\lambda \in [0, \lambda_0]$. If S is convex, then this is equivalent to the requirement that for every $x \in V$ there exists some $\lambda_0 > 0$ such that $s + \lambda_0 x \in S$ holds.

If, in addition, V is a topological vector space, then every interior point of S is automatically an internal point, but the converse is not true.

Definition 1.21 (cf. [AT07, Lemma 1.7])**.** Let V be a real ordered vector space. An *order unit* is an element $u \in V^+$ satisfying any one (and therefore all) of the following equivalent criteria:

(1) For every $x \in V$ there is some $\alpha > 0$ such that $x \leq \alpha u$ holds;

(2) For every $x \in V$ there is some $\alpha > 0$ such that $-\alpha u \leq x \leq \alpha u$ holds;

(3) One has $V = \bigcup_{\alpha > 0} [-\alpha u, \alpha u]$;

(4) u is an internal point of V^+.

If V is a complex ordered vector space, then we say that $u \in V^+$ is an *order unit* if it is an order unit in the real ordered space $\mathrm{Re}(V)$.[4]

Clearly a cone with order units is automatically generating. The converse is true if V is finite dimensional (cf. [AT07, Lemma 3.2]), but not in general. A counterexample is given in Example 4.2.

If V is a real ordered topological vector space, then clearly every interior point of V^+ is an order unit. The converse is not true: order units need not be interior points; see Example 4.3. Remarkably, if the topology is completely metrisable, then the converse is also true, so the order units are precisely the interior points of the positive cone (cf. [AT07, Theorem 2.8]).

1.8 Locally full topologies

We examine one way for a cone and a topology to interact.

Definition 1.22. Let V be a real ordered topological vector space. We say that V is *locally full* if there is a neighbourhood base of 0 consisting of full sets.

Alternatively, we say that a cone K in a topological vector space $(V, \mathcal{T})$ is *normal* if $(V, \mathcal{T}, K)$ is locally full.

Adhering pedantically to our framework of complex ordered vector spaces, where all order theory is carried out inside $\mathrm{Re}(V)$, we have no concept of full sets outside $\mathrm{Re}(V)$. The obstacle is purely notational. However, for our purposes the real case suffices.

We list some of the basic properties of locally full spaces (without proof).

Theorem 1.23 ([AT07, Corollary 2.21])**.** *Let V be a real ordered topological vector space which is locally full. Then there is a neighbourhood base of 0 consisting of balanced, full open sets.*

If the topology is also locally convex, then there is a neighbourhood base of 0 consisting of balanced, full, convex open sets.

Theorem 1.24 ([AT07, Theorem 2.23])**.** *Let V be a real ordered topological vector space. Then the following are equivalent:*

(a) V is locally full;

(b) If $\{x_\lambda\}_{\lambda\in\Lambda}$, $\{y_\lambda\}_{\lambda\in\Lambda}$, $\{z_\lambda\}_{\lambda\in\Lambda}$ are nets in V (indexed by the same set) satisfying $x_\lambda \leq y_\lambda \leq z_\lambda$ for all $\lambda \in \Lambda$, and if $\{x_\lambda\}_{\lambda\in\Lambda}$ and $\{z_\lambda\}_{\lambda\in\Lambda}$ both converge to the same $v \in V$, then $\{y_\lambda\}_{\lambda\in\Lambda}$ also converges to v.

Furthermore, if V is metrisable, then the nets can be replaced by sequences.

Theorem 1.25 ([AT07, Corollary 2.24])**.** *Let V be a real or complex topological vector space, and let $K \subseteq V$ be a normal cone. Then $\overline{K}$ is a cone (as opposed to a wedge), and $\overline{K}$ is once again normal.*

Of particular interest are locally full norms.

Definition 1.26. Let V be a real ordered vector space, and let $\mu : V \to \mathbb{R}$ be a seminorm. We say that μ is

(a) *monotone* if $0 \leq y \leq z$ implies $\mu(y) \leq \mu(z)$;

(b) *fully monotone*[5] if $x \leq y \leq z$ implies $\mu(y) \leq \max(\mu(x), \mu(z))$.

Clearly a fully monotone seminorm is monotone, but the converse is not true. (Consider $V = \mathbb{R}^2$ with the coordinate-wise cone $\mathbb{R}^2_{\geq 0}$, the ℓ^p-norm $\|\cdot\|_p$ for some $p \in [1,\infty)$, and the vectors $x = (0,-1)$, $y = (1,-1)$, $z = (1,0)$.)

Furthermore, it is easy to see that a seminorm is fully monotone if and only if its closed (or open) unit ball is full. This is complemented by the following result regarding Minkowski functionals.

Proposition 1.27. *Let V be a real ordered vector space. Let $S \subseteq V$ be a full, convex, absorbing and balanced subset, and let $\mu_S : V \to \mathbb{R}$ be the Minkowski functional of S. Then μ_S is a fully monotone seminorm.*

Proof. It follows from [Rud91, Theorem 1.35] that μ_S is a seminorm. Now let $x \leq y \leq z$ be given. Since αS is full for all $\alpha > 0$, it is clear that $x, z \in \alpha S$ implies $y \in \alpha S$, so we find

$$\begin{aligned}\{\alpha > 0 \,:\, y \in \alpha S\} &\supseteq \{\alpha > 0 \,:\, x, z \in \alpha S\}\\ &= \{\alpha > 0 \,:\, x \in \alpha S\} \cap \{\alpha > 0 \,:\, z \in \alpha S\}.\end{aligned}$$

Since each of these sets is upwards closed (because S is balanced), it follows that

$$\begin{aligned}\mu_S(y) &= \inf\{\alpha > 0 \,:\, y \in \alpha S\}\\ &\leq \inf\{\alpha > 0 \,:\, x, z \in \alpha S\}\\ &= \inf\big(\{\alpha > 0 \,:\, x \in \alpha S\} \cap \{\alpha > 0 \,:\, z \in \alpha S\}\big)\\ &= \max\big(\mu_S(x), \mu_S(z)\big).\end{aligned}$$ ■

Note that a fully monotone norm is locally full: if $B \subseteq V$ denotes the open unit ball, then the set $\{\frac{1}{n}B : n \in \mathbb{N}^+\}$ is a full neighbourhood base of 0. A well-known result from the literature is that every locally full norm is equivalent to a fully monotone norm,

Theorem 1.28 (cf. [AT07, Theorem 2.38])**.** *Let V be a real ordered vector space. For a norm $\|\cdot\|$ on V, the following are equivalent:*

(1) *$\|\cdot\|$ is locally full;*

(2) *$\|\cdot\|$ is equivalent to a monotone norm;*

(3) *$\|\cdot\|$ is equivalent to a fully monotone norm.*

The implications (3) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (1) are trivial. The other implications are proven in [AT07, Theorem 2.38], together with various other equivalent criteria.[6] Furthermore, we give another proof of the implication (1) $\Longrightarrow$ (3) in Proposition 3.15, as part of the study of order unitisations.

For much more on the theory of normed ordered spaces, the interested reader is referred to [AT07, Section 2.5].

1.9 The (semi)norm generated by an order unit

In a real ordered vector space V with order unit $u \in V^+$, the order interval $[-u, u]$ is convex, balanced and absorbing, so we may consider its Minkowski functional μ_u, given by

$$\mu_u(x) \;=\; \inf\left\{\lambda \in \mathbb{R}_{>0} \,:\, -\lambda u \leq x \leq \lambda u\right\}.$$

Then μ_u defines a seminorm on V (cf. [Rud91, Theorem 1.35]). Furthermore, since $[-u, u]$ is full, it follows from Proposition 1.27 that μ_u is fully monotone. It should be pointed out that μ_u is not generally a norm; see Example 4.4.

Theorem 1.29 (cf. [Kad51a, Lemma 2.3] and [AT07, Theorem 2.55])**.** *Let V be a real ordered vector space containing an order unit $u \in V^+$. Then:*

(a) For every $x \in V$ we may define $\alpha_u(x) := \sup\{\alpha \in \mathbb{R} : \alpha u \leq x\}$ and $\omega_u(x) := \inf\{\omega \in \mathbb{R} : x \leq \omega u\}$; these numbers are well-defined and satisfy $\alpha_u(x) \leq \omega_u(x)$;

(b) For $x, y \in V$ with $x \leq y$ one has $\alpha_u(x) \leq \alpha_u(y)$ and $\omega_u(x) \leq \omega_u(y)$;

(c) For $x \in V$ one has $\mu_u(x) = \max(-\alpha_u(x), \omega_u(x)) = \max(|\alpha_u(x)|, |\omega_u(x)|)$;

Additionally, assume that V is Archimedean. Then:

(d) For every $x \in V$ one has $\alpha_u(x) \cdot u \leq x \leq \omega_u(x) \cdot u$ (so the maximum and the minimum in part (a) are attained);

(e) μ_u defines a fully monotone norm on V;

(f) The closed unit ball of μ_u is precisely the order interval $[-u, u]$;

(g) The positive cone V^+ is μ_u-closed.

Proof.

(a) Since u is an order unit, we may choose $\lambda_0 > 0$ such that $-\lambda_0 u \leq x \leq \lambda_0 u$ holds. Note that the set $\{\omega \in \mathbb{R} : x \leq \omega u\}$ is non-empty (it contains λ_0) and bounded below by $-\lambda_0$ (after all, if $x \leq \omega u$ holds, then we have $-\lambda_0 u \leq x \leq \omega u$, hence $-\lambda_0 \leq \omega$). It follows that $\omega_u(x)$ is well-defined. An analogous argument shows that $\alpha_u(x)$ is well-defined.

If $\alpha u \leq x$ holds, then α is a lower bound for the set $\{\omega \in \mathbb{R} : x \leq \omega u\}$, by the preceding argument, so we find $\alpha \leq \omega_u(x)$. We see that $\omega_u(x)$ is an upper bound for the set $\{\alpha \in \mathbb{R} : \alpha u \leq x\}$, so it follows that $\alpha_u(x) \leq \omega_u(x)$ holds.

(b) Clearly we have $\{\alpha \in \mathbb{R} : \alpha u \leq x\} \subseteq \{\alpha \in \mathbb{R} : \alpha u \leq y\}$, hence $\alpha_u(x) \leq \alpha_u(y)$. The inequality $\omega_u(x) \leq \omega_u(y)$ follows analogously.

(c) First of all, note that we have

$$-\alpha_u(x) \;=\; \inf\{\alpha \in \mathbb{R} : -\alpha u \leq x\}.$$

Secondly, note that we may write

$$\big\{\lambda \in \mathbb{R} : -\lambda u \leq x \leq \lambda u\big\} \;=\; \big\{\alpha \in \mathbb{R} : -\alpha u \leq x\big\} \cap \big\{\omega \in \mathbb{R} : x \leq \omega u\big\}.$$

Since both sets in the right-hand side are upwards closed, so is their intersection, and we have

$$\inf\big\{\lambda \in \mathbb{R} : -\lambda u \leq x \leq \lambda u\big\} \;=\; \max\big(-\alpha_u(x)\,,\,\omega_u(x)\big).$$

Of course the set in the left-hand side does not contain any negative numbers. Furthermore, possibly removing the least element of an upwards closed set doesn't affect the infimum, so we find

$$\mu_u(x) \;=\; \max\big(-\alpha_u(x)\,,\,\omega_u(x)\big).$$

In order to prove the second expression for $\mu_u(x)$, recall from part (a) that we have $\alpha_u(x) \leq \omega_u(x)$, and therefore also $-\alpha_u(x) \geq -\omega_u(x)$. As such, we find

$$\begin{aligned}
\mu_u(x) \;&=\; \max\big(-\alpha_u(x)\,,\,\omega_u(x)\big)\\
&= \max\big(\alpha_u(x)\,,\,-\alpha_u(x)\,,\,\omega_u(x)\,,\,-\omega_u(x)\big)\\
&= \max\big(|\alpha_u(x)|\,,\,|\omega_u(x)|\big).
\end{aligned}$$

(d) For all $n \in \mathbb{N}^+$ we have $x \leq (\omega_u(x) + \frac{1}{n})u$, that is, $n(x - \omega_u(x)\cdot u) \leq u$. By the Archimedean property, we find $x - \omega_u(x)\cdot u \leq 0$, or equivalently: $x \leq \omega_u(x)\cdot u$. The inequality $\alpha_u(x)\cdot u \leq x$ follows analogously.

(e) It was already established that μ_u is a fully monotone seminorm. To show that it is a norm in the present setting, let $x \in V$ be given with $\mu_u(x) = 0$. By part (c) we have $\alpha_u(x) = \omega_u(x) = 0$, so now it follows from part (d) that $0 \leq x \leq 0$ holds. This shows that x must be zero, so μ_u is a norm.

(f) For $x \in [-u, u]$ we clearly have $\mu_u(x) \leq 1$. Conversely, if $\mu_u(x) \leq 1$ holds, then we have $-u \leq x \leq u$ by parts (c) and (d).

(g) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in V^+ converging to some $x \in V$. Let $n \in \mathbb{N}^+$ be given, then we may choose some $m \in \mathbb{N}^+$ with $-\frac{1}{n}u \leq x - x_m \leq \frac{1}{n}u$. Now we have

$$x \,=\, (x - x_m) + x_m \,\geq\, x - x_m \,\geq\, -\tfrac{1}{n}u.$$

Thus, for all $n \in \mathbb{N}^+$ we have $-nx \leq u$. Using the Archimedean property, we find $-x \leq 0$, or equivalently, $x \geq 0$. ∎

Whenever μ_u is a norm, we usually denote it by $\|\cdot\|_u$ instead of μ_u. One of the useful properties of this norm (in the Archimedean setting) is that positive linear maps from $(V, \|\cdot\|_u)$ tend to be continuous.

Proposition 1.30. *Let V be a real Archimedean ordered vector space containing an order unit $u \in V^+$, let W be a (real) locally full topological ordered vector space, and let $f : V \to W$ be a positive linear map. Then f is continuous.*

Proof. Let $T \subseteq W$ be a neighbourhood of 0 in W, then we may choose a full neighbourhood $S \subseteq T$ of 0. Since every neighbourhood of 0 is absorbing, we may choose some $\varepsilon > 0$ such that $-\varepsilon f(u), \varepsilon f(u) \in S$ holds. By positivity of f, for all $x \in [-\varepsilon u, \varepsilon u]$ we have $-\varepsilon f(u) \leq f(x) \leq \varepsilon f(u)$. Since S is full, it follows that $f(x) \in S$ holds as well. Therefore we find $[-\varepsilon u, \varepsilon u] \subseteq f^{-1}(S) \subseteq f^{-1}(T)$. Since $[-u, u]$ coincides with the closed unit ball of V, it is clear that $f^{-1}(T)$ is a neighbourhood of 0 in V. ∎

Corollary 1.31. *Let V be a real Archimedean ordered vector space containing an order unit $u \in V^+$. Then all positive linear functionals $V \to \mathbb{R}$ are continuous.*

Remark 1.32. Note that the definition of μ_u (and $\|\cdot\|_u$) only works in the real case. In the complex case, the order interval $[-u, u]$ is not absorbing (as it is a subset of the real subspace $\mathrm{Re}(V) \subsetneq V$), so an order unit only defines a seminorm on $\mathrm{Re}(V)$ in this case. There are ways to extend it to a seminorm on all V, but there is no canonical way to do so.

If V is a complex vector space with a complex conjugation $\bar{\ } : V \to V$ and a norm $\|\cdot\|_1$ on $\mathrm{Re}(V)$, then we say that a norm $\|\cdot\|_2$ defined on all of V is a *reasonable complexification of* $\|\cdot\|_1$ if it extends $\|\cdot\|_1$ and furthermore makes the complex conjugation isometric.

Every norm $\|\cdot\|_1$ on $\mathrm{Re}(V)$ admits a reasonable complexification norm. All reasonable complexifications of $\|\cdot\|_1$ are equivalent, and each is complete if and only if $\|\cdot\|_1$ is complete. There is a smallest and a largest reasonable complexification. The smallest is given by

$$\|v\|_\epsilon \,:=\, \sup_{|\lambda|=1} \|\mathrm{Re}(\lambda v)\|_1,$$

and the largest is given as the Minkowski functional of the balanced convex hull (inside V) of the open unit ball of $\|\cdot\|_1$.

The theory of reasonable complexification norms has received some attention in the literature; see for instance [MST99]. The problem can be seen as a special case of the theory of tensor product norms (cf. [Rya02]), where V is interpreted as the (algebraic) tensor product $\mathrm{Re}(V) \otimes_{\mathbb{R}} \mathbb{C}$ of real vector spaces.

Complexification norms are considered to be beyond the scope of this thesis, so we will not say much more about this. (In particular, we do not investigate what might be the best choice of complexification in certain special cases.) As a result, some of the theory developed in this thesis is incomplete in the complex case. This might be a direction for further study.

1.10 End notes

1. (page 3) The intersection of a *non-empty* family of cones is again a cone, but the family of cones containing S might be empty. For instance, if the vector space V is non-zero, then there is no cone containing all of V.

2. (page 5) We avoid the word *real* in the context of the induced conjugation. This terminology would introduce ambiguity, because a real linear map $V \to W$ is usually understood to be a map which is $\mathbb{R}$-linear rather than $\mathbb{C}$-linear. Similarly, we should not call these maps *self-adjoint* or *Hermitian*, for this creates ambiguity in the case where $V = W$ is a finite-dimensional Hilbert space (so that $L(V, W) \cong M_n(\mathbb{C})$ is a C^*-algebra). Indeed, the induced conjugation on $M_n(\mathbb{C})$ is simply the entry-wise conjugation, so a self-conjugate linear map need not be self-adjoint (or vice versa).

3. (page 6) A word of warning: while every ordered vector space is in particular an ordered group, the Archimedean property commonly used in the theory of ordered groups is a different one! In the setting of ordered groups, it is usually assumed that the order is linear (that is, total), and the Archimedean property is that every non-zero element is an order unit. A classical result in this context is that every Archimedean, linearly ordered, abelian group is isomorphic (as an ordered group) to a subgroup of $(\mathbb{R}, \mathbb{R}_{\geq 0})$. While this is very far from our terminology, the content of this classical result is closely related to Proposition 2.7.

4. (page 8) In the complex setting, note that we have $V^+ \subseteq \mathrm{Re}(V)$, so V^+ cannot contain internal points (relative to all of V). Instead, we have that u is an internal point of V^+ relative to the subspace $\mathrm{Re}(V) \supseteq V^+$.

5. (page 9) Terminology invented by the author ("fully monotone seminorm").

6. (page 10) The concept of fully monotone norms is not used in [AT07], so Theorem 1.28 is slightly stronger than [AT07, Theorem 2.38]. However, this stronger result follows immediately from the proof of [AT07, Theorem 2.38, implication (1) $\implies$ (2)]. It is merely a matter of missing terminology.

2 Semisimple ordered vector spaces

A common theme in functional analysis is to represent various types of spaces as a space of (continuous) functions. This entails finding an injective linear map $\psi : V \to \mathbb{F}^\Omega$ (or $\psi : V \to C(\Omega, \mathbb{F})$ if V is a normed space) preserving some of the additional structure of V (e.g. multiplication or lattice operations). In this chapter we examine questions of this type for ordered vector spaces.

The question in this setting is whether or not V admits an injective positive linear map $V \to \mathbb{F}^\Omega$ to some space of functions. Similarly, we ask which spaces admit a bipositive representation $V \to \mathbb{F}^\Omega$. If V carries a topology, then we equip $\mathbb{F}^\Omega$ with the product topology, and we restrict our attention to *continuous* representations $V \to \mathbb{F}^\Omega$ of the aforementioned types (injective and positive, or bipositive). If V is a normed space admitting such a continuous representation, then we show how to choose a Hausdorff topological space Ω' and a continuous representation $V \to C(\Omega', \mathbb{F})$ of the same type as the representation $V \to \mathbb{F}^\Omega$.

Our treatment constitutes an extension of the ideas and techniques from [Kad51a, Section 2] to a more general setting.

The results are proven only for real ordered vector spaces; the modifications needed in the complex case are listed in Section 2.9 below.

2.1 Ideals and quotients of ordered vector spaces

Let V be an ordered vector space. An *order ideal* is a linear subspace $I \subseteq V$ such that $y \in I$ and $0 \leq x \leq y$ imply $x \in I$.[1] It is easy to see that the order ideals are precisely the full subspaces of V. We will simply call these *ideals* if no ambiguity can arise (i.e. the space V does not have additional algebraic structure). The terminology is explained, in part, by Lemma 2.2 below.

Clearly $\{0\}$ and V are ideals in every ordered vector space. We call these the *trivial ideals*, so an ideal $I \subseteq V$ is *non-trivial* if we have $\{0\} \subsetneq I \subsetneq V$. Furthermore, an ideal $I \subseteq V$ is called *proper* if $I \neq V$ holds.

Note that the intersection of a non-empty collection of ideals is once again an ideal. Thus, for every non-empty set $S \subseteq V$ there is a smallest ideal containing S (namely the intersection of all ideals containing S). We call this the *ideal generated by* S. Ideals generated by a singleton are called *principal*.

If $y \in V$ is incomparable with 0, then the principal ideal generated by y is simply $\operatorname{span}(y)$. If y is positive or negative, then the principal ideal generated by y is $\bigcup_{\alpha \in \mathbb{R}} [-\alpha y, \alpha y]$. In particular, if $\dim(V) > 1$ holds, then the principal ideal generated by y is proper if and only if neither y nor $-y$ is an order unit.[2]

Proposition 2.1. *Let V be a real ordered vector space with positive cone V^+, and let $I \subseteq V$ be an order ideal. Then the image of V^+ under the natural map $\pi : V \to V/I$ is a cone (as opposed to a wedge).*

Proof. Clearly the image of a wedge under a linear map is again a wedge. We prove that $\pi(V^+)$ is a cone. Suppose that $z \in \pi(V^+) \cap -\pi(V^+)$ holds, that is, we can write $z = \pi(x) = -\pi(y)$ with $x, y \in V^+$. Then we have $\pi(x + y) = 0$, hence $x + y \in I$. Furthermore we have $0 \leq x \leq x + y$ and $0 \leq y \leq x + y$, so we find $x, y \in I$ (since I is an ideal). It follows that $z = 0$ holds, proving that $\pi(V^+)$ is a cone. ∎

It should be pointed out that the linear image of a pointed cone is in general merely a wedge. In fact, the order ideals are precisely those subspaces $I \subseteq V$ for which the image of V^+ under the natural map $\pi : V \to V/I$ is a cone. Indeed, if $0 \leq x \leq y$ are such that $y \in I$ and $x \notin I$ hold, then we have $\pi(x)+\pi(y-x) = 0$, where both $\pi(x)$ and $\pi(y - x)$ are non-zero elements of $\pi(V^+)$, so in this case the image wedge contains both $\pi(x)$ and $-\pi(x) = \pi(y - x)$.

We can say something stronger: the ideals are precisely the subspaces which occur as kernels of positive linear maps, which explains their name.

Lemma 2.2. *A subset $I \subseteq V$ is an order ideal if and only if I is the kernel of some positive linear map.*

Proof. It is easy to see that the kernel of a positive linear map is an order ideal. For the converse, let $I \subseteq V$ be an order ideal, and consider the natural map $\pi : V \to V/I$. Clearly π is positive as a map $(V, V^+) \to (\pi(V), \pi(V^+))$, and we have $I = \ker(\pi)$. ■

If V is an ordered vector space and I is an ideal, we will always understand the quotient space V/I to be ordered with the *quotient cone*, that is, the image of the cone V^+ under the natural map $V \to V/I$ (as in Proposition 2.1).

Some naturally occurring examples of order ideals are given in Section 4.2.

The homomorphism and isomorphism theorems for ordered vector spaces are not quite as well-behaved as their counterparts for algebras (or rings). The first isomorphism theorem is weaker in this setting, and the second isomorphism theorem fails.[3] We state the remaining results without proof; they are analogous to the corresponding statements for vector spaces (or modules, rings, etcetera).

Theorem 2.3 (Homomorphism theorem)**.** *Let $\phi : V \to W$ be a positive linear map between real ordered vector spaces V and W, and let $I \subseteq V$ be an ideal with $I \subseteq \ker(\phi)$. Then there is a unique positive linear map $\tilde{\phi} : V/I \to W$ such that ϕ is equal to the composition*

$$V \xrightarrow{\pi} V/I \xrightarrow{\tilde{\phi}} W.$$

Theorem 2.4 (First isomorphism theorem)**.** *Let $\phi : V \to W$ be a positive linear map between real ordered vector spaces V and W. Then the natural linear isomorphism $V/\ker(\phi) \cong \operatorname{ran}(\phi)$ defines an order isomorphism*

$$(V/\ker(\phi), V^+/\ker(\phi)) \xrightarrow{\sim} (\phi(V), \phi(V^+)).$$

Remark 2.5. Note that we pass to a different cone in $\operatorname{ran}(\phi)$. We cannot expect the linear isomorphism $V/\ker(\phi) \cong \operatorname{ran}(\phi)$ to be bipositive as a map

$$(V/\ker(\phi), V^+/\ker(\phi)) \to (\operatorname{ran}(\phi), W^+ \cap \operatorname{ran}(\phi)),$$

which is perhaps a more straightforward choice of cone in the codomain. This is because a positive linear isomorphism is not necessarily bipositive: the cone in the codomain can be larger.

Theorem 2.6 (Third isomorphism theorem)**.** *Let V be a real ordered vector space, and let $I, J \subseteq V$ be ideals with $I \subseteq J \subseteq V$. Then J/I is an ideal of V/I. Furthermore, every ideal of V/I is of the form J/I for some ideal $I \subseteq J \subseteq V$, so the ideals of V/I are in bijective correspondence with the ideals $I \subseteq J \subseteq V$. Finally, the natural isomorphism $(V/I)/(J/I) \cong V/J$ is bipositive, that is, an isomorphism of ordered vector spaces.*

2.2 Maximal ideals and simple ordered spaces

An order ideal $I \subseteq V$ is called *maximal* if I is proper and there are no ideals lying strictly between I and V. A real ordered vector space V is called *simple* if the zero ideal is maximal, or equivalently: V has exactly two ideals.

Proposition 2.7. *A real ordered vector space V is simple if and only if V is one-dimensional.*

Proof. Clearly every one-dimensional ordered vector space (regardless of its cone) is simple: the only subspaces are $\{0\}$ and V, which are both ideals. If V is zero-dimensional, then the zero ideal is not proper and therefore not maximal. Assume now that V has dimension at least 2. We distinguish two cases:

- If V has an element $y \in V$ which is neither positive nor negative, then $\operatorname{span}(y)$ is a non-trivial ideal, so V is not simple.
- Assume that every element of V is either positive or negative, that is, V is linearly ordered. Choose linearly independent vectors $x, y \in V$ with $x \geq 0$ and $y \leq 0$. We consider the line segment joining x and y. Since V^+ and $-V^+$ are convex, there is some $\alpha_0 \in [0,1]$ such that $\alpha x + (1-\alpha)y$ is positive for all $\alpha > \alpha_0$ and negative for all $\alpha < \alpha_0$. We may assume without loss of generality that the choice of $\alpha = \alpha_0$ also yields a positive element (if it is negative, pass to $(x', y') := (-y, -x)$ instead). But now $z := \alpha_0 x + (1-\alpha_0)y$ is a positive element which is not an internal point of V^+. Consequently, z is not an order unit. Since z is furthermore non-zero (x and y are linearly independent), it follows that the principal ideal generated by z is non-trivial. ∎

Corollary 2.8. *An ideal $I \subseteq V$ is maximal if and only if V/I is one-dimensional.*

Proof. By Theorem 2.6, the ideals of V/I are in bijective correspondence with the ideals $I \subseteq J \subseteq V$, so we see that V/I is simple if and only if I is maximal. Consequently, by Proposition 2.7, I is maximal if and only if V/I is one-dimensional. ∎

Corollary 2.9. *If $f : V \to \mathbb{R}$ is a non-zero positive linear functional, then $\ker(f)$ is a maximal ideal.*

Proof. Clearly $\ker(f)$ is an ideal. By the first isomorphism theorem, we have an isomorphism $V/\ker(f) \cong \mathbb{R}$ of vector spaces. Therefore $V/\ker(f)$ is one-dimensional, and it follows from Corollary 2.8 that $\ker(f)$ is maximal. ∎

We give a partial converse of Corollary 2.9. Note that there are two non-isomorphic simple ordered vector spaces: $(\mathbb{R}, \{0\})$ and $(\mathbb{R}, \mathbb{R}_{\geq 0})$. In general, the quotient of V by a maximal ideal can be order isomorphic with either. However, since the identity $(\mathbb{R}, \{0\}) \to (\mathbb{R}, \mathbb{R}_{\geq 0})$ is positive, every maximal ideal also gives rise to a positive linear functional, which is unique up to a scalar.[4] Thus, we see that the maximal ideals are precisely the kernels of positive linear functionals. Rephrased in terms of convex geometry, we find that the maximal ideals are precisely the *supporting hyperplanes* of the positive cone V^+.

One particular consequence of the above is worth mentioning: a real ordered vector space V admits a non-zero positive linear functional $V \to \mathbb{R}$ if and only if there is a maximal order ideal $I \subseteq V$. We show that this is the case if V has an order unit.

Proposition 2.10. *Let V be a real ordered vector space containing an order unit $u \in V^+$. Then every proper order ideal is contained in a maximal order ideal.*

Proof. Let $I \subseteq V$ be a proper ideal, and consider the set $\mathscr{I}$ of all proper ideals $J \supseteq I$. We show that every chain in $\mathscr{I}$ has an upper bound (in $\mathscr{I}$). To that end, let $\mathcal{C} \subseteq \mathscr{I}$ be a chain. If $\mathcal{C}$ is empty, then $I \in \mathscr{I}$ is an upper bound for $\mathcal{C}$. Assume that $\mathcal{C}$ is non-empty, then we may define $M := \bigcup \mathcal{C}$. Note that M is an ideal containing I. In order to see that M is proper, observe that we have $u \notin M$, since u does not belong to any of the proper ideals $J \in \mathcal{C}$. It follows that $M \in \mathscr{I}$ is an upper bound for C, so we see that every chain in $\mathscr{I}$ has an upper bound in $\mathscr{I}$. Therefore $\mathscr{I}$ has a maximal element by Zorn's lemma. ∎

Corollary 2.11. *Let V be a non-zero real ordered vector space with order unit. Then V has a maximal order ideal, or equivalently: there is a non-zero positive linear functional $V \to \mathbb{R}$.*

Proof. The zero ideal is proper (since V is non-zero), so it is contained in a maximal ideal. ∎

The preceding results are analogous to Krull's theorem for commutative rings with unit. It should be pointed out that the conclusion fails if V does not have an order unit; see examples 4.8 and 4.15. A similar situation occurs in ring theory: a ring without unit does not necessarily have a maximal ideal.

We conclude this section with a simple application of the theory developed so far, relating the dual wedge $(V^+)'$ to ideals containing V^+. Note that the subspace $V^+ - V^+$ generated by V^+ is trivially an ideal, so it is also the ideal generated by V^+. In this setting, we have the following.

Theorem 2.12. *For a real ordered vector space V, the following are equivalent:*

(1) *The dual wedge $(V^+)'$ is a cone;*

(2) *For every maximal ideal I, the quotient V/I is isomorphic with $(\mathbb{R}, \mathbb{R}_{\geq 0})$.*

(3) *The positive cone V^+ is not contained in a maximal ideal.*

Proof. (1) $\Longrightarrow$ (2). Let $I \subseteq V$ be a maximal ideal, and choose a positive linear functional $f : V \to \mathbb{R}$ with kernel I. (Recall that f is determined uniquely by I, up to a scalar.) Since $(V^+)'$ is a cone, we know that $-f$ is not a positive linear functional. This is not possible if V/I is isomorphic with $(\mathbb{R}, \{0\})$, so we conclude that V/I is isomorphic with $(\mathbb{R}, \mathbb{R}_{\geq 0})$ instead.

(2) $\Longrightarrow$ (3). For any given maximal ideal $I \subseteq V$, the image of V^+ under the natural map $V \to V/I$ is non-zero, so in particular V^+ is not contained in I.

(3) $\Longrightarrow$ (1). Let $f : V \to \mathbb{R}$ be a non-zero positive linear functional. Then $\ker(f)$ is a maximal ideal, so by assumption we have $V^+ \not\subseteq \ker(f)$. It follows that there exists some $v \in V^+$ with $f(v) > 0$, so we see that $-f$ is not positive. As a consequence, we have $(V^+)' \cap -(V^+)' = \{0\}$. ∎

We see that the question posed in Proposition 2.10 is of some interest: it is useful to know whether or not $V^+ - V^+$ is contained in a maximal ideal. However, the result of Proposition 2.10 is of no use here: if V has an order unit, then V^+ is automatically generating, so the ideal $V^+ - V^+$ generated by V^+ is not proper.

The following well-known result follows immediately from Theorem 2.12.

Corollary 2.13 (cf. [AT07, Corollary 2.14])**.** *Let V be a real ordered vector space with generating cone. Then the dual wedge $(V^+)'$ is a cone.*

2.3 The order radical and order semisimplicity

In this section we come to our main question of this chapter: which spaces admit a positive and injective representation as a space of functions? We characterise these in terms of the order radical, as defined in Definition 2.14 below.

For an arbitrary index set S, we let $\mathbb{R}^S$ denote the ordered vector space of all functions $S \to \mathbb{R}$, equipped with the pointwise cone.

Definition 2.14. Let V be a real ordered vector space and let $(V^+)' \subseteq V'$ be the dual wedge. The *order radical* of V is the set

$$\operatorname{ora}(V) := \big\{x \in V \,:\, f(x) = 0 \text{ for all } f \in (V^+)'\big\}.$$

Proposition 2.15. *The order radical is equal to the intersection of all maximal order ideals (where the empty intersection is understood to mean all of V). In particular,* $\operatorname{ora}(V)$ *is an ideal.*

Proof. In light of the bijective correspondence between maximal order ideals and positive linear functionals (up to a scalar), we have

$$\begin{aligned}\operatorname{ora}(V) &= \big\{x \in V \,:\, x \in \ker(f) \text{ for all } f \in (V^+)'\big\}\\ &= \big\{x \in V \,:\, x \in I \text{ for every maximal ideal } I \subseteq V\big\}\\ &= \bigcap_{\substack{I \subseteq V\\ \text{maximal}}} I.\end{aligned}$$

The second conclusion follows since the intersection of a (possibly empty) collection of ideals is once again an ideal. ■

Proposition 2.16. *Let V be a real ordered vector space. Then the following are equivalent:*

(1) *There exists a set S and an injective positive linear map $V \to \mathbb{R}^S$;*

(2) *The order radical of V is zero;*

(3) *The dual wedge $(V^+)' \subseteq V'$ separates points.*

Proof. (1) $\Longrightarrow$ (2). Let $\varphi : V \to \mathbb{R}^S$ be injective and positive. For every $s \in S$ we get a positive linear functional $f_s(v) := \varphi(v)_s$. Since φ is injective, we have $f_s(v) = 0$ for all $s \in S$ if and only if $v = 0$. Therefore we have $\operatorname{ora}(V) = \{0\}$.

(2) $\implies$ (3). If $x \in V$ is non-zero, then we have $x \notin \mathrm{ora}(V)$, so there exists some $f \in (V^+)'$ with $f(x) \neq 0$.

(3) $\implies$ (1). Set $S := (V^+)'$, and let $\varphi : V \to \mathbb{R}^S$ be the evaluation map $\varphi(v) := (s \mapsto s(v))$. If $v \in V$ is positive, then so is $s(v)$ for all $s \in S$, so we see that φ is positive. Furthermore, since $(V^+)'$ separates points, it is clear that φ is injective. ■

It should be pointed out that the set S in the preceding proposition is far from being unique. In fact, the proof reveals that choosing a set S and an injective positive linear map $V \to \mathbb{R}^S$ is the same as choosing a separating multiset of positive linear functionals.[5]

An ordered vector space satisfying any (and therefore all) of the criteria from Proposition 2.16 will be called *order semisimple*.[6] Clearly every subspace of $\mathbb{R}^S$ is order semisimple; we will encounter more examples later on.

While the presence of an order unit is enough to ensure that the dual wedge $(V^+)'$ is non-empty (Corollary 2.11), it does not guarantee that $(V^+)'$ also separates points. An example of this is the non-Archimedean order unitisation from Section 3.1 below.

2.4 Topologically order semisimple spaces

Until now we only considered abstract ordered vector spaces without topology. The next step is to consider what happens when V is at the same time a (real) ordered vector space and a (real) topological vector space. Recall: we assume all topological vector spaces to be Hausdorff, and we do not assume any form of compatibility between the topology and the positive cone.

Where general order ideals are precisely the kernels of positive linear maps (cf. Lemma 2.2), we have that *closed* order ideals are precisely the kernels of *continuous* positive linear maps.[7] In particular, if $I \subseteq V$ is a maximal ideal, then the positive linear functional $V \to \mathbb{R}$ determined by I (up to a scalar) is continuous if and only if I is closed (cf. [Rud91, Theorem 1.18]). We give an example of a discontinuous positive linear functional in Example 4.9; it follows that maximal ideals are not necessarily closed. In this context, we mention the following theorem (without proof).

Theorem 2.17 ([AT07, Corollary 2.34])**.** *Let V be a real topological ordered vector space. If the topology is completely metrisable and the positive cone is closed and generating, then every positive linear functional is continuous.*

For an arbitrary index set S, let $\mathbb{R}^S$ be equipped with the (locally convex) topology of pointwise convergence. Our goal will be to formulate topological analogues of the theory from Section 2.3, where we restrict our attention to *closed* ideals, *continuous* positive linear functionals, and a *continuous* positive and injective representation $V \to \mathbb{R}^S$. A more general question is to find a continuous representation $V \to W$ to a space of functions $W \subseteq \mathbb{R}^S$ carrying a stronger topology; this is addressed in Section 2.7.

Definition 2.18. Let V be a topological ordered vector space and let $(V^+)^*$ be the topological dual wedge, i.e. the set of all continuous positive linear functionals $V \to \mathbb{R}$. The *topological order radical* of V is the set

$$\mathrm{tora}(V) := \big\{ v \in V \,:\, f(v) = 0 \text{ for all } f \in (V^+)^* \big\}.$$

Propositions 2.15 and 2.16 have the following topological counterparts.

Proposition 2.19. *The topological order radical is equal to the intersection of all closed maximal order ideals (where the empty intersection is understood to mean all of V). In particular,* $\operatorname{tora}(V)$ *is a closed ideal.*

Proposition 2.20. *For a topological ordered vector space V the following are equivalent:*

(1) *There exists a set S and a continuous, injective and positive linear map $V \to \mathbb{R}^S$; where $\mathbb{R}^S$ is equipped with the product topology (i.e. the topology of pointwise convergence);*

(2) *The topological order radical of V is zero;*

(3) *The topological dual wedge $(V^+)^* \subseteq V^*$ separates points.*

In the proof of (3) $\implies$ (1), one has to use the universal property of the product topology: a map $\varphi : V \to \mathbb{R}^S$ is continuous if and only if for every $s \in S$ the map $\varphi_s : V \to \mathbb{R}$, $v \mapsto \varphi(v)_s$ is continuous. Apart from that, the proofs can be copied verbatim.

Perhaps not surprisingly, we say that a topological ordered vector space V is *topologically order semisimple* if it meets any (and therefore all) of the properties of Proposition 2.20.

Remark 2.21. Just like in the non-topological setting, we have that choosing a set S and a continuous, injective and positive linear map $\varphi : V \to \mathbb{R}^S$ is the same as choosing a separating multiset of continuous positive linear functionals $V \to \mathbb{R}$. After removing all superfluous duplicates, we may view S as a subset of V^*, so that we can equip S with the relative weak-$*$ topology $\sigma(V^*, V)$. For every $v \in V$, we have that $\varphi(v)$ is the evaluation map $\widehat{v} : S \to \mathbb{R}$, $s \mapsto s(v)$, which is continuous as a map $(S, \sigma(V^*, V)) \to \mathbb{R}$. It follows that $\operatorname{ran}(\varphi)$ is contained in $C(S, \mathbb{R})$, so we automatically get a representation as a space of *continuous* functions. We do not pursue this idea any further until Section 2.7.

Remark 2.22. If V is a vector space without topology, then its algebraic dual V' separates points.[8] As such, the $\sigma(V, V')$-topology turns V into a (Hausdorff) locally convex space whose topological dual V^* is equal to the algebraic dual V' (cf. [Rud91, Theorem 3.10]). It follows that a real ordered vector space is order semisimple if and only if it is topologically order semisimple with respect to the $\sigma(V, V')$-topology.

Assume now that we introduce another vector space topology $\mathcal{T}$ on an order semisimple space V. If $\mathcal{T}$ is stronger than the $\sigma(V, V')$-topology ($\sigma(V, V') \subseteq \mathcal{T}$), then all linear functionals remain continuous, so V remains topologically order semisimple. However, if $\mathcal{T}$ is weaker than $\sigma(V, V')$, or incomparable, then it has fewer continuous linear functionals, so the topological dual wedge becomes smaller. (This is because $\sigma(V, V')$ is the weakest topology making all linear functionals continuous.) As such, $(V^+)^*$ might fail to separate points, so we see that V is not necessarily *topologically* order semisimple. Examples of spaces like this are given in examples 4.8 and 4.10. Additionally, in Section 4.6 we exhibit a class of order semisimple spaces which do not admit a topologically order semisimple norm.

2.5 Duality and the bipolar theorem for wedges

Now that we have addressed positive injective representations in some detail, we shift our attention towards bipositive representations. These turn out to be closely related to duality theory. In this section we state and prove the bipolar theorem for wedges, which will be used to study bipositive representations in Section 2.6 below.

Let V and W be real vector spaces. Recall that a *dual pairing* or *duality* is a bilinear map $\langle\,\cdot\,,\,\cdot\,\rangle : V\times W\to\mathbb{R}$ which is both left and right non-degenerate:

(2.23.1) if $\langle x_0, y\rangle = 0$ holds for all $y\in W$, then $x_0 = 0$;

(2.23.2) if $\langle x, y_0\rangle = 0$ holds for all $x\in V$, then $y_0 = 0$.

Everything about the duality is symmetric in V and W, so all the results we prove have analogues where the roles of V and W are reversed.

Given a duality $\langle\,\cdot\,,\,\cdot\,\rangle : V\times W\to\mathbb{R}$, we obtain a linear map $\psi : W\to V'$ which associates with a vector $y_0\in W$ the evaluation functional $\widehat{y_0} : V\to\mathbb{R}$ given by $x\mapsto\langle x, y_0\rangle$. It follows from (2.23.2) that ψ is injective, so we can think of W as being a subspace of the (algebraic) dual of V.

Conversely, if W is a separating space of linear functionals on V, then the bilinear map $V\times W\to\mathbb{R}$, $(x,f)\mapsto f(x)$ is a dual pairing. After all, (2.23.1) follows from the assumption that W separates points, and (2.23.2) follows from what it means for two functions to be different. The following two special cases are of interest to us:

- If V is a vector space without topology, then its algebraic dual separates points (by Remark 2.22), so we have a natural dual pairing $V\times V'\to\mathbb{R}$.
- Likewise, if V is a topological vector space whose topological dual V^* separates points, then we have a natural dual pairing $V\times V^*\to\mathbb{R}$. The condition is always met if V is locally convex; this is a consequence of the Hahn–Banach separation theorems (cf. [Rud91, unnamed corollary after Theorem 3.4]). For general topological vector spaces, however, the topological dual V^* does not always separate points; see Example 4.8.

Central to our theory of bipositive representations is the wedge analogue of the (one-sided) bipolar theorem.

Definition 2.24. Let $\langle\,\cdot\,,\,\cdot\,\rangle : V\times W\to\mathbb{R}$ be a dual pairing. For a subset $S\subseteq V$, define the *polar wedge*[9] to be the set

$$S^{\smile} := \big\{y\in W\,:\,\langle s, y\rangle\geq 0\text{ for all }s\in S\big\}.$$

Similarly, for a subset $T\subseteq W$ we define the *prepolar wedge* to be the set

$$^{\smile}T := \big\{x\in V\,:\,\langle x, t\rangle\geq 0\text{ for all }t\in T\big\}.$$

It is readily verified that every (pre)polar wedge is indeed a wedge, as the name suggests. Furthermore, if S (resp. T) is a wedge, then $S^{\smile}$ (resp. $^{\smile}T$) coincides with the dual wedge as defined in [AT07, Section 2.2], and also with the one-sided polar, as defined, for instance, in [Bou87, page II.44].[10] In particular, the dual wedge $(V^+)'$ of a real ordered vector space (V, V^+) is equal to the polar wedge $(V^+)^{\smile}$ obtained from the dual pairing $V\times V'\to\mathbb{R}$. A similar statement holds in the topological case (provided that V^* separates points).

It follows from elementary set theory that $S_1 \subseteq S_2$ implies $S_1^\smile \supseteq S_2^\smile$, and that every set $S \subseteq V$ satisfies $S \subseteq {}^\smile(S^\smile)$. Analogous statements hold for subsets $T \subseteq W$.

If $S \subseteq V$ is any set, then by $S \subseteq {}^\smile(S^\smile)$ we also have $S^\smile \supseteq ({}^\smile(S^\smile))^\smile$. However, for $T := S^\smile$ we also have $T \subseteq ({}^\smile T)^\smile$, which shows that we have equality: $S^\smile = ({}^\smile(S^\smile))^\smile$.

We prove that polar wedges are always closed in the $\sigma(W,V)$-topology. Note that we have

$$S^\smile = \bigcap_{s\in S} \widehat{s}^{\,-1}[\mathbb{R}_{\geq 0}],$$

where $\widehat{s} \in W'$ denotes the evaluation functional $y \mapsto \langle s, y\rangle$, which is continuous by the definition of $\sigma(W,V)$. Of course, the inverse image of a closed set under a continuous map is closed, and the intersection of closed sets is closed, so it follows at once that $S^\smile$ is $\sigma(W,V)$-closed. Analogously, prepolars are closed in the $\sigma(V,W)$-topology.

The bipolar theorem has a straightforward analogue for polar wedges, which we prove for the sake of completeness (this is not difficult).

Theorem 2.25 (The bipolar theorem for wedges; cf. [AT07, Theorem 2.13(3)])**.** *Let $\langle\,\cdot\,,\,\cdot\,\rangle : V\times W \to \mathbb{R}$ be a dual pairing, and let $S \subseteq V$ be an arbitrary subset. Then ${}^\smile(S^\smile)$ coincides with the $\sigma(V,W)$-closed wedge generated by S.*

Proof. Let $S_1 \subseteq V$ denote the $\sigma(V,W)$-closed wedge generated by S. It follows from the preceding remarks that ${}^\smile(S^\smile)$ is a $\sigma(V,W)$-closed wedge containing S, so we have $S_1 \subseteq {}^\smile(S^\smile)$.

For the converse, let $x_0 \in V \setminus S_1$ be given. Now S_1 and $\{x_0\}$ are disjoint, closed, convex and non-empty, and $\{x_0\}$ is furthermore compact. Since $\sigma(V,W)$ turns V into a locally convex space with topological dual $V^* = W$, it follows from the Hahn–Banach separation theorems (cf. [Rud91, Theorem 3.4(b)]) that there exist $y_0 \in W$ and $\gamma_1,\gamma_2 \in \mathbb{R}$ such that

$$\langle x_0, y_0\rangle < \gamma_1 < \gamma_2 < \langle s, y_0\rangle, \qquad (\text{for all } s \in S_1).$$

We show that $y_0 \in S_1^\smile$ holds. To that end, let $s \in S_1$ be given. Since S_1 is a wedge, for all $\alpha > 0$ we have $\alpha s \in S_1$ as well, so we find

$$\alpha\langle s, y_0\rangle = \langle \alpha s, y_0\rangle > \gamma_2, \qquad (\text{for all } \alpha > 0).$$

In particular, for all $n \in \mathbb{N}^+$ we have $n\cdot -\langle s, y_0\rangle < -\gamma_2$. As such, it follows from the Archimedean property (of $\mathbb{R}$) that we have $-\langle s, y_0\rangle \leq 0$, or equivalently, $\langle s, y_0\rangle \geq 0$. This holds for all $s \in S_1$, so we find $y_0 \in S_1^\smile$, proving our claim.

Since we have $0 \in S_1$, it is clear that $\langle x_0, y_0\rangle < \gamma_1 < \gamma_2 < 0$ holds, so we find $x_0 \notin {}^\smile(S_1^\smile)$. But we have $S \subseteq S_1$, hence $S^\smile \supseteq S_1^\smile$, and finally, ${}^\smile(S^\smile) \subseteq {}^\smile(S_1^\smile)$, so we conclude that $x_0 \notin {}^\smile(S^\smile)$ holds. ∎

Various similar theorems can be found in the literature: the one-sided bipolar theorem (cf. [Bou87, Theorem 1 on page II.44], or [Sch99, Theorem IV.1.5]), the absolute bipolar theorem (cf. [Con07, Theorem V.1.8]), and the wedge duality theorem (cf. [AT07, Theorem 2.13(3)]).

Recall that a convex subset of a locally convex space is weakly closed if and only if it is originally closed (cf. [Rud91, Theorem 3.12]). In this setting, the bipolar theorem has the following immediate consequence.

Corollary 2.26. *Let $(V, \mathcal{T})$ be a (real) locally convex space, and consider the natural dual pairing $V \times V^* \to \mathbb{R}$. For an arbitrary subset $S \subseteq V$, the bipolar ${}^{\smile}(S^{\smile})$ coincides with the $\mathcal{T}$-closed wedge generated by S.*

Remark 2.27. It should be noted that this statement is not symmetric in the dual pair. Even if V is a normed space, so that V^* also has an "original" topology (besides the weak-$*$ topology), it is not necessarily true that a convex subset $T \subseteq V^*$ is originally closed if and only if it is weak-$*$ closed. This is because the $\sigma(V^*, V)$-topology need not coincide with the $\sigma(V^*, V^{**})$-topology. Indeed, if V is not reflexive, then there is some continuous linear functional $f \in V^{**}$ not of the form $\widehat{x}$ for any $x \in V$. Then, by [Rud91, Theorem 3.10], f is not weak-$*$ continuous, so now it follows from [Rud91, Theorem 1.18] that $\ker(f)$ is not weak-$*$ closed. On the other hand, it is clear that $\ker(f)$ is convex and originally closed.

We conclude this section with an application of the bipolar theorem to order semisimplicity.

Theorem 2.28. *Let V be a (real) topological ordered vector space such that V^* separates points. Then V is topologically order semisimple if and only if the weak closure of V^+ is a cone (as opposed to a wedge).*

Proof. By the bipolar theorem, the weak closure of V^+ is ${}^{\smile}((V^+)^{\smile})$. Furthermore, we have

$$\begin{aligned}
\operatorname{tora}(V) &= \bigcap_{f \in (V^+)^{\smile}} \{v \in V : f(v) = 0\} \\
&= \bigcap_{f \in (V^+)^{\smile}} \Big(\{v \in V : f(v) \geq 0\} \cap \{v \in V : f(v) \leq 0\} \Big) \\
&= \left(\bigcap_{f \in (V^+)^{\smile}} \{v \in V : f(v) \geq 0\} \right) \cap \left(\bigcap_{f \in (V^+)^{\smile}} \{v \in V : f(v) \leq 0\} \right) \\
&= {}^{\smile}((V^+)^{\smile}) \cap -{}^{\smile}((V^+)^{\smile}).
\end{aligned}$$

In particular, we see that ${}^{\smile}((V^+)^{\smile})$ is a cone if and only if V is topologically order semisimple. ∎

If V is locally convex, then V^* automatically separates points, and the weak closure of V^+ coincides with its original closure (because it is convex). Therefore we have an even simpler conclusion in this case.

Corollary 2.29. *Let V be a (real) locally convex ordered vector space. Then V is topologically order semisimple if and only if $\overline{V^+}$ is a cone (as opposed to a wedge).*

The non-topological variant follows immediately (use Remark 2.22).

Corollary 2.30. *Let V be a (real) ordered vector space with algebraic dual V'. Then V is order semisimple if and only if the $\sigma(V, V')$-closure of V^+ is a cone (as opposed to a wedge).*

2.6 Bipositive representations

Using the duality theory developed in the previous section, we obtain simple characterisations of bipositive representations.

Theorem 2.31. *Let V be a real ordered vector space, W a real vector space, and $\langle\,\cdot\,,\,\cdot\,\rangle : V \times W \to \mathbb{R}$ a dual pairing. For any subset $T \subseteq W$, the following are equivalent:*

(1) $\widehat{T} := \big\{\widehat{t} : x \mapsto \langle x, t\rangle \;\mid\; t \in T\big\} \subseteq V'$ *is a separating set of positive linear functionals and the associated representation $V \to \mathbb{R}^T$ is bipositive;*

(2) *one has ${}^{\smile}T = V^+$;*

(3) *V^+ is $\sigma(V, W)$-closed, and $(V^+)^{\smile}$ is the $\sigma(W, V)$-closed wedge generated by T.*

Note that V is automatically order semisimple if property (1) holds, so the theorem is vacuous whenever V is not order semisimple.

Proof. (1) $\Longrightarrow$ (2). Since the representation is bipositive, we have $\langle x, t\rangle \geq 0$ for all $t \in T$ if and only if $x \in V^+$ holds. It follows at once that ${}^{\smile}T = V^+$ holds.

(2) $\Longrightarrow$ (1). Consider the natural map $\varphi : V \to \mathbb{R}^T$, which assigns to $v \in V$ the evaluation map $\widehat{v} : T \to \mathbb{R}$, $t \mapsto \langle v, t\rangle$. By the definition of ${}^{\smile}T$, we have $\langle v, t\rangle \geq 0$ for all $t \in T$ if and only if $v \in {}^{\smile}T = V^+$ holds. In other words, the representation φ is bipositive. That T separates points follows *a posteriori*, since bipositivity implies injectivity.

(2) $\Longrightarrow$ (3). If ${}^{\smile}T = V^+$ holds, then we have $(V^+)^{\smile} = ({}^{\smile}T)^{\smile}$, so it follows from the bipolar theorem that $(V^+)^{\smile}$ is the $\sigma(W, V)$-closed wedge generated by T. Furthermore, V^+ is $\sigma(V, W)$-closed because it is a prepolar.

(3) $\Longrightarrow$ (2). By the second assumption (and the bipolar theorem), we have $({}^{\smile}T)^{\smile} = (V^+)^{\smile}$, hence ${}^{\smile}T = {}^{\smile}(({}^{\smile}T)^{\smile}) = {}^{\smile}((V^+)^{\smile})$. Since V^+ is a $\sigma(V, W)$-closed wedge, it follows from the bipolar theorem that ${}^{\smile}((V^+)^{\smile}) = V^+$ holds, so we find ${}^{\smile}T = V^+$. ■

If W is an ordered topological vector space, $T \subseteq W$ a subset (or a multiset), and $T_1 \subseteq W$ the closed wedge generated by T, then we will find it convenient to say that T is a *topological generating (multi)set* for T_1.

Corollary 2.32. *Let V be a (real) topological ordered vector space such that V^* separates points. Then V admits a continuous bipositive representation as a space of functions ($V \to \mathbb{R}^T$ for some set T) if and only if V^+ is weakly closed.*

If this is the case, then choosing such a representation $V \to \mathbb{R}^T$ is the same as choosing a weak-$*$ topological generating multiset for the dual wedge $(V^+)^*$.

Proof. Consider the natural dual pairing $\langle\,\cdot\,,\,\cdot\,\rangle : V \times V^* \to \mathbb{R}$. If V admits a bipositive continuous representation $V \to \mathbb{R}^T$, then V^+ is weakly closed by Theorem 2.31(3). Conversely, if V^+ is weakly closed, then the choice of $T := (V^+)^{\smile}$ meets requirement (3) of Theorem 2.31, so it follows that the associated continuous representation $V \to \mathbb{R}^T$ is bipositive.

The second conclusion also follows immediately from Theorem 2.31. ■

Like before, the conclusion is even simpler if V is locally convex.

Corollary 2.33. *Let V be a (real) locally convex ordered vector space. Then V admits a continuous bipositive representation as a space of functions if and only if V^+ is closed.*

If this is the case, then choosing such a representation $V \to \mathbb{R}^T$ is the same as choosing a weak-$*$ topological generating multiset for the dual wedge $(V^+)^*$.

Again the non-topological variant follows immediately.

Corollary 2.34. *Let V be a (real) ordered vector space with algebraic dual V'. Then V admits a bipositive representation as a space of functions if and only if V^+ is $\sigma(V, V')$-closed.*

If this is the case, then choosing such a representation $V \to \mathbb{R}^T$ is the same as choosing a $\sigma(V', V)$-topological generating multiset for the dual wedge $(V^+)'$.

The results obtained in this section can also be applied to the theory of injective and positive representations. If V is topologically order semisimple, then V^+ is contained in a weakly closed cone $\mathcal{K}$ (for instance $\mathcal{K} := {}^\smile((V^+)^\smile)$, but there might be others), and every bipositive representation $(V, \mathcal{K}) \to \mathbb{R}^T$ gives rise to an injective and positive representation $(V, V^+) \to (V, \mathcal{K}) \to \mathbb{R}^T$. Conversely, every injective and positive representation $V \to \mathbb{R}^T$ is bipositive for some weakly closed cone $\mathcal{K} \supseteq V^+$, so all injective and positive representations can be understood through the theory of bipositive representations.

2.7 Representations on $C(\Omega)$ spaces

Now that we have addressed questions regarding continuous representations of V as a space of functions, we focus our attention on representations of V as a space of *continuous* functions on some *compact Hausdorff* space Ω. It turns out that this is a purely topological question, so we present the results without any reference to the order structure. Furthermore, as questions regarding injectivity have been sufficiently addressed in the ordered setting as well, we will not require injectivity here.

Remark 2.35. Let V be a real or complex topological vector space and let V^* denote its (topological) dual. Unlike before, we no longer require that V^* separates points. Since V separates points on V^* (two linear functionals are equal if and only if they take the same value at every $x \in V$), we can always equip V^* with the weak-$*$ topology, and this turns V^* into a (Hausdorff) locally convex topological vector space. If $S \subseteq V^*$ is a weak-$*$ compact subset, then we have a natural map $\varphi : V \to C(S, \mathbb{F})$, which associates to every $v \in V$ its evaluation function $\widehat{v} : s \mapsto s(v)$. (For every $v \in V$, the function $\widehat{v} : S \to \mathbb{F}$ is continuous by definition of the weak-$*$ topology.)

We claim that the obtained representation $\varphi : V \to C(S, \mathbb{F})$ is continuous if $C(S, \mathbb{F})$ is equipped with the (locally convex) topology of pointwise convergence. Indeed, by the characteristic property of the product topology, it suffices to check that for each $s \in S$ the map $\varphi_s : V \to \mathbb{F}$, $v \mapsto \varphi(v)_s$ is continuous. But have $\varphi(v)_s = \widehat{v}(s) = s(v)$, hence $\varphi_s = s$, which is a continuous linear functional. This proves our claim.

We prove a converse of this observation in Proposition 2.36. The question whether φ remains continuous if $C(S,\mathbb{F})$ is equipped with the (stronger) norm topology is addressed in Remark 2.38.

Proposition 2.36. *Let V be a real or complex topological vector space, Ω a compact Hausdorff space, and $\psi : V \to C(\Omega,\mathbb{F})$ a linear map which is continuous with respect to the topology of pointwise convergence on $C(\Omega,\mathbb{F})$. Then the set $S := \{f_\omega : v \mapsto \psi(v)_\omega \mid \omega \in \Omega\} \subseteq V^*$ is weak-$*$ compact and Hausdorff.*

Proof. Note that for every $\omega \in \Omega$ the function $f_\omega : v \mapsto \psi(v)_\omega$ defines a continuous linear functional on V. Consider the map $\psi^t : \Omega \to V^*$ given by $\omega \mapsto f_\omega$. We claim that ψ^t is continuous, where V^* is understood to be equipped with the weak-$*$ topology. By the characteristic property of weak topologies, ψ^t is continuous if and only if $\widehat{v} \circ \psi^t : \Omega \to \mathbb{F}$ is continuous for every $v \in V$. But we have $(\widehat{v} \circ \psi^t)(\omega) = \widehat{v}(f_\omega) = f_\omega(v) = \psi(v)_\omega$, so we see that $\widehat{v} \circ \psi^t$ is simply the function $\psi(v) \in C(\Omega,\mathbb{F})$. This is of course continuous, so it follows that ψ^t is continuous.

It is clear from the definition that S is the image of Ω under the map $\psi^t : \Omega \to V^*$. Since Ω is compact and ψ^t continuous, it follows that S is weak-$*$ compact. Furthermore, S is Hausdorff since it is a subspace of a (Hausdorff) topological vector space. ∎

Remark 2.37. In the setting of Proposition 2.36, it is easy to see that ψ factors as the composition

$$V \xrightarrow{\varphi} C(S,\mathbb{F}) \xrightarrow{\chi} C(\Omega,\mathbb{F}),$$

where $\varphi : V \to C(S,\mathbb{F})$ is as in Remark 2.35 and $\chi : C(S,\mathbb{F}) \to C(\Omega,\mathbb{F})$ is the map $f \mapsto f \circ \psi^t$, with $\psi^t : \Omega \to V^*$ as in the proof of Proposition 2.36. (Indeed, for $v \in V$ we have $(\chi \circ \varphi)(v) = \chi(\widehat{v}) = \widehat{v} \circ \psi^t = \psi(v)$, so we find $\chi \circ \varphi = \psi$.)

Note that χ preserves both pointwise and uniform convergence, and that both φ and χ can be recovered only from the continuous function $\psi^t : \Omega \to V^*$. As such, it follows that choosing a compact Hausdorff space Ω and a linear map $\psi : V \to C(\Omega,\mathbb{F})$ which is continuous with respect to the topology of pointwise convergence is essentially the same as choosing a compact Hausdorff space Ω and a continuous function $\psi^t : \Omega \to V^*$. (This continuous function is the topological analogue of the multisets we encountered before.)

Remark 2.38. In light of Remark 2.37, it is now not so hard to see that choosing a compact Hausdorff space Ω and a linear map $\psi : V \to C(\Omega,\mathbb{F})$ which is continuous with respect to the *norm* topology is the same as choosing a compact Hausdorff space Ω and a continuous function $\psi^t : \Omega \to V^*$ such that the set

$$\{x \in V \,:\, |s(x)| < 1 \text{ for all } s \in \operatorname{ran}(\psi^t)\}$$

is a neighbourhood of 0 in V, or equivalently, has non-empty interior. If V is such that V^* separates points, then the latter criterion is equivalent to the more concise statement that the absolute prepolar ${}^\circ\operatorname{ran}(\psi^t)$ has non-empty interior. (If $\langle\,\cdot\,,\,\cdot\,\rangle : V \times W \to \mathbb{F}$ is a dual pairing, then the *absolute prepolar* of a subset $T \subseteq W$ is defined as ${}^\circ T := \{x \in V \,:\, |\langle x,t\rangle| \leq 1 \text{ for all } t \in T\}$; cf. [Sch99, page 125]. Others call this simply the *(pre)polar*; cf. [Con07, Definition V.1.6].) Yet another way to put this is that $\operatorname{ran}(\psi^t)$ should be an *equicontinuous* set of linear functionals (see also [Sch99, remark 5 on page 125]).

The results from this section can easily be combined with the results from Section 2.6 to find criteria for injective (bi)positive representations of a (real) topological ordered vector space as a subspace of a $C(\Omega,\mathbb{R})$ space. In particular, the following theorem is immediate.

Theorem 2.39. *Let V be a (real) topological ordered vector space such that V^* separates points. Then the following are equivalent:*

(1) *there exists a compact Hausdorff space Ω and a continuous bipositive representation $V \to C(\Omega,\mathbb{R})$;*

(2) *there exists a weak-$*$ compact subset $T \subseteq V^*$ with ${}^{\smile}T = V^+$ such that ${}^{\circ}T$ has non-empty interior;*

(3) *V^+ is weakly closed and $(V^+)^{\smile}$ admits a weak-$*$ topological generating set which is both weak-$*$ compact and equicontinuous.*

As in Section 2.6, the result can be applied to positive and injective representations as well, by passing to a weakly closed cone $\mathcal{K} \supseteq V^+$.

Comparable results regarding representations as $C(\Omega)$ spaces are proven in theorems 2.44 and 2.45 below.

2.8 Order semisimple normed spaces

We conclude the representation theory of real ordered vector spaces with a brief investigation of normed ordered spaces.

If V is a normed ordered space, then it follows from Corollary 2.29 that V is topologically order semisimple if and only if $\overline{V^+}$ is a cone (and not a wedge). We assume for the remainder of this section that this is the case. Furthermore, like before, we let $\langle\,\cdot\,,\,\cdot\,\rangle : V \times V^* \to \mathbb{R}$ denote the natural dual pairing. We show that V admits a continuous bipositive representation $\varphi : (V, \overline{V^+}) \to C(\Omega,\mathbb{R})$ for some compact Hausdorff space Ω. There are two natural ways of doing so.

Construction 2.40. By the Banach–Alaoglu theorem, the closed unit ball $B \subseteq V^*$ is weak-$*$ compact. Furthermore, the polar wedge $(V^+)^{\smile}$ is weak-$*$ closed, so it follows that $T := B \cap (V^+)^{\smile}$ is also weak-$*$ compact. Thirdly, note that ${}^{\circ}B$ is simply the closed unit ball of V. Since we have $T \subseteq B$, it follows that ${}^{\circ}T \supseteq {}^{\circ}B$ holds, so ${}^{\circ}T$ has non-empty interior. Finally, since B is absorbing, it is clear that T is a generating set of $(V^+)^{\smile}$.

All in all, we see that T is a weak-$*$ compact equicontinuous generating set of $(V^+)^{\smile}$. It follows from Theorem 2.39 that the natural representation $V \to \mathbb{F}^T$ becomes a continuous representation $\varphi : V \to C(\Omega,\mathbb{R})$.

Continuity of φ is also easy to verify directly: for $v \in V$ and $f \in T \subseteq B$ we have $|f(v)| \leq \|f\|\|v\| \leq \|v\|$, so we find $\|\varphi(v)\|_\infty = \sup_{f\in T}|f(v)| \leq \|v\|$. ⊠

Construction 2.41. Let $T \subseteq V^*$ be as in Construction 2.40, and let $\Omega \subseteq T$ be the weak-$*$ closure of the set of all extreme points of T. Then Ω is weak-$*$ compact (a closed subset of a compact set is compact), and it follows from the Krein–Milman theorem that T is the weak-$*$ closed convex hull of Ω. As a consequence, Ω is a weak-$*$ topological generating set of $(V^+)^{\smile}$. Furthermore, we have $\Omega \subseteq T$, so clearly Ω is equicontinuous as well.

The remainder of Construction 2.40 can be carried out verbatim, and we get a continuous, bipositive representation $\psi : (V, \overline{V^+}) \to C(\Omega,\mathbb{R})$. ⊠

It should be noted that the set of extreme points of T is not weak-$*$ closed in general, so it really is necessary to take the weak-$*$ closure in Construction 2.41. For instance, in [Gli60, Theorem 2.8] the following is shown: if V is a UHF algebra (i.e. a certain type of C^*-algebra), then the weak-$*$ closure of the set of extreme points of T is equal to all of T.

We mention one consequence of the preceding results.

Corollary 2.42. *Let V be a (real) Archimedean ordered vector space containing an order unit $u \in V^+$. Then there exists a compact Hausdorff space Ω and a bipositive continuous representation $\varphi : V \to C(\Omega, \mathbb{R})$.*

Proof. Let $\|\cdot\|_u$ denote the norm associated with u (cf. Theorem 1.29). Then V^+ is closed, so the result follows from constructions 2.40 and 2.41. ■

Remark 2.43. While we showed that every topologically order semisimple normed space can be represented as a $C(\Omega)$ space, we did not choose any particular representation. The most common choice of representation is the one from Construction 2.41. Similar constructions, using the machinery of the Hahn–Banach, Banach–Alaoglu and Krein–Milman theorems, occur time and again in various branches of functional analysis. Examples include the Gelfand representation for commutative Banach algebras, as well as the Stone–Krein–Kakutani–Yosida representation theorem for Riesz spaces with an order unit. In general, the question is to represent a certain class of normed spaces as subspaces of spaces of continuous functions. In a 1951 paper of Richard V. Kadison, [Kad51a], the relations between many of these theorems are studied in detail. In this paper, the representation theorem for Archimedean ordered vector spaces with an order unit is used as a basis for the other representation theorems.

We close this section with the following question: given an order semisimple space V without any topological structure, when is it possible to equip V with a norm such that V becomes *topologically* order semisimple? Using the techniques from this chapter, the following result is now easy to prove.

Theorem 2.44. *Let V be a (real) ordered vector space. Then the following are equivalent:*

(1) *there exists a norm $\|\cdot\|$ on V that turns V into a topologically order semisimple space;*

(2) *there exists a norm $\|\cdot\|$ on V such that $\overline{V^+}$ is a cone (as opposed to a wedge);*

(3) *there exists a monotone norm $\|\cdot\|$ on V;*

(4) *there exists a fully monotone norm $\|\cdot\|$ on V;*

(5) *there exists a compact Hausdorff space Ω and an injective and positive linear map $\varphi : V \to C(\Omega, \mathbb{R})$.*

Proof. (1) $\Longleftrightarrow$ (2). This follows from Corollary 2.29

(1) $\Longrightarrow$ (5). This follows from the constructions from this section.

(5) $\Longrightarrow$ (4). If $\varphi : V \to C(\Omega, \mathbb{R})$ is positive and injective, then clearly $\|v\| := \|\varphi(v)\|_\infty$ defines a fully monotone norm on V.

(4) $\Longrightarrow$ (3). Trivial.

(3) $\Longrightarrow$ (2). This follows from theorems 1.25 and 1.28. ■

Adding closed cones, we get the following bipositive version.

Theorem 2.45. *Let V be a (real) ordered vector space. Then the following are equivalent:*

(1) *there exists a norm $\|\cdot\|$ on V such that V^+ is closed;*

(2) *there exists a monotone norm $\|\cdot\|$ on V such that V^+ is closed;*

(3) *there exists a fully monotone norm $\|\cdot\|$ on V such that V^+ is closed;*

(4) *there exists a compact Hausdorff space Ω and a bipositive linear map $\varphi : V \to C(\Omega, \mathbb{R})$.*

Proof. (1) $\Longrightarrow$ (4). This follows from the constructions from this section.

(4) $\Longrightarrow$ (3). If $\varphi : V \to C(\Omega, \mathbb{R})$ is bipositive, then $\|v\| := \|\varphi(v)\|_\infty$ defines a fully monotone norm on V. Furthermore, since φ is bipositive and isometric, we find that $V^+ = \varphi^{-1}[C(\Omega, \mathbb{R})^+]$ is closed.

(3) $\Longrightarrow$ (2) $\Longrightarrow$ (1). Trivial. ■

In Section 4.6 we exhibit a class of ordered vector spaces which are order semisimple but nevertheless fail to meet the criteria of Theorem 2.44 (let alone Theorem 2.45).

2.9 Modifications for the complex case

So far, all the theory developed in this chapter was developed for real ordered spaces only. We briefly list the modifications needed to be made in order to translate the theory to the complex setting.

2.9.1 Ideals in complex ordered spaces

Recall that we require the cone of a complex ordered vector spaces to be real with respect to some complex conjugation. If we define ideals in a complex ordered vector space simply as full (complex) subspaces, then some strange things happen. The following example is instructive.

Example 2.46. Consider $V = \mathbb{C}^2$ with the entry-wise conjugation and the cone $\mathbb{R}^2_{\geq 0} \subseteq \mathbb{R}^2 = \mathrm{Re}(V)$. It is easily seen that the subspace $I := \mathrm{span}\{(1, i)\}$ is full, for it contains no positive elements. Therefore the quotient V/I can be equipped with the quotient cone $(V/I)^+$. However, V/I is two-dimensional as a real vector space, and the cone $(V/I)^+$ is not contained in a one-dimensional real subspace. Consequently, there is no conjugation on V/I that turns it into a complex ordered vector space. ⊠

In light of the preceding example, we will say that an *ideal* in a complex ordered vector space is a self-conjugate full subspace. In this setting, it is easy to see that the conjugation of V carries over to V/I, so the latter becomes a complex ordered vector space.

Recall from Remark 1.12 that positive linear maps are not necessarily self-conjugate. The following example shows that the kernel of a positive linear map might fail to be an ideal.

Example 2.47. Let V and I be as in Example 2.46, except that we equip V with the zero cone this time. Then V/I also carries the zero cone, so the natural map $V \to V/I$ gives rise to a positive linear functional $f : V \to \mathbb{C}$. Now I is the kernel of a positive linear functional, but I is not self-conjugate. ⊠

Therefore it seems that we chose the "wrong" definition of positive linear maps in the complex case.[11] Consequently, we will focus most of our attention to *self-conjugate* positive linear maps.

Analogously to Lemma 2.2, we find that ideals of a complex ordered vector space are precisely the kernels of *self-conjugate* positive linear maps. Note: this should not be construed as saying that a positive linear map $f : V \to W$ is self-conjugate if and only if $\ker(f)$ is self-conjugate; this is not true.[12]

In order for the isomorphism theorems to hold in the complex case, they should be rephrased in terms of *self-conjugate* positive linear maps.

2.9.2 Maximal ideals in complex ordered spaces

Let V be a complex ordered vector space. Note that a subspace $X \subseteq V$ is self-conjugate if and only if it is of the form $X = Y + iY$ for some real subspace $Y \subseteq \mathrm{Re}(V)$. (Concretely, this Y can be found as $Y = \mathrm{Re}(X) := X \cap \mathrm{Re}(V)$.) In particular, this applies to ideals, so we see that the ideals of V are in bijective correspondence with the ideals of $\mathrm{Re}(V)$.

It follows at once that an ideal $I \subseteq V$ is maximal if and only if $\mathrm{Re}(I)$ is a maximal ideal in $\mathrm{Re}(V)$. Consequently, we find that V is simple if and only if $\mathrm{Re}(V)$ is simple, if and only if $\mathrm{Re}(V)$ is one-dimensional (as a real space), if and only if V is one-dimensional (as a complex space). Like in the real case, we have exactly two non-isomorphic simple spaces: $(\mathbb{C}, \{0\})$ and $(\mathbb{C}, \mathbb{R}_{\geq 0})$. Similarly, like in the real case, every maximal ideal gives rise to a non-zero positive linear functional $V \to (\mathbb{C}, \mathbb{R}_{\geq 0})$ by composing the natural map $V \to V/I$ with a positive linear isomorphism $V/I \to (\mathbb{C}, \mathbb{R}_{\geq 0})$.

The remaining results from Section 2.2 hold in the complex case as well; the required modifications are straightforward.

Remark 2.48. Before we move on to order semisimplicity, we pause for a moment to point out one of the major reasons for defining complex ordered vector spaces as we did, with a complex conjugation rather than merely a cone in a complex vector space. In the latter setting, there are more than two non-isomorphic simple complex ordered spaces (any cone in $\mathbb{C}$ suffices). Some cones in $\mathbb{C}$ are larger than the standard cone $\mathbb{R}_{\geq 0}$, so the quotient of V by a maximal ideal cannot always be identified with a non-zero positive linear functional $V \to (\mathbb{C}, \mathbb{R}_{\geq 0})$ (modulo scalar). Indeed, examples like the one given in Example 2.46 show that larger cones can occur in the quotient of V by a maximal ideal.

In fact, the situation is much worse: if we define complex ordered vector spaces without the additional structure provided by the complex conjugation, then a simple ordered vector space need not be one-dimensional! We give an example in Example 4.7.

This shows that the complex conjugation and the requirement $V^+ \subseteq \mathrm{Re}(V)$ serve a clear purpose: to retain the inherently real nature of ordered vector spaces, and to circumvent strange counterexamples like the one presented in Example 4.7.

2.9.3 Order semisimplicity of complex ordered spaces

As a slight modification of what was pointed out in Remark 1.12, note that $f \in V'$ is positive if and only if $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ and $-\mathrm{Im}(f)$ are positive. (This is because $v \in V^+$ and $f \in (V^+)'$ together imply $\overline{f}(v) = \overline{f(\overline{v})} = \overline{f(v)} = f(v)$, hence $\mathrm{Re}(f)(v) = f(v)$ and $\mathrm{Im}(f)(v) = 0$.) As such, we may define

$$\begin{aligned} \mathrm{ora}(V) &:= \big\{ x \in V \,:\, f(x) = 0 \text{ for all } f \in (V^+)' \big\} \\ &= \big\{ x \in V \,:\, f(x) = 0 \text{ for all } f \in (V^+)' \cap \mathrm{Re}(V') \big\}, \end{aligned}$$

as these two sets are easily seen to be equal.

The remaining results of Section 2.3 are easily extended to the complex case. By the above, $(V^+)'$ separates points on V if and only if $(V^+)' \cap \mathrm{Re}(V')$ separates points on V. Therefore it is clear that V admits an injective and positive representation $V \to \mathbb{C}^S$ if and only if it admits a *self-conjugate* injective and positive representation $V \to \mathbb{C}^T$. Like in the real case, choosing such a representation $V \to \mathbb{C}^S$ is the same as choosing a separating multiset S of positive linear functionals. Furthermore, the representation $V \to \mathbb{C}^S$ is self-conjugate if and only if S is real (that is, $S \subseteq \mathrm{Re}(V')$).

2.9.4 Topological order semisimplicity of complex ordered spaces

Let V be a complex topological ordered vector space. Recall that this means that V is equipped with a *continuous* complex conjugation $^- : V \to V$. Before extending the notion of order semisimplicity to the complex topological case, we briefly study questions related to the continuity of the conjugation. First of all, we show that the induced conjugation on V' can be restricted to V^*.

Proposition 2.49. *Let V be a complex topological vector space equipped with a continuous complex conjugation $^- : V \to V$. Then its topological dual V^* is self-conjugate as a subset of the algebraic dual V'. Consequently, the induced conjugation on V' restricts to a well-defined complex conjugation on V^*.*

Proof. The conjugate of a continuous linear functional $f : V \to \mathbb{C}$ is once again continuous, as it can be written as the composition $\overline{f} = {}^- \circ f \circ {}^-$ of continuous functions. Therefore V^* is self-conjugate, and we get an induced conjugation $f \mapsto {}^- \circ f \circ {}^-$ on V^*. ∎

Here we see one reason for the requirement that the complex conjugation should be continuous: this ensures that the induced conjugation is well-defined on V^*. The converse is true for weak topologies, as the following result shows.

Proposition 2.50. *Let V be a complex vector space equipped with a complex conjugation $^- : V \to V$, and let V' be its algebraic dual, equipped with the induced conjugation. If $W \subseteq V'$ is a separating space of linear functionals, then $^- : V \to V$ is continuous with respect to the $\sigma(V, W)$-topology if and only if W is self-conjugate ($\overline{W} = W$).*

Proof. If $^- : V \to V$ is continuous with respect to the $\sigma(V, W)$-topology, then it follows from Proposition 2.49 and the fact that the (topological) dual of $(V, \sigma(V, W))$ coincides with W that W is self-conjugate.

Assume now that W is self-conjugate, and equip V with the $\sigma(V,W)$-topology. From the characteristic property of initial topologies, we know that a net $\{x_\lambda\}_{\lambda\in\Lambda}$ in V converges to some $x \in V$ if and only if $\{f(x_\lambda)\}_{\lambda\in\Lambda}$ converges to $f(x)$ for all $f \in W$. Assume that this is the case, then for all $f \in W$ we have

$$\lim_{\lambda\in\Lambda} f\left(\overline{x_\lambda}\right) = \lim_{\lambda\in\Lambda} \overline{\overline{f}\left(x_\lambda\right)} = \overline{\lim_{\lambda\in\Lambda} \overline{f}(x_\lambda)} = \overline{\overline{f}(x)} = f(\overline{x}),$$

where we used that W is self-conjugate and $^{-} : \mathbb{C} \to \mathbb{C}$ is continuous. It follows that $\{\overline{x_\lambda}\}_{\lambda\in\Lambda}$ converges to $\overline{x}$, which shows that $^{-} : V \to V$ is continuous. ∎

Bearing this in mind, it is easy to extend the results from Section 2.4 to the complex case, in the same way that we extended the results fromSection 2.3 to the complex case.

In Remark 2.22, we showed how to interpret questions about a bare ordered vector space V in terms of the analogous topological questions with respect to the $\sigma(V,V')$-topology. It follows from Proposition 2.50 that the $\sigma(V,V')$-topology makes any complex conjugation on V continuous; therefore we may use Remark 2.22 in the complex case as well.

2.9.5 The bipolar theorem for wedges in complex spaces

Let V and W be complex vector spaces each carrying a complex conjugation. We say that a dual pairing $\langle\,\cdot\,,\,\cdot\,\rangle : V \times W \to \mathbb{C}$ is *of self-conjugate type* if we have $\overline{\langle x,y\rangle} = \langle \overline{x},\overline{y}\rangle$ for all $x \in V$ and $y \in W$. This is equivalent to the requirement that the natural map $W \hookrightarrow V'$ is self-conjugate (i.e. conjugation-preserving). Another equivalent condition is that $\langle\,\cdot\,,\,\cdot\,\rangle$ restricts to a dual pairing $\operatorname{Re}(V) \times \operatorname{Re}(W) \to \mathbb{R}$.

If V is a complex topological vector space carrying a continuous complex conjugation, then it follows from Proposition 2.49 that V^* can be equipped with the induced conjugation. Clearly the natural dual pairing $V \times V^* \to \mathbb{C}$ is of self-conjugate type. The same holds, of course, for the natural dual pairing $V \times V' \to \mathbb{C}$ in the non-topological setting.

The definition of (pre)polar wedges is a bit different in the complex case. Consider a dual pairing $\langle\,\cdot\,,\,\cdot\,\rangle : V \times W \to \mathbb{C}$ (not necessarily of self-conjugate type) and subsets $S \subseteq V$ and $T \subseteq W$, then we define

$$\begin{aligned} S^{\smallsmile} &:= \big\{ y \in W \,:\, \operatorname{Re}(\langle s,y\rangle) \geq 0 \text{ for all } s \in S \big\};\\ {}^{\smallsmile}T &:= \big\{ x \in V \,:\, \operatorname{Re}(\langle x,t\rangle) \geq 0 \text{ for all } t \in T \big\}. \end{aligned}$$

Note: in case the dual pairing is of self-conjugate type and $V^+ \subseteq \operatorname{Re}(V)$ is a cone, then $(V^+)^{\smallsmile}$ does *not* coincide with the dual wedge $(V^+)'$. Indeed, for any subset $S \subseteq \operatorname{Re}(V)$ and $s \in S$, $y \in W$ we have $\overline{\langle s,y\rangle} = \langle \overline{s},\overline{y}\rangle = \langle s,\overline{y}\rangle$, hence

$$S^{\smallsmile} = \big\{ y \in W \,:\, \langle s, \operatorname{Re}(y)\rangle \geq 0 \text{ for all } s \in S \big\}.$$

In particular, if we take $S = V^+$, then we find

$$(V^+)^{\smallsmile} = \big\{ y \in W \,:\, \operatorname{Re}(y) \in (V^+)' \big\} = \big((V^+)' \cap \operatorname{Re}(W)\big) + i\operatorname{Re}(W).$$

Of course, this is not generally equal to $(V^+)'$; this already fails in the case $(V,V^+) = (\mathbb{C},\mathbb{R}_{\geq 0})$.

The reason for this modification to the definitions of $S^\smile$ and ${}^\smile T$ is that the bipolar theorem fails if we define $S^\smile$ and ${}^\smile T$ like in the real case (i.e. with $\langle s, y\rangle \geq 0$ instead of $\mathrm{Re}(\langle s, y\rangle) \geq 0$). Consider the following example.

Example 2.51. Consider $V = W = \mathbb{C}$ with standard conjugation and the natural dual pairing $\langle x, y\rangle := xy$, and consider the set $S := \mathbb{R} = \mathrm{Re}(V)$. If $y \in \mathbb{C}$ is such that $xy \geq 0$ holds for all $x \in S$, then in particular we have $1 \cdot y \geq 0$ and $-1 \cdot y \geq 0$, so we must have $y = 0$. Therefore the "naive" polar wedge of S is zero, and the "naive" bipolar of S is all of $\mathbb{C}$. In particular, it is larger than the closed wedge generated by S. ⊠

The present definitions of $S^\smile$ and ${}^\smile T$ correspond to the common definition of one-sided polars in the complex case; see for instance [Sch99, page 125].

Complex polar wedges satisfy the same basic properties as in the real case, and the bipolar theorem remains true here as well. (In fact, the dual pairing does not have to be of self-conjugate type for this.) The proof of the bipolar theorem requires only one modification: when applying the complex version of the Hahn–Banach separation theorem, we get some $y_0 \in W$ such that

$$\mathrm{Re}(\langle x_0, y_0\rangle) < \gamma_1 < \gamma_2 < \mathrm{Re}(\langle s, y_0\rangle), \qquad (\text{for all } s \in S_1),$$

and these real parts need to be carried along for the remainder of the proof.

Since the dual wedge $(V^+)'$ and the polar wedge $(V^+)^\smile$ no longer coincide in the complex case, the proof of Theorem 2.28 becomes a little trickier. However, it is still relatively straightforward to prove that one has

$$\mathrm{tora}(V) = \Big({}^\smile((V^+)^\smile) \cap -{}^\smile((V^+)^\smile)\Big) + i\Big({}^\smile((V^+)^\smile) \cap -{}^\smile((V^+)^\smile)\Big).$$

Note: since a dual pairing of self-conjugate type restricts to a dual pairing $\mathrm{Re}(V) \times \mathrm{Re}(W) \to \mathbb{R}$, the bipolar of a subset $S \subseteq \mathrm{Re}(V)$ can also be computed with respect to this real dual pairing. The real and complex bipolars coincide: in both cases it is the $\sigma(V, W)$-closed wedge generated by S. (Here we use that $\mathrm{Re}(V)$ is $\sigma(V, W)$-closed, because of continuity of the conjugation.)

2.9.6 Bipositive representations of complex ordered spaces

In attempting to formulate the complex analogue of Theorem 2.31, we run into trouble. If $T \subseteq W$ is a subset such that ${}^\smile T = V^+$ holds, then the corresponding linear functionals $\widehat{t} : x \mapsto \langle x, t\rangle$ need not be positive: in general we merely have $\mathrm{Re}(\widehat{t}(v)) \geq 0$ for $v \in V^+$ and $t \in T$. We solve this by identifying W with a subspace of V', and only considering subsets $T \subseteq (V^+)' \cap W$.

Theorem 2.52. *Let V, W be complex vector spaces each equipped with a complex conjugation, let $V^+ \subseteq \mathrm{Re}(V)$ be a cone, and let $\langle\,\cdot\,,\,\cdot\,\rangle : V \times W \to \mathbb{C}$ be a dual pairing of self-conjugate type. For any subset $T \subseteq (V^+)' \cap W$, the following are equivalent:*

(1) $\widehat{T} := \big\{\widehat{t} : x \mapsto \langle x, t\rangle \ \big|\ t \in T\big\} \subseteq (V^+)'$ *is a separating set of positive linear functionals and the associated representation $V \to \mathbb{R}^T$ is bipositive;*

(2) *one has ${}^\smile(T + i\,\mathrm{Re}(W)) = V^+$;*

(3) *V^+ is $\sigma(V, W)$-closed, and $(V^+)^\smile$ is the $\sigma(W, V)$-closed wedge generated by $T + i\,\mathrm{Re}(W)$.*

The proof of Theorem 2.52 is analogous to the real case.

In applying Theorem 2.52, recall that the representation $V \to \mathbb{R}^T$ is self-conjugate if and only if $T \subseteq \mathrm{Re}(V')$ holds.

The remaining corollaries from Section 2.6 can be translated to the complex setting with straightforward modifications.

2.9.7 Representations on $C(\Omega)$ spaces

Everything in Section 2.7 was already proved for real *and* complex spaces, so there is nothing to be done here.

2.9.8 Representations of complex normed spaces

In the complex analogue of Construction 2.40, we set $T := B \cap (V^+)^\vee \cap \mathrm{Re}(V^*)$, so that the representation $V \to C(T, \mathbb{C})$ will be self-conjugate (i.e. conjugation-preserving). If this choice is carried over to Construction 2.41, then the same holds for the representation $V \to C(\Omega, \mathbb{C})$ obtained therein.

If V is a complex Archimedean ordered vector space with an order unit, then we have to choose a complexification of the order unit norm $\| \cdot \|_u$, and the obtained representation depends on this choice (see also Remark 1.32).

In the complex versions of theorems 2.44 and 2.45, the representations $\varphi : V \to C(\Omega, \mathbb{C})$ can be required to be self-conjugate (i.e. conjugation-preserving).

2.10 End notes

1. (page 15) The term *order ideal* dates back to [Kad51a, Definition 2.2], and possibly even further. Other authors refer to these as *full* (or *order-convex*) *subspaces*.

A word of warning: a different definition of order ideals is given in [vGK08], but it is not equivalent. The goal of the latter is to generalise the notion of Riesz ideals to pre-Riesz spaces, so the ideals defined there are precisely the kernels of Riesz homomorphisms rather than positive linear maps.

2. (page 15) As a matter of convention, order units are commonly required to be positive, as in Definition 1.21. Our view in terms of ideals suggests that we should also allow negative elements to be order units, but this is not customary. We shall therefore stick to our original definition.

3. (page 16) The second isomorphism theorem would read: *Let V be a real ordered vector space, and let $W \subseteq V$ be a subspace and $I \subseteq V$ an ideal. Then $W + I$ is a subspace of V, $W \cap I$ is an ideal of W, and the natural linear isomorphism $W/(W \cap I) \cong (W + I)/I$, $w + (W \cap I) \mapsto w + I$ is bipositive.* Unfortunately, this is not true: all we get is a positive linear isomorphism $W/(W \cap I) \to (W + I)/I$. To see why this is not generally bipositive, consider $V = \mathbb{R}^2$ with a non-zero cone $V^+ \subseteq V$, and let $W, I \subseteq V$ be different one-dimensional subspaces satisfying $W \cap V^+ = I \cap V^+ = \{0\}$. Note that I is trivially an ideal (there is no $y \in I$ with $0 \leq y$, so the situation $0 \leq x \leq y$ does not occur). However, $W/(W \cap I)$ is isomorphic with $(\mathbb{R}, \{0\})$, whereas $(W + I)/I$ is isomorphic with $(\mathbb{R}, \mathbb{R}_{\geq 0})$.

4. (page 17) We distinguish two cases. On the one hand, if V/I is isomorphic with $(\mathbb{R}, \{0\})$, then the positive linear functional defined by I is unique up to a non-zero scalar. On the other hand, if V/I is isomorphic with $(\mathbb{R}, \mathbb{R}_{\geq 0})$, then the positive linear functional defined by I is unique up to a positive scalar.

5. (page 20) Given a representation $V \to \mathbb{R}^S$, different $s_1, s_2 \in S$ might give rise to the same linear functional $f_{s_1} = f_{s_2}$, so we get a multiset instead of a set. (Of course, we can remove all superfluous duplicates if the situation allows us to do so.)

6. (page 20) Terminology invented by the author. The term *order semisimple* is chosen because the concept is similar in spirit to semisimplicity of Banach algebras. It should be pointed out, however, that the concept is not directly analogous to semisimplicity of rings in abstract algebra, but rather to Jacobson semisimplicity of rings and modules. Perhaps a better name would be *order Jacobson semisimple* (or *order semiprimitive*), to stay closer to the terminology of rings and algebras, but this terminology will sound alien to most in functional analysis. (Furthermore, no concept of order semisimplicity exists to date, so the chosen terminology is not particularly ambiguous.)

7. (page 20) If $\varphi : V \to W$ is a continuous linear map to a topological vector space W, then $\{0\} \subseteq W$ is closed (we assume topological vector spaces to be Hausdorff), so $\ker(\varphi)$ is closed as well (by continuity). Conversely, if $I \subseteq V$ is a closed subspace, then the quotient topology turns V/I into a (Hausdorff) topological vector space; cf. [Rud91, Definition 1.40] or [Con07, Exercise IV.1.16].

8. (page 21) Let $x \in V$ be non-zero, then the singleton $\{x\}$ can be extended to a basis $\mathcal{B}$. Now there is a unique linear functional $\varphi : V \to \mathbb{F}$ given by $x \mapsto 1$ and $b \mapsto 0$ for all $b \in \mathcal{B} \setminus \{x\}$. In particular, now we have $\varphi(x) \neq 0$. (The set of functionals derived from the basis $\mathcal{B}$ in this way form the *dual basis* of $\mathcal{B}$, but it is only a basis of V' if V is finite-dimensional.)

9. (page 22) A word of warning: this is not common terminology or notation.

10. (page 22) Some authors define the one-sided polar on the other side; see e.g. [Sch99, page 125]. Of course, this does not matter much: you just end up with the same set reflected in the origin.

11. (page 31) The present notion of positive linear maps is commonly used throughout functional analysis, in particular in the context of positive linear *functionals*, so we chose not to deviate too much from existing terminology.

12. (page 31) The kernel of a self-conjugate map is self-conjugate, but the converse is not true (even for positive linear maps). Indeed, consider a linear map $f : (\mathbb{C}, \{0\}) \to (\mathbb{C}, \mathbb{R}_{\geq 0})$ which is not self-conjugate (i.e. of the form $x \mapsto \lambda x$ for $\lambda \notin \mathbb{R}$), then f is positive and $\ker(f) = \{0\}$ is self-conjugate.

3 Order unitisations

A common scenario in abstract algebra and operator theory is that a given (Banach) algebra does not have an algebraic unit, in which case it might be desirable to adjoin one. The same happens in ordered vector spaces: we saw in the previous chapter that having order units presents some advantages. It is therefore of great interest to determine which spaces admit an "extension" (more precisely, an injective positive linear map) to a space with order unit, preferably an Archimedean one.

In this chapter we present two ways of adjoining order units to certain classes of ordered vector spaces, and we study the properties of these order unitisations. One of the main results is the extension theorem from Section 3.5.

As in the previous chapter, we build the theory only for real ordered vector spaces, and we briefly list the modifications needed for the complex case in Section 3.6 below.

3.1 A non-Archimedean order unitisation

There is one straightforward way to adjoin an order unit to a (real) ordered vector space V: consider the direct sum $\mathring{V} := V \oplus \mathbb{R}$ equipped with the reverse lexicographic cone

$$\mathring{V}^+ \ := \ \{(v,\lambda) \in \mathring{V} \ : \ \lambda > 0 \text{ or } (\lambda = 0 \text{ and } v \geq 0)\}.$$

The embedding $v \mapsto (v,0)$ is bipositive, and every element $(v,\lambda) \in \mathring{V}$ with $\lambda > 0$ is an order unit.

We will see that this unitisation is not particularly useful, in that it has various undesirable properties. First of all, we show that $\mathring{V}$ is far form being Archimedean.

Proposition 3.1. *Let V, W be ordered vector spaces and let $\phi : \mathring{V} \to W$ be injective and positive. If $V \neq \{0\}$, then W is not Archimedean.*

Proof. Since ϕ is injective and $V \neq \{0\}$, the image of $V \oplus \{0\}$ under ϕ defines a non-zero subspace $X \subseteq W$. Since $X^+ = X \cap W^+$ is a cone in a non-zero space, we may choose some $x \in X \setminus X^+$. Let $v \in V$ be such that $\phi(v,0) = x$ holds. Note that we have $n \cdot (-v,0) \leq (0,1)$ in $\mathring{V}$ for all $n \in \mathbb{N}^+$, so by positivity of ϕ we also have $-nx \leq \phi(0,1)$ for all $n \in \mathbb{N}^+$. However, we have $-x \not\leq 0$, since we assumed $x \notin X^+$, so we find that W is not Archimedean. ∎

In particular, for $V \neq 0$ we have that $\mathring{V}$ itself is not Archimedean, and cannot be extended to an Archimedean space. This shows that the order unitisation $\mathring{V}$ constructed here is, in a sense, maximally non-Archimedean.

Since $\mathring{V}$ is a non-zero ordered vector space with an order unit, it follows from Corollary 2.11 that $\mathring{V}$ admits a maximal ideal, and therefore a non-zero positive linear functional $f : \mathring{V} \to \mathbb{R}$. However, we could have noticed that ourselves: clearly the map $(v,\lambda) \mapsto \lambda$ is non-zero and positive. In fact, the situation is much worse: the positive scalar multiples of this map are the only positive linear functionals!

Proposition 3.2. *Let V be a (real) ordered vector space. Then $V \oplus \{0\}$ is the unique maximal ideal of $\mathring{V}$.*

Proof. Clearly $V \oplus \{0\}$ is a maximal ideal, as it is the kernel of the positive linear functional $(v, \lambda) \mapsto \lambda$.

Now let $I \subseteq \mathring{V}$ be an arbitrary maximal ideal. Since order units (and their negatives) cannot belong to a proper ideal, we must have $I \subseteq V \oplus \{0\}$. Then, by maximality of I, we have $I = V \oplus \{0\}$. ■

The preceding results show that $\mathring{V}$ has none of the advantages of spaces with an order unit, so we will not study this unitisation any further.

3.2 An Archimedean order unitisation

If the ordered vector space V is equipped with a sufficiently nice norm $\|\cdot\|$, then we can carry out a different construction, which turns out to be more fruitful. The assumption we make is that V is topologically order semisimple with respect to $\|\cdot\|$. By Corollary 2.29, this is equivalent to the requirement that $\overline{V^+}$ is a cone, as opposed to a wedge.

Definition 3.3. Let V be a (real) normed ordered space which is topologically order semisimple. Then we define $\tilde{V}$ to be the space $\tilde{V} := V \oplus \mathbb{R}$ equipped with the cone

$$\tilde{V}^+ \;:=\; \big\{(v, \lambda) \in \tilde{V} \;:\; \lambda \geq d(v, V^+)\big\},$$

where $d(v, V^+) := \inf_{w \in V^+} \|v - w\|$ denotes the distance between v and V^+.

We shall shortly see that $\tilde{V}$ is indeed an Archimedean unitisation of V. Furthermore, we prove a fundamental property of this unitisation in Section 3.5. However, it is not a universal property, so $\tilde{V}$ is merely *an* Archimedean order unitisation of V (see Remark 3.20).

Before proving any additional properties of this unitisation, we first have to prove that $\tilde{V}$ is Archimedean, or that $\tilde{V}^+$ even defines a cone at all.

Proposition 3.4. *Let V be a (real) normed ordered space which is topologically order semisimple. Then the set $\tilde{V}^+$ from Definition 3.3 is a cone.*

Proof. Since V^+ is a cone, for $v \in V$ and $\alpha > 0$ we have

$$d(\alpha v, V^+) \;=\; \inf_{x \in V^+} \|\alpha v - x\| \;=\; \inf_{y \in V^+} \|\alpha v - \alpha y\| \;=\; \alpha d(v, V^+).$$

Similarly, for $v, w \in V$ we have

$$\begin{aligned} d(v + w, V^+) \;&=\; \inf_{x, y \in V^+} \|v + w - x - y\| \\ &\leq\; \inf_{x, y \in V^+} \|v - x\| + \|w - y\| \\ &=\; d(v, V^+) + d(w, V^+). \end{aligned}$$

From this it easily follows that $\tilde{V}^+$ is a wedge.

In order to show that $\tilde{V}^+$ is a cone, let $(v, \lambda) \in \tilde{V}^+ \cap -\tilde{V}^+$ be given. Note that we have $\lambda \geq d(v, V^+) \geq 0$ as well as $-\lambda \geq d(-v, V^+) \geq 0$, so we find $\lambda = 0$ as well as $d(v, V^+) = d(-v, V^+) = 0$. From the latter it follows that $v, -v \in \overline{V^+}$ holds. Since V is topologically order semisimple, we know that $\overline{V^+}$ is a cone, so we must have $v = 0$. ■

Proposition 3.5. *Let V be a (real) normed ordered space which is topologically order semisimple. Then the cone $\tilde{V}^+$ defined in Definition 3.3 is Archimedean, and $(0,1) \in \tilde{V}^+$ is an order unit.*

Proof. Note that the function $V \to \mathbb{R}$, $v \mapsto d(v, V^+)$ is uniformly continuous: if $v, w \in V$ lie distance ε apart, then $d(w, V^+)$ is at most $d(v, V^+) + \varepsilon$, by the triangle inequality. Analogously, we have $d(v, V^+) \leq d(w, V^+) + \varepsilon$, so we find $|d(v, V^+) - d(w, V^+)| \leq \|v - w\|$, proving our claim.

Now equip $\tilde{V}$ with the norm $\|(v, \lambda)\|_1 := \|v\| + |\lambda|$. It follows from the preceding paragraph that the function $f : \tilde{V} \to \mathbb{F}$, $(v, \lambda) \mapsto \lambda - d(v, V^+)$ is continuous. Since $\tilde{V}^+$ is the inverse image of $\mathbb{R}_{\geq 0}$ under f, it follows that $\tilde{V}^+$ is closed. As a consequence, $\tilde{V}$ is Archimedean.

In order to show that $(0,1) \in \tilde{V}$ is an order unit, assume that $(v, \lambda) \in \tilde{V}$ is given. We claim that $(v, \lambda) \leq \|(v, \lambda)\|_1 \cdot (0,1)$ holds. Indeed, note that we have $d(-v, V^+) \leq \|-v\| = \|v\|$ (because $0 \in V^+$), hence

$$\|(v, \lambda)\|_1 - \lambda \;=\; \|v\| + |\lambda| - \lambda \;\geq\; \|v\| \;\geq\; d(-v, V^+),$$

which proves our claim. ■

The norm $\| \cdot \|_1$ defined in the proof of Proposition 3.5 is only temporary. Now that it is established that $\tilde{V}$ is an Archimedean space with order unit $u = (0,1) \in \tilde{V}^+$, we will understand $\tilde{V}$ to be normed with the norm $\| \cdot \|_u$ corresponding to this order unit (cf. Theorem 1.29).

Proposition 3.6. *Let V be a (real) normed ordered space which is topologically order semisimple. Then the map $\phi : V \to \tilde{V}$, $v \mapsto (v, 0)$ is injective, positive, and norm-decreasing. Furthermore, ϕ is bipositive as a map $(V, \overline{V^+}) \to \tilde{V}$.*

Proof. Injectivity of ϕ is a simple algebraic fact, and positivity follows because $v \in V^+$ implies $d(v, V^+) = 0$.

To show that ϕ is continuous, recall from the proof of Proposition 3.5 that we have $(v, \lambda) \leq \|(v, \lambda)\|_1 \cdot (0,1)$ for all $(v, \lambda) \in \tilde{V}$. Replacing (v, λ) by $(-v, -\lambda)$, we also find $-\|(v, \lambda)\|_1 \cdot (0,1) \leq (v, \lambda)$, so it follows that $\|(v, \lambda)\|_u \leq \|(v, \lambda)\|_1$ holds. In particular, for $\lambda = 0$ we find $\|(v, 0)\|_u \leq \|v\|$, which shows that ϕ is continuous with $\|\phi\| \leq 1$.

Finally, it is clear from the definition that $\tilde{V}^+ \cap (V \oplus \{0\}) = \overline{V^+} \oplus \{0\}$ holds, so we see that ϕ is bipositive as a map $(V, \overline{V^+}) \to \tilde{V}$. ■

It follows that $\tilde{V}$ really defines an Archimedean order unitisation of V. The remainder of this chapter is devoted to studying the properties of this unitisation.

First of all, we use Theorem 1.29 in order to obtain a simple formula for the norm of $\tilde{V}$.

Proposition 3.7. *Let V be a (real) normed ordered space which is topologically order semisimple, and let $\tilde{V}$ be as in Definition 3.3. Now the functions $\alpha_u, \omega_u : \tilde{V} \to \mathbb{R}$, as defined in Theorem 1.29, are given by*

$$\alpha_u\big((v,\lambda)\big) \;=\; \lambda - d(v, V^+) \qquad \textit{and} \qquad \omega_u\big((v,\lambda)\big) \;=\; \lambda + d(-v, V^+).$$

Consequently, the norm $\|\cdot\|_u$ on $\tilde{V}$ is given by

$$\|(v,\lambda)\|_u \;=\; \max\big(d(v,V^+) - \lambda\,,\; d(-v,V^+) + \lambda\big).$$

Proof. By definition we have

$$\begin{aligned}
\omega_u\big((v,\lambda)\big) \;&=\; \inf\big\{\mu \in \mathbb{R} \;:\; (v,\lambda) \leq (0,\mu)\big\} \\
&=\; \inf\big\{\mu \in \mathbb{R} \;:\; (-v, \mu - \lambda) \in \tilde{V}^+\big\} \\
&=\; \inf\big\{\mu \in \mathbb{R} \;:\; \mu - \lambda \geq d(-v, V^+)\big\} \\
&=\; \lambda + d(-v, V^+).
\end{aligned}$$

The formula for $\alpha_u((v,\lambda))$ follows analogously. The formula for $\|\cdot\|_u$ follows immediately from Theorem 1.29(c). ■

3.3 Special cases of the construction

Before we proceed to study the general properties of the order unitisation from Section 3.2, it is helpful to look at two motivating examples, which arise as special cases of the current order unitisation.

The first example comes from the theory of ordered vector spaces.

Example 3.8. If V is a (real) normed space without order structure, then we may consider the zero cone $\{0\}$ in V. It is automatically closed, so it turns V into a topologically order semisimple normed space. The unitisation of V is equipped with the cone

$$\tilde{V}^+ \;:=\; \big\{(v,\lambda) \in \tilde{V} \;:\; \lambda \geq \|v\|\big\}.$$

The cone thus constructed is a special case of the so-called *ice cream cones*; a larger class of cones of this type is studied in [AT07, Section 2.6]. ⊠

Another example arises in the theory of C^*-algebras.

Example 3.9. Let $\mathcal{A}$ be a non-unital C^*-algebra, and let V be its self-adjoint part. Then V is a real normed vector space with a closed cone $V^+ = \mathcal{A}^+$. As such, V is topologically order semisimple, so we may consider its order unitisation $\tilde{V}$. We compare it with the C^*-algebra unitisation of $\mathcal{A}$, defined as $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ and subsequently equipped with a C^*-algebra structure in such a way that $(0,1)$ is the (algebraic) unit (cf. [Mur90, Theorem 2.1.6]). We show that the natural map $\tilde{V} \to \tilde{\mathcal{A}}$, $(v,\lambda) \mapsto (v,\lambda)$ is bipositive, so the order unitisation of the self-adjoint part of $\mathcal{A}$ is equipped simply with the positive cone of $\tilde{\mathcal{A}}$.

Let $a \in \tilde{\mathcal{A}}^{\mathrm{sa}}$ be self-adjoint, and let $a^+, a^- \in \tilde{\mathcal{A}}^+$ denote its positive and negative parts, respectively. Note that we have

$$-\|a^-\| \;=\; \min \sigma(a) \;=\; \max\{\lambda \in \mathbb{R} \,:\, a - \lambda \geq 0\}.$$

(The first equality can be seen by taking the Gelfand representation of the C^*-subalgebra generated by 1 and a, and the second follows from the easy observation that $\sigma(a+\lambda) = \sigma(a) + \lambda$ holds for every $\lambda \in \mathbb{C}$.)

Note furthermore that we have $\|a\| = \max(\|a^-\|, \|a^+\|)$ for any $a \in \tilde{\mathcal{A}}^{\mathrm{sa}}$; again this follows easily by looking at the Gelfand representation of the C^*-subalgebra generated by 1 and a.

Let $a \in \tilde{\mathcal{A}}^{\mathrm{sa}}$ be given, then for any positive $b \in \tilde{\mathcal{A}}^+$ we have

$$\begin{aligned} -\|(a-b)^-\| &= \max\{\lambda \in \mathbb{R} \,:\, a-b-\lambda \geq 0\} \\ &\leq \max\{\lambda \in \mathbb{R} \,:\, a-\lambda \geq 0\} \\ &= -\|a^-\|, \end{aligned}$$

and therefore $\|a-b\| \geq \|(a-b)^-\| \geq \|a^-\|$. This holds for every $b \geq 0$, so it follows that $d(a, \tilde{\mathcal{A}}^+) \geq \|a^-\|$ holds. The choice of $b = a^+$ gives equality, so we have $d(a, \tilde{\mathcal{A}}^+) = \|a^-\|$.

Now suppose that a belongs to the C^*-subalgebra $\mathcal{A} \subseteq \tilde{\mathcal{A}}$, then we clearly have $d(a, \mathcal{A}^+) \geq d(a, \tilde{\mathcal{A}}^+) = \|a^-\|$. Since a^- and a^+ also belong to $\mathcal{A}$, we once again have equality: $d(a, \mathcal{A}^+) = \|a^-\|$.

By what was established before, we have $a - \lambda \geq 0$ if and only if $\lambda \leq -\|a^-\|$ holds. Equivalently, we have $a + \lambda \geq 0$ if and only if $\lambda \geq \|a^-\| = d(a, \mathcal{A}^+)$ holds. This shows that the cones of $\tilde{\mathcal{A}}$ and $\tilde{V}$ coincide. ⊠

3.4 The associated fully monotone norm

In this section we use the order unitisation to define a different norm on V, and we study the properties of this new norm. Throughout this section, V denotes a (real) normed ordered space which is topologically order semisimple, and $\tilde{V}$ denotes the order unitisation that was constructed in Section 3.2.

Since the natural map $\phi : V \to \tilde{V}$ is injective, we can define a second norm $\|\cdot\|_u$ on V by setting

$$\|v\|_u \;:=\; \|\phi(v)\|_u \;=\; \|(v,0)\|_u.$$

Since the inclusion $\phi : V \hookrightarrow \tilde{V}$ is positive and $\|\cdot\|_u$ is a fully monotone norm on $\tilde{V}$ (cf. Theorem 1.29(e)), it is easy to see that $\|\cdot\|_u$ defines a fully monotone norm on V as well. We call this the *fully monotone norm associated with* $\|\cdot\|$.[1] It follows from Proposition 3.7 that $\|\cdot\|_u$ is given by

$$\|v\|_u \;=\; \max\big(d(v, V^+), d(-v, V^+)\big).$$

Remark 3.10. Like before, let $\phi : V \hookrightarrow \tilde{V}$ denote the natural inclusion. Recall from the proof of Proposition 3.6 that $\phi^{-1}[\tilde{V}^+] = \overline{V^+}$ holds. Since $\tilde{V}^+$ is closed with respect to $\|\cdot\|_u$ (cf. Theorem 1.29(g)) and ϕ is isometric as a map $(V, \|\cdot\|_u) \to \tilde{V}$, it follows that $\overline{V^+}$ is closed not only with respect to $\|\cdot\|$ but also with respect to $\|\cdot\|_u$. In particular, it follows that $\overline{V^+}$ and $\overline{V^+}^u$ coincide, and that $(V, \|\cdot\|_u)$ is also topologically order semisimple.

For the remainder of this section, we study the relationship between $\|\cdot\|_u$ and $\|\cdot\|$.

Recall that we have $\|v\|_u \leq \|v\|$ for all $v \in V$. The following proposition gives a necessary and sufficient condition for these two norms to be equal.

Proposition 3.11. *Let V be a (real) normed ordered space which is topologically order semisimple. Then the norms $\|\cdot\|$ and $\|\cdot\|_u$ on V coincide if and only if $\|\cdot\|$ is fully monotone.*

Proof. If $\|\cdot\|$ and $\|\cdot\|_u$ coincide, then clearly $\|\cdot\|$ is fully monotone.

For the converse, suppose that $\|\cdot\|$ is fully monotone, and let $v \in V$ be given. For every choice of $x, y \in V^+$, to be determined later, we have $v - x \leq v \leq v + y$, hence $\|v\| \leq \max(\|v - x\|, \|v + y\|)$.

Now let $\mu > \max(d(v, V^+), d(-v, V^+))$ be given, then we have $\mu > d(v, V^+)$ and $\mu > d(-v, V^+)$, so we may choose $x, y \in V^+$ with $\|v - x\|, \|v + y\| < \mu$. By the preceding paragraph, we have $\|v\| < \mu$ in this case. But this holds for all $\mu > \max(d(v, V^+), d(-v, V^+))$, so we find

$$\|v\| \;\leq\; \max\big(d(v, V^+), d(-v, V^+)\big) \;=\; \|v\|_u.$$

The inequality $\|v\|_u \leq \|v\|$ was already established in the general case, so here we have equality. ■

Remark 3.12. We note an interesting consequence of Proposition 3.11. Let $\|\cdot\|$ be a norm making V topologically order semisimple, and let $\|\cdot\|_u$ be the fully monotone norm associated with it. By Remark 3.10, V is also topologically order semisimple with respect to $\|\cdot\|_u$, so we may take it one step further and consider the corresponding order unitisation $\tilde{V}_u$ and the fully monotone norm $\|\cdot\|_{uu}$ associated with it. It follows from Proposition 3.11 that $\|\cdot\|_u$ and $\|\cdot\|_{uu}$ are the same! In other words, taking the associated fully monotone norm is idempotent. We prove a stronger statement in Theorem 3.14: the order unitisations $\tilde{V}$ and $\tilde{V}_u$ are the same as well.

Proposition 3.13. *Let V be a (real) normed ordered space which is topologically order semisimple. Then the open unit ball of $(V, \|\cdot\|_u)$ is equal to the full hull of the open unit ball of $(V, \|\cdot\|)$.*

Proof. For convenience, let $B, B_u \subseteq V$ denote the open unit balls of $(V, \|\cdot\|)$ and $(V, \|\cdot\|_u)$, respectively, and let $\mathrm{fh}(B)$ denote the $\|\cdot\|$-closed full hull of B.

Since we have $\|v\|_u \leq \|v\|$ for all $v \in V$, it follows that $B \subseteq B_u$ holds. Note that B_u is full, since $\|\cdot\|_u$ is fully monotone, so we have $B \subseteq \mathrm{fh}(B) \subseteq B_u$.

Recall from Proposition 1.19 that the full hull of a convex and balanced set is again convex and balanced. Furthermore, since we have $B \subseteq \mathrm{fh}(B)$, it is clear that $\mathrm{fh}(B)$ is absorbing. As such, the Minkowski functional $p : V \to \mathbb{R}_{\geq 0}$ associated with $\mathrm{fh}(B)$ defines a seminorm. As we have $B \subseteq \mathrm{fh}(B) \subseteq B_u$, it is clear that $\|v\|_u \leq p(v) \leq \|v\|$ holds for all $v \in V$. An immediate consequence of this is that p is a norm. Furthermore, it follows from Proposition 1.27 that p is fully monotone.

Now let p_u denote the fully monotone norm associated with p. Then, by Proposition 3.11 we have $p_u = p$. Furthermore, since we have $p(v) \leq \|v\|$, we also have $d_p(v, V^+) \leq d_{\|\cdot\|}(v, V^+)$, and therefore $p_u(v) \leq \|v\|_u$. All in all, for every $v \in V$ we have

$$p(v) = p_u(v) \leq \|v\|_u \leq p(v),$$

so we must have equality throughout. It follows from [Rud91, Theorem 1.35(d)] that $B_u \subseteq \mathrm{fh}(B)$ holds. ■

Theorem 3.14. *Let V be a (real) normed ordered space which is topologically order semisimple, and let $\tilde{V}$ and $\tilde{V}_u$ denote the order unitisations of $(V, \|\cdot\|)$ and $(V, \|\cdot\|_u)$, respectively. Then one has $\tilde{V}^+ = \tilde{V}_u^+$.*

Proof. Like in the proof of Proposition 3.13, let $B, B_u \subseteq V$ denote the open unit balls of $(V, \|\cdot\|)$ and $(V, \|\cdot\|_u)$, respectively. We prove the following: for $v \in V$ and $\alpha > 0$, we have $(v+\alpha B) \cap V^+ \neq \varnothing$ if and only if $(v+\alpha B_u) \cap V^+ \neq \varnothing$.

Since we have $B \subseteq B_u$, it is immediately clear that $(v + \alpha B) \cap V^+ \neq \varnothing$ implies $(v+\alpha B_u) \cap V^+ \neq \varnothing$. For the converse, suppose that $y \in (v+\alpha B_u) \cap V^+$ is given. Then we have $\frac{1}{\alpha}(y - v) \in B_u$. Since B_u is the full hull of B, we may choose $x, z \in B$ with $x \leq \frac{1}{\alpha}(y-v) \leq z$. In particular, we have $v+\alpha z \in v+\alpha B$, but also $v + \alpha z \geq y \geq 0$, so it follows that $(v + \alpha B) \cap V^+ \neq \varnothing$ holds. This proves our claim.

Now, to prove the theorem, note that we may write

$$d(v, V^+) \;=\; \inf\big\{\alpha > 0 \;:\; (v + \alpha B) \cap V^+ \neq \varnothing\big\},$$

and similarly for $\|\cdot\|_u$. It follows from our claim that $d(v, V^+) = d_u(v, V^+)$ holds for all $v \in V$. In particular, we have $\lambda \geq d(v, V^+)$ if and only if $\lambda \geq d_u(v, V^+)$, and the conclusion follows. ∎

Recall that a locally full norm $\|\cdot\|$ is always topologically order semisimple (cf. Theorem 1.25) and equivalent to a fully monotone norm (cf. Theorem 1.28). This leads one to ask whether $\|\cdot\|$ and $\|\cdot\|_u$ are automatically equivalent in this case. Perhaps not surprisingly, the answer is yes.

Proposition 3.15. *Let V be a (real) ordered vector space equipped with a locally full norm $\|\cdot\|$, and let $\|\cdot\|_u$ denote its associated fully monotone norm. Then $\|\cdot\|$ and $\|\cdot\|_u$ are equivalent.*

Proof. Like in the proof of Proposition 3.13, let $B, B_u \subseteq V$ denote the open unit balls of $(V, \|\cdot\|)$ and $(V, \|\cdot\|_u)$, respectively. There exists a neighbourhood base of 0 consisting of full sets, so in particular we may choose a full neighbourhood F of 0 satisfying $F \subseteq B$. But now we also have $\varepsilon B \subseteq F$ for some $\varepsilon > 0$, since F is a neighbourhood of 0. Therefore we have $B \subseteq \frac{1}{\varepsilon}F$, hence $B_u = \operatorname{fh}(B) \subseteq \frac{1}{\varepsilon}F$. All in all, we have $B \subseteq B_u \subseteq \frac{1}{\varepsilon}F \subseteq \frac{1}{\varepsilon}B$, so it follows that $\varepsilon\|v\| \leq \|v\|_u \leq \|v\|$ holds for all $v \in V$. In other words, $\|\cdot\|$ and $\|\cdot\|_u$ are equivalent. ∎

Corollary 3.16. *Let V be a (real) normed ordered space which is topologically order semisimple. Then the norms $\|\cdot\|$ and $\|\cdot\|_u$ on V are equivalent if and only if $\|\cdot\|$ is locally full.*

We saw in Theorem 3.14 that $\|\cdot\|$ and $\|\cdot\|_u$ always give rise to exactly the same order unitisations, even if these two norms are inequivalent. We might ask whether this holds in certain other cases as well. However, even if we have two equivalent norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on a topologically semisimple ordered space, their order unitisations $\tilde{V}_a^+$ and $\tilde{V}_b^+$ need not be equal. In fact, they might even fail to be order isomorphic. For instance, if we consider $V = \mathbb{R}^2$ with the coordinate-wise cone $\mathbb{R}^2_{\geq 0}$, then the ℓ_2-norm gives rise to a cone $\tilde{V}_2^+ \subseteq \mathbb{R}^3$ with a curved side, whereas the ℓ_∞-norm gives rise to a polyhedral (and even a lattice) cone. In general, the only relation that remains between the cones $\tilde{V}_a^+$ and $\tilde{V}_b^+$ corresponding to equivalent norms $\|\cdot\|_a$ and $\|\cdot\|_b$ is that there are $\alpha, \omega \in \mathbb{R}_{>0}$ such that $\alpha\tilde{V}_a \subseteq \tilde{V}_b \subseteq \omega\tilde{V}_a$ holds.

3.5 Extension of positive linear maps

In this section, we consider the following question: when can a continuous positive linear map $f : V \to W$ be extended to a continuous positive linear map $\tilde{V} \to W$? We show that this can be done whenever W is an Archimedean space with an order unit.

Theorem 3.17. *Let V be a (real) normed ordered space which is topologically order semisimple, and let W be a (real) Archimedean space with an order unit $w \in W^+$, equipped with the order unit norm $\|\cdot\|_w$. If $f : V \to W$ is a continuous positive linear map, then the map $\tilde{f} : \tilde{V} \to W$ given by*

$$\tilde{f}(v, \lambda) \;:=\; f(v) + \lambda \cdot \|f\| \cdot w.$$

is a continuous positive linear extension of f satisfying $\|\tilde{f}\| = \|f\|$.

Proof. Let $x = (v, \lambda) \in \tilde{V}$ be positive, that is, satisfying $\lambda \geq d(v, V^+)$. Since f is positive, we have

$$\begin{aligned} d_w(f(v), W^+) \;&=\; \inf_{y \in W^+} \|f(v) - y\| \\ &\leq\; \inf_{x \in V^+} \|f(v) - f(x)\| \\ &\leq\; \|f\| \cdot d(v, V^+) \\ &\leq\; \|f\| \cdot \lambda. \end{aligned}$$

Recall that the closed unit ball of $\|\cdot\|_w$ coincides with the order interval $[-w, w]$. Consequently, for all $\mu > \|f\| \cdot \lambda$ we have $(f(v) + [-\mu w, \mu w]) \cap W^+ \neq \varnothing$, and therefore $f(v) + \mu w \geq 0$. In particular, if we take $\mu = \|f\| \cdot \lambda + \frac{1}{n}$ for some $n \in \mathbb{N}^+$, then we find $-\frac{1}{n} w \leq f(v) + \|f\| \cdot \lambda \cdot w$, or equivalently:

$$n \cdot \big(-f(v) \;-\; \|f\| \cdot \lambda \cdot w \big) \;\leq\; w, \qquad \text{(for all } n \in \mathbb{N}^+\text{)}.$$

It follows from the Archimedean property that $-f(v) - \|f\| \cdot \lambda \cdot w \leq 0$ holds. This shows that we have $\tilde{f}(v, \lambda) \geq 0$, proving that $\tilde{f}$ is positive.

Clearly $\tilde{f}$ extends f. In order to see that $\tilde{f}$ is continuous, let $(v, \lambda) \in \tilde{V}$ be given with $\|(v, \lambda)\|_u \leq 1$, so that we have $(0, -1) \leq (v, \lambda) \leq (0, 1)$. Since $\tilde{f}$ is positive, it follows that we have $-\|f\|w \leq \tilde{f}(v, \lambda) \leq \|f\|w$, and therefore $\|\tilde{f}(v, \lambda)\|_w \leq \|f\|$. This shows that $\tilde{f}$ is continuous with $\|\tilde{f}\| \leq \|f\|$. Plugging in $(v, \lambda) = (0, 1)$ shows that we have equality: $\|\tilde{f}\| = \|f\|$. ∎

It should be noted that the extension $\tilde{f}$ constructed in Theorem 3.17 is not generally unique; there might be another continuous positive extension $g : \tilde{V} \to W$ satisfying $\|g\| = \|f\|$. Consider the following example.

Example 3.18. Set $V := (\mathbb{R}, \mathbb{R}_{\geq 0})$ with the standard norm, $W := (\mathbb{R}^2, \mathbb{R}^2_{\geq 0})$ with the order unit $w := (1, 1) \in W^+$, and let $f : V \to W$ be the isometric order embedding $f(x) := (x, 0)$. Theorem 3.17 now gives us an extension $\tilde{f} : \tilde{V} \to W$ satisfying $\|\tilde{f}\| = \|f\| = 1$, given by $(x, \lambda) \mapsto (x + \lambda, \lambda)$.

On the other hand, we can also take the identity $g : V \to V$, $x \mapsto x$ and extend it to a positive linear map $\tilde{g} : \tilde{V} \to V$ satisfying $\|\tilde{g}\| = \|g\| = 1$, given by $(x, \lambda) \mapsto x + \lambda$. If this is subsequently composed with the isometric order embedding $f : V \to W$, $x \mapsto (x, 0)$, then we get another continuous positive linear map $f \circ \tilde{g} : \tilde{V} \to W$. The latter is given concretely by $(x, \lambda) \mapsto (x + \lambda, 0)$; clearly it extends f. Since f is isometric, we have $\|f \circ \tilde{g}\| = \|\tilde{g}\| = 1$, showing that the extension is not unique.

Generally speaking, there might be an order unit w' relative to some subspace $W' \subseteq W$ containing $\operatorname{ran}(f)$. If the order unit norms $\|\cdot\|_w$ and $\|\cdot\|_{w'}$ coincide on W', then the extensions $\tilde{V} \to W$ and $\tilde{V} \to W' \hookrightarrow W$ differ. ⊠

The extension theorem has some interesting consequences. First of all, if W is a normed space, then we note that a continuous linear map $(V, \|\cdot\|_u) \to W$ is also continuous as a map $(V, \|\cdot\|) \to W$, since we have $\|v\|_u \leq \|v\|$ for all $v \in V$. Let $\|f\|_u$ and $\|f\|$ denote the norm of f as a map $(V, \|\cdot\|_u) \to W$ or $(V, \|\cdot\|) \to W$, respectively, then we clearly have $\|f\| \leq \|f\|_u$. Remarkably, for certain *positive* continuous linear maps, we have the following converse.

Corollary 3.19. *Let V be a (real) normed ordered space which is topologically order semisimple, and let W be a (real) Archimedean space with an order unit $w \in W^+$, equipped with the corresponding order unit norm $\|\cdot\|_w$. Then a linear map $f : V \to W$ is continuous and positive as a map $(V, V^+, \|\cdot\|) \to W$ if and only if it is continuous and positive as a map $(V, \overline{V^+}, \|\cdot\|_u) \to W$. Furthermore, if this is the case, then one has $\|f\| = \|f\|_u$.*

Proof. Since the identity $(V, V^+, \|\cdot\|) \to (V, \overline{V^+}, \|\cdot\|_u)$ is positive and norm-decreasing, it is clear that a continuous positive linear map $(V, \overline{V^+}, \|\cdot\|_u) \to W$ is also continuous and positive as a map $(V, V^+, \|\cdot\|) \to W$. Furthermore, in this case we clearly have $\|f\| \leq \|f\|_u$.

Conversely, if f is continuous and positive as a map $(V, V^+, \|\cdot\|) \to W$, then it follows from Theorem 3.17 that f can be extended to a continuous positive linear map $\tilde{f} : \tilde{V} \to W$ with $\|f\| = \|\tilde{f}\|$. Since the natural map $\phi : V \to \tilde{V}$ defines a bipositive isometry $(V, \overline{V^+}, \|\cdot\|_u) \to \tilde{V}$, it follows that f is also continuous and positive as a map $(V, \overline{V^+}, \|\cdot\|_u) \to W$, and that we have $\|f\|_u = \|\tilde{f} \circ \phi\|_u \leq \|\tilde{f}\| = \|f\|$. ■

The special case $W = \mathbb{R}$ is worth mentioning: we find that $(V, V^+, \|\cdot\|)$ and $(V, \overline{V^+}, \|\cdot\|_u)$ have the same continuous positive linear functionals! As such, a closed maximal order ideal $I \subseteq V$ for the cone $V^+ \subseteq V$ and the norm $\|\cdot\|$ is also full with respect to the larger cone $\overline{V^+}$, and similarly remains closed for the smaller (i.e. topologically weaker/coarser) norm $\|\cdot\|_u$.

Remark 3.20. We note that the property from Theorem 3.17 does not constitute a universal property. For this it would be required that $\tilde{V}$ is uniquely determined by the property from Theorem 3.17, up to a unique isomorphism. However, it might happen that the subspace $V \oplus \{0\} \subseteq \tilde{V}$ already contains an order unit u' which furthermore induces the same norm $\|\cdot\|_{u'} = \|\cdot\|_u$ on $V \oplus \{0\}$. If this is the case, then $V \oplus \{0\} \subseteq \tilde{V}$ also satisfies the property from Theorem 3.17: this follows at once from Corollary 3.19. (Note: as a vector space, $V \oplus \{0\}$ is isomorphic with V, but as a normed ordered space it is understood to be equipped with the positive cone $\overline{V^+} \oplus \{0\}$ and the norm $\|\cdot\|_u$.)

It is easy to come up with examples where this happens: let V be an Archimedean space with an order unit $u \in V$, equipped with the order unit norm $\|\cdot\|_u$. Consider a subcone $\mathcal{K} \subseteq V^+$ whose closure equals V^+, then the space $(V, \mathcal{K}, \|\cdot\|_u)$ has a smaller Archimedean order unitisation than $\tilde{V}$.

A second consequence of Theorem 3.17 is that continuous positive linear maps $V \to W$ can be extended to continuous positive linear maps $\tilde{V} \to \tilde{W}$.

Corollary 3.21. *Let V and W be (real) normed ordered spaces which are topologically order semisimple. If $f : V \to W$ is a continuous positive linear map, then the map $\tilde{f} : \tilde{V} \to \tilde{W}$ given by $(v, \lambda) \mapsto (f(v), \|f\| \cdot \lambda)$ is positive and continuous with $\|\tilde{f}\| = \|f\|$.*

Proof. We apply Theorem 3.17 to the composition

$$V \xrightarrow{f} (W, \|\cdot\|) \xrightarrow{\phi_W} (W, \|\cdot\|_u)$$

of continuous positive linear maps. It follows at once that $\tilde{f}$ is positive and continuous with $\|\tilde{f}\| = \|\phi_W \circ f\| \leq \|f\|$. Since we have $\tilde{f}(0, 1) = (0, \|f\|)$, it is clear that $\|\tilde{f}\| = \|f\|$ holds. ∎

3.6 Modifications for the complex case

We briefly list the modifications needed to be made in the complex case.

The unitisations from sections 3.1 and 3.2 can simply be constructed inside the real part of V. For instance, the complex analogue of Definition 3.3 is to set $\tilde{V} := V \oplus \mathbb{C}$, equipped with the coordinate-wise conjugation and the cone

$$\tilde{V}^+ \;:=\; \big\{(v, \lambda) \in \mathrm{Re}(\tilde{V}) \;:\; \lambda \geq d(v, V^+)\big\}.$$

Most interesting properties of this cone can be understood by passing to $\mathrm{Re}(\tilde{V})$, so much of the order unitisation can be understood through the real theory.

There is exactly one obstacle in the complex setting, and it is a major one. The order unit $u \in \tilde{V}^+$ only defines a norm on $\mathrm{Re}(\tilde{V})$, and there is no canonical way to extend it to a norm on all of $\tilde{V}$; see Remark 1.32. Even in the case where V is a (non-commutative) C^*-algebra (cf. Example 3.9), it is not at all clear whether the norm of its C^*-algebra unitisation can be derived purely from the norm of V and the norm of $\mathrm{Re}(\tilde{V})$.

Due to the apparent difficulty in choosing an appropriate complexification norm, the results in this chapter regarding the norm $\|\cdot\|_u$ do not have straightforward analogues in the complex case. The study of complexification norms is considered to be beyond the scope of this thesis; this might be a direction for further research.

3.7 End notes

1. (page 41) Terminology invented by the author. Another possible name would be *order spectral radius*, as it is similar in spirit to the spectral radius from operator theory. However, this might lead to confusion about the properties of either of the two, as the similarities are not particularly strong. (For instance, the spectral radius does not generally define a norm.)

4 Examples and counterexamples

The purpose of this chapter is to collect some motivating examples and counterexamples relating to the previous chapters. (Some have been moved here so as not to interrupt the flow of ideas.)

4.1 Basic properties of ordered vector spaces

The following examples relate to the basic theory of Chapter 1.

Example 4.1 (An Archimedean cone which is not closed)**.** Let $V := c(\mathbb{N}^+)$ be the space of all convergent sequences with its usual pointwise cone. Consider the linear map $\varphi : c(\mathbb{N}^+) \to c_0(\mathbb{N}^+)$ given by $\varphi(f)(k) = \frac{1}{k}(f(k) + f(k+1))$. Note that φ is injective and positive; therefore we may define a monotone norm on V by setting $\|f\| := \|\varphi(f)\|_\infty$. Define $g, f_1, f_2, \ldots \in V$ by

$$g(k) := \begin{cases} -1, & \text{if } k = 1; \\ 1, & \text{if } k > 1; \end{cases}$$

$$f_n(k) := \begin{cases} (-1)^{k+1}, & \text{if } k \leq n; \\ 0, & \text{if } k > n. \end{cases}$$

Note that we have

$$\varphi(f_n)(k) = \begin{cases} \frac{1}{k} \cdot (-1)^{k+1} & \text{if } k = n; \\ 0, & \text{if } k \neq n. \end{cases}$$

Therefore we have $\|f_n\| = \frac{1}{n}$, so the sequence $\{f_n\}_{n=1}^\infty$ converges to 0. Note that $f_n + g$ is positive for all $n \in \mathbb{N}^+$, so $\{f_n + g\}_{n=1}^\infty$ is a sequence in V^+ converging to g. But g is not positive, so we see that V^+ is not closed.

Another non-closed Archimedean cone is given in Example 4.10. ⊠

Example 4.2 (A generating cone without order units)**.** Fix some $p \in [1, \infty)$, and consider the sequence space $V := \ell^p(\mathbb{N}^+)$ with its usual pointwise cone. Clearly the positive cone of V is generating. We claim that V has no order units. To that end, consider an arbitrary positive element $f \in V^+$. Since the sequence $\{f(k)\}_{k=1}^\infty$ is p-summable, in particular it converges to zero. Therefore we may choose $x_1 < x_2 < \cdots$ such that $f(x_n) \leq \frac{1}{n^3}$ holds for all $n \in \mathbb{N}^+$. Define $g : \mathbb{N}^+ \to \mathbb{R}$ with $g(x_n) = \frac{1}{n^2}$ for all $n \in \mathbb{N}^+$, and zero elsewhere. Note that we have $g \in \ell^p$, since

$$\sum_{k=1}^\infty |g(k)|^p = \sum_{n=1}^\infty \frac{1}{n^{2p}} = \zeta(2p) < +\infty.$$

For every constant $\lambda > 0$ we have $\lambda f(x_n) = \frac{\lambda}{n^3}$ as well as $g(x_n) = \frac{1}{n^2}$ for all $n \in \mathbb{N}^+$. Therefore it is clear that there is no $\lambda > 0$ such that $g \leq \lambda f$ holds. It follows that f is not an order unit. But this holds for all positive $f \in V^+$, so we conclude that V has no order units.

Similar arguments can be used to show that spaces such as c_0 and $C_0(\mathbb{R})$ have no order units (this is even easier). ⊠

Example 4.3 (An order unit which is not an interior point of the cone)**.** Consider the space $V := C([0,1],\mathbb{R})$ with its usual, point-wise cone. Clearly the constant function $\mathbb{1} : [0,1] \to \mathbb{R}$, $x \mapsto 1$ is an order unit. Equip V with the norm $\|f\|_{\text{alt}} := \|x \cdot f(x)\|_\infty$. Then the closed unit ball of $(V, \|\cdot\|_{\text{alt}})$ contains arbitrarily large positive functions, such as the sequence $\{f_n\}_{n=1}^\infty$ given by

$$f_n(x) := \begin{cases} n, & \text{if } x < \frac{1}{n}; \\ \frac{1}{x}, & \text{if } x \geq \frac{1}{n}. \end{cases}$$

Therefore every neighbourhood of $\mathbb{1}$ contains non-negative functions (e.g. of the form $\mathbb{1} - \varepsilon f_n$ for appropriate choices of $\varepsilon > 0$ and $n \in \mathbb{N}^+$). In particular, it follows that $\mathbb{1}$ is not an interior point of V^+. ⊠

Example 4.4 (The order unit seminorm μ_u might not be a norm)**.** Consider $V := \mathbb{R}^2$ with the reverse lexicographic order:

$$V^+ := \big\{(x,y) \in \mathbb{R}^2 \,:\, y > 0 \text{ or } (y = 0 \text{ and } x \geq 0\big\}.$$

In other words, V is the non-Archimedean order unitisation of $(\mathbb{R}, \mathbb{R}_{\geq 0})$, in the sense of Section 3.1. Recall that $u := (0,1)$ is an order unit in this setting. The order interval $[-u, u]$ contains the entire x-axis, so for all $x \in \mathbb{R}$ we have $\mu_u((x,0)) = 0$. Therefore μ_u is not a norm but merely a seminorm. ⊠

4.2 Two natural examples of order ideals

We show that order ideals occur naturally in two well-known settings.

Example 4.5 (The positive cone in L^p spaces)**.** Recall that L^p spaces are not actually spaces of functions, but rather quotients of spaces of functions. Specifically, if $(\Omega, \mathcal{A}, \mu)$ is a measure space, then $L^p(\mu)$ is defined as the quotient of $\mathscr{L}^p(\mu)$ by the subspace $\mathcal{N}$ of functions which are a.e. equal to zero. Note that $\mathcal{N}$ is an ideal: we have $g \in \mathcal{N}$ if and only if $\overline{g} \in \mathcal{N}$ (in the complex case), and if $0 \leq f \leq g$ holds with g a.e. equal to zero, then f is a.e. equal to zero as well. Hence it follows that $L^p(\mu)$ is an ordered vector space (as opposed to a pre-ordered space).

Since $L^p(\mu)^+$ is a closed cone in a locally convex space, it is clear that $L^p(\mu)$ is topologically order semisimple (cf. Corollary 2.29). We will see in Example 4.8 that a quotient of a space of functions might fail to be order semisimple. ⊠

Example 4.6 (Hereditary C^*-subalgebras)**.** Let $\mathcal{A}$ be a C^*-algebra. Recall that a C^*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is said to be *hereditary* if $0 \leq a \leq b$ and $b \in \mathcal{B}$ imply $a \in \mathcal{B}$ (cf. [Mur90, Section 3.2]). Since C^*-subalgebras are self-adjoint, it follows that every hereditary subalgebra is an order ideal.

It can be shown ([Mur90, Theorem 3.1.2 and Corollary 3.2.3]) that every closed ideal in a C^*-algebra is a hereditary subalgebra. It follows that a closed ideal is automatically an order ideal. (Then again, this is not very surprising, since the natural map $\mathcal{A} \to \mathcal{A}/I$ is a $*$-homomorphism, and therefore positive.)

We will see in Example 4.11 that a self-adjoint algebra ideal (not necessarily closed) in a C^*-algebra might fail to be an order ideal. ⊠

4.3 A note on complex ordered vector spaces

Recall from Remark 2.48 that a major reason for our definition of complex ordered spaces (with a complex conjugation) is to ensure that every *simple* space V admits a positive linear isomorphism $V \to (\mathbb{C}, \mathbb{R}_{\geq 0})$, so that every maximal ideal $I \subseteq V$ gives rise to a positive linear functional $V \to (\mathbb{C}, \mathbb{R}_{\geq 0})$.

For this section only, let us assume that a *complex ordered vector space* is defined simply as a complex vector space V equipped with a cone $V^+ \subseteq V$, and that an *ideal* is simply a full subspace of V. To show just how strange the theory becomes in this setting, we show that there are simple ordered spaces of dimension larger than one.

Example 4.7. Let $n \in \mathbb{N}^+$ be a positive integer, and consider $\mathbb{R}^{2n}$ with the lexicographic cone $K \subseteq \mathbb{R}^{2n}$:

$$K := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n} \end{pmatrix} \in \mathbb{R}^{2n} \;\middle|\; \begin{array}{cc} x_1 > 0, & \text{or} \\ (x_1 = 0 \text{ and } x_2 > 0), & \text{or} \\ \vdots & \text{or} \\ (x_1 = \ldots = x_{2n-1} = 0 \text{ and } x_{2n} \geq 0) & \end{array} \right\}.$$

Identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via the real linear isomorphism $\psi : \mathbb{R}^{2n} \xrightarrow{\sim} \mathbb{C}^n$ given by

$$(x_1, x_2, \ldots, x_{2n}) \mapsto \big(x_1 + ix_{n+1}\,,\; x_2 + ix_{n+2}\,,\; \ldots\,,\; x_n + ix_{2n}\big).$$

Now $\psi(K)$ defines a cone in $\mathbb{C}^2$. We prove that $\{0\}$ and $\mathbb{C}^n$ are the only full complex subspaces. Suppose that $I \subseteq \mathbb{C}^n$ is a non-zero full subspace, then I contains some non-zero vector $y = (y_1, \ldots, y_n) \in I$ as well as all *complex* multiples of y. Polarising with respect to some non-zero coefficient y_i gives a vector $\lambda y \in I$ with at least one coordinate y_i in $\mathbb{R}_{>0}$. Consequently, the corresponding real coefficient x_i of λy is strictly positive. Therefore it is clear that the real ideal generated by λy (i.e. the full hull of $\operatorname{span}_{\mathbb{R}}\{\lambda y\}$) must contain the standard basis vectors $e_j \in \mathbb{R}^{2n}$ for all $j > i$. In particular, I contains each of the vectors $(i, 0, \ldots, 0), (0, i, 0, \ldots, 0), \ldots, (0, \ldots, 0, i)$. Since I is a (complex) subspace, it follows that $I = \mathbb{C}^n$ must hold, proving that the space is simple. ⊠

4.4 Discontinuous positive linear functionals

We show various examples relating to discontinuous positive linear functionals.

Example 4.8 (A space with no continuous linear functionals)**.** Let $p \in (0, 1)$ be fixed, and consider the space $V := L^p[0, 1]$ with its usual topology, that is, the topology given by the metric

$$d(f, g) \;=\; \int_0^1 |f(t) - g(t)|^p \, dt.$$

(Note that we do not take the p-th root, as in the spaces L^p for $p \geq 1$, for this would violate the triangle inequality.[1]) This topology turns V into a topological vector space which does not admit any non-zero continuous linear map to a locally convex space; cf. [Rud91, Example 1.47]. In particular, there are no non-zero continuous linear functionals.

It follows that V fails to be topologically order semisimple for every possible choice of a positive cone $V^+ \subseteq V$. If we choose an order semisimple cone $V^+ \subseteq V$ (for instance, the zero cone suffices), then there are plenty of positive linear functionals (enough to separate the points), none of which is continuous. Similarly, there are many maximal order ideals, but none of these is closed.

It is interesting to note that the standard cone $L^p[0,1]^+$ fails to be order semisimple in this case. We outline a proof of this statement. It can be shown that the metric d is complete and that the positive cone is closed, analogously to the familiar case $p \geq 1$. But then it follows from Theorem 2.17 that every positive linear functional is continuous, so the only positive linear functional is the zero functional. In conclusion: $L^p[0,1]$ fails to be order semisimple for all $p \in (0,1)$. This reminds us once again that $L^p[0,1]$ is not a space of functions but a *quotient* of a space of functions. ⊠

While the preceding example borders on the pathological, the following examples exhibit discontinuous positive linear functionals on *normed* ordered spaces with Archimedean and generating cones.

Example 4.9 (Another discontinuous positive linear functional)**.** Consider the algebra $V := \mathbb{R}[X]$ of real-valued polynomials in one variable. Equip V with the norm obtained from the inclusion $\mathbb{R}[X] \hookrightarrow C[0,1]$ and the coefficient-wise cone, that is, the cone obtained from the natural linear isomorphism $\mathbb{R}[X] \cong c_{00}$. For a positive polynomial $f(X) \in V^+$ we have $\|f\| = f(1)$, from which it follows that the norm is monotone. Now it follows from theorems 1.25 and 1.28 that V is topologically order semisimple. This could also be seen directly: clearly the point evaluations $\{\widehat{\omega} : f \mapsto f(\omega) \mid \omega \in [0,1]\}$ are continuous and positive.

Now fix some $\ell \in \mathbb{N}^+$ and let $\pi_\ell : V \to \mathbb{R}$ be the linear map that extracts the ℓ-th coefficient, that is, $\pi_\ell(\alpha_0 + \alpha_1 X + \cdots + \alpha_n X^n) = \alpha_\ell$. Clearly π_ℓ is positive. We show that π_ℓ is discontinuous. Note that we have $\|1 - X\| = 1$, and more generally $\|(1-X)^n\| = 1$ for all $n \in \mathbb{N}^+$. However, by the binomial theorem, we have

$$(1-X)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} \cdot (-X)^k,$$

hence

$$\big|\pi_\ell((1-X)^n)\big| \;=\; \binom{n}{\ell}.$$

Since we assumed $\ell \geq 1$, it follows that π_ℓ is unbounded. ⊠

Example 4.10 (Order semisimple but not topologically order semisimple)**.** Once again, consider the algebra $V := \mathbb{R}[X]$ of real-valued polynomials in one variable. Now let $a, b \in \mathbb{R}$ be real numbers satisfying $1 \leq a < b$, and equip V with the positive cone obtained from the inclusion $\mathbb{R}[X] \hookrightarrow C[a,b]$. Clearly this turns V into an Archimedean and semisimple ordered vector space.

Suppose that V is equipped with the coefficient-wise supremum norm $\|\cdot\|_{00}$ (i.e. the norm obtained from the natural linear isomorphism $\mathbb{R}[X] \cong c_{00} \subseteq c_0$). We show that V is not *topologically* order semisimple in this case. In fact, we prove something stronger: V^+ is dense in V. Note that it follows from this that V does not admit a non-zero continuous positive linear functional; this can either be seen as a consequence of (the proof of) Theorem 2.28, or directly by noting that the inverse image of $\mathbb{R}_{\geq 0}$ under a continuous positive linear functional is a closed set containing V^+, and therefore must be all of V.

To prove our claim, let $g \in V$ be any polynomial, and let $\|g\|_{[a,b]}$ denote the maximum of $|g|$ on the interval $[a, b]$. Suppose that $\varepsilon > 0$ is given. Since we assumed $a \geq 1$, the series $\sum_{n=0}^{\infty} a^n$ diverges, so we may choose some $N \in \mathbb{N}$ such that $\sum_{n=0}^{N} a^n > \frac{\|g\|_{[a,b]}}{\varepsilon}$ holds. By monotonicity we have $\sum_{n=0}^{N} x^n > \frac{\|g\|_{[a,b]}}{\varepsilon}$ for all $x \in [a, b]$, so it follows that the polynomial $g + \varepsilon \sum_{n=0}^{N} x^n$ is positive. This shows that we have $d(g, V^+) \leq \varepsilon$, where the distance is taken with respect to the $\| \cdot \|_{00}$-norm on V. As this holds for every $\varepsilon > 0$, it follows that $g \in \overline{V^+}$ holds. We conclude that V^+ is dense in V, which proves our claim. ⊠

4.5 Algebra versus order ideals in $C_0(\Omega)$

Central to our theory are $C(\Omega, \mathbb{F})$ spaces, i.e. spaces of continuous functions from a compact Hausdorff space Ω to the ground field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Here the connections between operator theory and ordered vector spaces are at their clearest. For instance, the representation theorem for normed, topologically order semisimple spaces obtained in Section 2.8 is very similar to the Gelfand representation for commutative Banach algebras.

More generally, for a locally compact Hausdorff space Ω, the space $C_0(\Omega, \mathbb{F})$ is a normed space naturally equipped with order *and* algebra structure. In this section we briefly compare these structures, though admittedly there is much more to be said. We focus on the comparison between order ideals and algebra ideals.

It is well-known that every closed algebra ideal $I \subseteq C_0(\Omega)$ is of the form $I = \{f \in C_0(\Omega) : f(\omega') = 0 \text{ for all } \omega' \in \Omega'\}$ for some closed subspace $\Omega' \subseteq \Omega$ (for instance, see [Con07, Exercise VII.8.8]). As such, it is easy to see that every closed algebra ideal is also an order ideal. (More generally, a closed algebra ideal in an arbitrary C^*-algebra is automatically an order ideal; see Example 4.6.) The following example shows that a non-closed algebra ideal need not be an order ideal.

Example 4.11. Consider $\Omega := [0, 1]$ and let $I \subseteq C(\Omega, \mathbb{F})$ be the principal ideal generated by the inclusion function $\iota : \Omega \to \mathbb{F}$, $\omega \mapsto \omega$. More concretely, I is the set of all functions of the form $x \cdot f(x)$ for some $f \in C(\Omega, \mathbb{F})$. Now consider $g(x) := x \cdot \sin(\frac{1}{x})$. Then g is continuous, self-adjoint, and satisfies $-\iota \leq g \leq \iota$. However, g cannot belong to I: if $f \in C(\Omega)$ is any function such that $g(x) = x \cdot f(x)$ holds for all $x \in [0, 1]$, then we must have $f(x) = \frac{g(x)}{x} = \sin(\frac{1}{x})$ for all $x \in (0, 1]$, but no such (continuous) function f exists. ⊠

Conversely, we show that a closed order ideal need not be an algebra ideal.

Example 4.12. Once again, we consider $\Omega := [0, 1]$. Now let $\varphi : C(\Omega, \mathbb{F}) \to \mathbb{F}$ be the (self-conjugate) continuous positive linear functional $f \mapsto \int_0^1 f(t)\, dt$. Then the set $I := \ker(\varphi)$ is a closed maximal order ideal. On the other hand, I is clearly not an algebra ideal, or even a subalgebra: for instance, we have $\sin(2\pi x) \in I$, but $\sin^2(2\pi x) \notin I$.

This example occurs because the characters of a commutative C^*-algebra are precisely the pure states, that is, the extreme points of the state space. The functional φ in this example defines a non-pure state. ⊠

However, if the order ideal contains sufficiently many positive elements, then it turns out also to be an algebra ideal.

Proposition 4.13. *Let Ω be a locally compact Hausdorff space and let $I \subseteq C_0(\Omega, \mathbb{F})$ be a (self-conjugate) order ideal such that $I^+ = I \cap C_0(\Omega, \mathbb{F})^+$ is generating in I (i.e. $I = \operatorname{span}(I^+)$). Then I is an algebra ideal.*

Proof. First of all, let $f \in C_0(\Omega, \mathbb{F})$ and $g \in I^+$ be given. Note that we have $-\|f\|_\infty \leq f \leq \|f\|_\infty$. Since multiplying by a *non-negative* constant preserves order, it is easy to see that $-\|f\|_\infty \cdot g(\omega) \leq f \cdot g(\omega) \leq \|f\|_\infty \cdot g(\omega)$ holds for every $\omega \in \Omega$. As such, we find $-\|f\|_\infty \cdot g \leq fg \leq \|f\|_\infty \cdot g$. Since I is an order ideal, it follows that $fg \in I$ holds.

Now let $f \in C_0(\Omega, \mathbb{F})$ and $g \in I$ be arbitrary, then we may write $g = g_1 - g_2$ or $g = g_1 - g_2 + ig_3 - ig_4$, depending on the base field $\mathbb{F}$, with $g_1, g_2, g_3, g_4 \in I^+$. By the preceding, we have $fg_1, fg_2, fg_3, fg_4 \in I$, so it follows that $fg \in I$ holds as well. ■

An example of a non-closed ideal of this form is $C_c(\Omega, \mathbb{F}) \subseteq C_0(\Omega, \mathbb{F})$. Indeed, it is easy to see that this is both an order ideal and an algebra ideal. It is proper if and only if Ω is not compact, and in this case it fails to be closed (in fact, it is dense in $C_0(\Omega, \mathbb{F})$; cf. [Con07, Exercise III.1.13]).

Note that the boundedness of functions in $C_0(\Omega, \mathbb{F})$ was used in a crucial way in the proof of Proposition 4.13. Indeed, the following example shows that a similar result does not hold in spaces of unbounded functions.

Example 4.14. Let Ω be a topological space such that $C(\Omega, \mathbb{F})$ contains unbounded functions (so in particular Ω is not compact). Then the subspace $C_b(\Omega, \mathbb{F}) \subseteq C(\Omega, \mathbb{F})$ is an order ideal with a generating cone, but it is not an algebra ideal. (In a sense, this is because the constant function $\mathbb{1} : \omega \mapsto 1$ is an algebraic unit but not an order unit in this setting.) ⊠

To conclude this section, we mention the following example regarding the existence of maximal ideals in an ordered vector space without order units. Recall from Proposition 2.10 and Corollary 2.11 that every proper order ideal in a space containing an order unit is contained in a maximal order ideal, so in particular a space with an order unit has at least one maximal ideal. The following example shows that this fails if the space does not have an order unit.

Example 4.15 (A non-zero ordered vector space with no maximal ideals)**.** Let Ω be a locally compact Hausdorff space which is not compact (e.g. $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{Z}^n$, but there are many others). Then $C_c(\Omega, \mathbb{F}) \subseteq C_0(\Omega, \mathbb{F})$ is a proper order ideal, so the quotient $V := C_0(\Omega, \mathbb{F})/C_c(\Omega, \mathbb{F})$ defines a non-zero ordered vector space.

We note that every positive linear functional on $C_0(\Omega, \mathbb{F})$ is continuous; this follows either from the well-known fact that positive linear functionals on a C^*-algebra are continuous (cf. [Mur90, Theorem 3.3.1]), or from the more general statement for complete ordered vector spaces with a closed and generating cone (cf. Theorem 2.17). Either way, it follows that every maximal order ideal of $C_0(\Omega, \mathbb{F})$ is closed. However, $C_c(\Omega, \mathbb{F})$ is dense in $C_0(\Omega, \mathbb{F})$, so none of the maximal order ideals of $C_0(\Omega, \mathbb{F})$ contains $C_c(\Omega, \mathbb{F})$. It follows that the proper order ideal $C_c(\Omega, \mathbb{F})$ is not contained in any maximal order ideal. Consequently, the quotient $V = C_0(\Omega, \mathbb{F})/C_c(\Omega, \mathbb{F})$ does not have a maximal order ideal. ⊠

4.6 Spaces of unbounded functions

Recall that an ordered vector space is *order semisimple* if it can be represented (injectively and positively) as a space of functions. By Remark 2.22, every order semisimple space admits a locally convex vector space topology making it *topologically order semisimple*. Finally, recall from abstract algebra that every real or complex vector space admits a norm.[2]

The goal of this section is to show that an order semisimple space does not always admit a norm making it topologically order semisimple. This is done by looking at certain spaces of unbounded functions. Spaces in this class include certain $C(\Omega)$ spaces (Ω not necessarily compact or Hausdorff), as well as certain L^∞ spaces.

We know from Theorem 2.44 that an ordered vector space can be equipped with a norm making it topologically order semisimple if and only if it admits a monotone norm. This turns out to be a convenient condition to check.

For an arbitrary set S, we let $\mathbb{F}^S$ denote the vector space of all functions $S \to \mathbb{F}$, equipped with pointwise complex conjugation $\overline{} : f \mapsto \overline{f}$ and the pointwise cone

$$(\mathbb{F}^S)^+ := \{f : S \to \mathbb{F} \,:\, f(s) \geq 0 \text{ for all } s \in S\},$$

which is both Archimedean and generating. We prove the following result.

Theorem 4.16. *Let V be a subspace of $\mathbb{F}^S$, equipped with the positive cone $V^+ := V \cap (\mathbb{F}^S)^+$. Suppose that V satisfies the following two properties:*

i. For every $f \in V$ we also have $|f| \in V$;

ii. If $f \in V$ is real-valued and $g : \mathbb{R} \to \mathbb{R}$ is piecewise linear, then $g \circ f \in V$.

Then V admits a monotone norm if and only if every function in V is bounded.

Note: the first assumption is needed only in the complex case; in the real case this follows directly from the second assumption.

Proof. If every function in V is bounded, then V can be normed with the supremum norm $\|\cdot\|_\infty$, which is monotone. For the converse, assume that $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ is a monotone norm. Suppose, for the sake of contradiction, that there exists an unbounded function $f \in V$. By assumption we also have $|f| \in V$, so let us assume without loss of generality that f is positive and unbounded. Define piecewise linear functions $g_n : \mathbb{R} \to \mathbb{R}$ by

$$g_n(x) \;:=\; ((x - n + 1) \vee 0) \wedge 1 \;=\; \begin{cases} 0, & \text{if } x < n - 1; \\ x - n + 1, & \text{if } n - 1 \leq x \leq n; \\ 1, & \text{if } x > n. \end{cases}$$

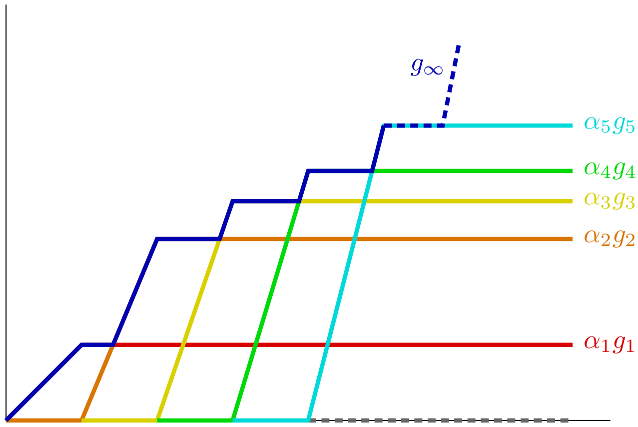
By assumption, we have $g_n \circ f \in V$ for all $n \in \mathbb{N}^+$. Furthermore, we have $g_n \circ f \geq 0$, and $g_n \circ f \neq 0$ since f is unbounded. Define $\{\alpha_n\}_{n=1}^\infty$ in $\mathbb{R}_{>0}$ by

$$\alpha_n \;:=\; \frac{n}{\|g_n \circ f\|}.$$

Note that, for every fixed $x \in \mathbb{R}$, at most finitely many g_n take a non-zero value at x. Hence we may define $g_\infty : \mathbb{R} \to \mathbb{R}$ by

$$g_\infty(x) \;:=\; \max_{n \in \mathbb{N}^+} \alpha_n g_n(x).$$

The situation is illustrated in the following figure.



This function g_∞ is again piecewise linear, so by assumption we have $g_\infty \circ f \in V$. But for all $n \in \mathbb{N}^+$ we have $0 \leq \alpha_n g_n \circ f \leq g_\infty \circ f$, hence

$$\|g_\infty \circ f\| \;\geq\; \|\alpha_n g_n \circ f\| \;=\; n.$$

This is a contradiction, so we conclude that every $f \in V$ must be bounded. ■

We discuss a few consequences of the above theorem. First of all, let us say that a topological space Ω is *pseudocompact* if every continuous function $\Omega \to \mathbb{R}$ is bounded. Clearly every compact topological space is pseudocompact, but there are non-compact spaces which are nevertheless pseudocompact. (Well-known examples include the least uncountable ordinal ω_1 equipped with the order topology, or $\mathbb{R}$ equipped with the left order topology.) Using this terminology, we have the following immediate consequence of Theorem 4.16.

Corollary 4.17. *Let Ω be a topological space. Then $C(\Omega, \mathbb{F})$ admits a monotone norm if and only if Ω is pseudocompact.*

For a second application, note first that Theorem 4.16 does not apply to $\mathscr{L}^p$ spaces for $1 \leq p < \infty$: the composition $g \circ f$ is again measurable, but there is no guarantee that it remains integrable. Indeed, there exist $\mathscr{L}^p$ spaces containing unbounded functions which nevertheless admit a monotone norm. For instance, consider the measure space $(\mathbb{N}^+, \mathcal{P}(\mathbb{N}^+), \mu)$, where μ is the weighted counting measure given by $\mu(\{n\}) := \frac{1}{4^n}$. Now there are no non-empty null sets, so we have $\mathscr{L}^p(\mu) = L^p(\mu)$, and the $\mathscr{L}^p$ seminorm $\|\cdot\|_p$ is in fact a (monotone) norm. But this function space nevertheless contains unbounded functions, for instance $f(n) = 2^{n/p}$.

Although the theorem does not apply to $\mathscr{L}^p$ spaces in general, it does apply to all $\mathscr{L}^\infty$ spaces: if f is almost everywhere bounded, then so is $g \circ f$ for any piecewise linear map $g : \mathbb{R} \to \mathbb{R}$. As a consequence, we find that $\mathscr{L}^\infty[0,1]$ does not admit a monotone norm, since it contains unbounded functions. (Of course, it does admit a monotone *seminorm*: the essential supremum seminorm.)

4.7 End notes

1. (page 49) Let f and g be the indicator functions for the intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1]$, respectively. Then we have

$$\left(\int_0^1 |f(t) + g(t)|^p \, dt\right)^{\frac{1}{p}} = 1;$$

$$\left(\int_0^1 |f(t)|^p \, dt\right)^{\frac{1}{p}} + \left(\int_0^1 |g(t)| \, dt\right)^{\frac{1}{p}} = \left(\tfrac{1}{2}\right)^{\frac{1}{p}} + \left(\tfrac{1}{2}\right)^{\frac{1}{p}} = 2^{1-\frac{1}{p}} < 1.$$

2. (page 53) Let V be a real or complex vector space. Choose a basis $\mathcal{B}$, then we have an isomorphism $\varphi : V \to \mathbb{F}^{\oplus \mathcal{B}}$ to the space of all functions $\mathcal{B} \to \mathbb{F}$ of finite support. The latter can be equipped for instance with the supremum norm.

Part II

The lattice-like structure of C^*-algebras

5 The quasi-lattice operations in a C^*-algebra

In this chapter we introduce the basic setting for the second part of this thesis: the lattice-like structure of C^*-algebras.

In Section 5.1 we briefly recall the relevant properties of C^*-algebra, as well as some basic order theory. In Section 5.2, we show how the Gelfand representation and functional calculus give rise to natural non-commutative analogues of the lattice operations in a commutative C^*-algebra. The remaining sections study some of the basic properties of these so-called *quasi-lattice operations*.

The quasi-lattice operations are studied in additional detail in the next chapters. In Chapter 6 we prove that the quasi-lattice operations might fail to be lattice operations, by proving that the self-adjoint part of $B(\mathcal{H})$ is in fact an *anti-lattice*. In Chapter 7 we dive a little deeper into this problem, resulting in a geometric understanding of the anti-lattice theorem. Finally, in Chapter 8 we show that a C^*-algebra is a lattice if and only if it is commutative.

5.1 Prerequisites

5.1.1 Order theory

We assume familiarity with the concept of partial orders, as well as the basics of ordered vector spaces (cf. Chapter 1). The following concepts are sufficiently important for the remainder of this thesis that we briefly recall their definitions.

Definition 5.1. Let $(P, \preceq)$ be a partially ordered set, and let $S \subseteq P$ be a non-empty subset. Then, for elements $s \in S$, $p \in P$ we say that

- s is a *minimal element* of S if for all $s' \in S$, $s' \preceq s$ implies $s' = s$;
- s is a *maximal element* of S if for all $s' \in S$, $s \preceq s'$ implies $s' = s$;
- s is the *least element* of S if for every $s' \in S$ one has $s \preceq s'$;
- s is the *greatest element* of S if for every $s' \in S$ one has $s' \preceq s$;
- p is a *lower bound* of S if for every $s \in S$ one has $p \preceq s$;
- p is an *upper bound* of S if for every $s \in S$ one has $s \preceq p$;
- p is the *infimum* of S if p is the greatest lower bound of S;
- p is the *supremum* of S if p is the least upper bound of S.

Note the subtle (but substantial) difference between *minimal* and *least* elements. For instance, if the partial order is not total, then we may choose two incomparable elements $p, q \in P$. Consequently, each of p and q is a minimal element of $\{p, q\}$, but neither is a least element.

If the set S has a least element, then it is necessarily unique, so it is justified that we speak of *the* least element (and similarly *the* greatest element, *the* infimum, *the* supremum). Furthermore, clearly a least element of S is the unique minimal element of S. The converse is not true; a unique minimal element need not be a least element.[1]

Definition 5.2. A *lattice* is a partially ordered set $(P, \preceq)$ satisfying any one (and therefore all) of the following equivalent properties:

(1) Every non-empty finite subset $S \subseteq P$ has a supremum and an infimum;

(2) Every two-element subset $S \subseteq P$ has a supremum and an infimum.

Definition 5.3. A real ordered vector space V which is lattice-ordered is called a *Riesz space* (or *vector lattice*). If V is a complex ordered vector space, then we say that V is a *complex Riesz space* if $\mathrm{Re}(V)$ is a Riesz space and for each $v \in V$ the set $\{\mathrm{Re}(\lambda v) : |\lambda| = 1\} \subseteq \mathrm{Re}(V)$ has a supremum (cf. [Zaa97, Chapter 6]).

A typical example of a complex Riesz space is $C_0(\Omega, \mathbb{C})$, where Ω is any locally compact Hausdorff space (cf. [Zaa97, examples 13.1 and 13.2]). Since any commutative C^*-algebra is isometrically $*$-isomorphic (and therefore also order isomorphic) to such a space, it follows that any commutative C^*-algebra is a complex Riesz space.

5.1.2 Positive square roots in a C^*-algebra

It is well-known that every positive element a in a C^*-algebra $\mathcal{A}$ has a positive square root; this follows at once by considering the Gelfand representation of the C^*-subalgebra $C^*(a) \subseteq \mathcal{A}$ generated by a. (This subalgebra may not be unital, so we use the fact that $\sqrt{}$ is continuous with $\sqrt{0} = 0$, so that the square root of a function vanishing at infinity also vanishes at infinity.)

The *existence* of positive square roots is important to our study, but equally important is its *uniqueness*. This is also a well-known result (see for instance [Mur90, Theorem 2.2.1]), but it is so important for what follows that we remind the reader of the underlying argument.

Proposition 5.4. *Let $\mathcal{A}$ be a C^*-algebra, let $a \in \mathcal{A}^+$ be positive, and let $b \in C^*(a)^+$ be the positive square root of a constructed inside $C^*(a)$. Suppose that $c \in \mathcal{A}^+$ is any positive square root of a, then one has $b = c$.*

Proof. Consider the C^*-subalgebra $C^*(c) \subseteq \mathcal{A}$ generated by c. Note that $C^*(c)$ contains $a = c^2$, so we have $C^*(a) \subseteq C^*(c)$. It follows that $b \in C^*(c)$ holds, so b and c are positive square roots of a in the commutative C^*-algebra $C^*(c)$. However, it is clear from the Gelfand representation that positive square roots in commutative C^*-algebras are unique, so we must have $b = c$. ■

5.1.3 Self-adjoint operators on a real Hilbert space

In chapters 6 and 7 we shall also be interested in operators on a *real* Hilbert space $\mathcal{H}$, but of course $B(\mathcal{H})$ does not form a C^*-algebra in this setting. While there is a theory of *real C^*-algebras* (cf. [Ing64, Pal70]), this is beyond the scope of this thesis. For our purposes it suffices that we have the following form of functional calculus.

Theorem 5.5. *Let $\mathcal{H}$ be a real Hilbert space and let $a \in B(\mathcal{H})$ be self-adjoint (i.e. real symmetric). Then there is a unital, isometric, bipositive algebra homomorphism $\varphi_a : C(\sigma(a), \mathbb{R}) \to B(\mathcal{H})$ satisfying $\varphi_a(\iota) = a$, where $\iota : \sigma(a) \to \mathbb{R}$ denotes the inclusion $z \mapsto z$. The range of φ_a is the closed subalgebra generated by 1 and a, that is, the closed linear span of $\{1, a, a^2, \ldots\}$.*

There are multiple ways to prove this. One is to pass to the complexification of $\mathcal{H}$ and show that the functional calculus can be restricted to the real operators to yield the above. A more direct proof, which settles the real and complex case simultaneously, is given in [Lan93, Chapter XVIII, §4].

From this theorem it is clear that a self-adjoint operator on a real Hilbert space also has a positive square root: consider $b := \varphi_a(\sqrt{})$. Furthermore, in this setting positive square roots are once again unique; the proof of this is analogous to the complex case.

5.2 The quasi-lattice operations

Let $\mathcal{A}$ be a C^*-algebra (or $B(\mathcal{H})$ for a real Hilbert space $\mathcal{H}$). For any self-adjoint element $a \in \mathcal{A}^{\mathrm{sa}}$, we have that a^2 is positive, so we may define $|a|$ to be the unique positive square root of a^2. We furthermore define $a^+ := \frac{1}{2}(|a| + a)$ and $a^- := \frac{1}{2}(|a| - a)$. Then it is easy to see from the Gelfand representation (or Theorem 5.5) that a^+ and a^- are positive elements satisfying $a = a^+ - a^-$, $|a| = a^+ + a^-$ and $a^+a^- = a^-a^+ = 0$.

Note that $|a|$ is an upper bound for a and $-a$, since we have $|a| = a + 2a^-$ as well as $|a| = -a + 2a^+$. Now the crucial observation is this: since the vector space order of $\mathcal{A}$ is translation-invariant, we can use the modulus to define upper bounds for arbitrary pairs of self-adjoint elements $a, b \in \mathcal{A}^{\mathrm{sa}}$! This leads to the following definition.

Definition 5.6. Let $\mathcal{A}$ be a C^*-algebra (or $B(\mathcal{H})$ for a real Hilbert space $\mathcal{H}$). For $a, b \in \mathcal{A}^{\mathrm{sa}}$ we define the *quasi-supremum* $a \curlyvee b$ and *quasi-infimum* $a \curlywedge b$ by

$$a \curlyvee b := \tfrac{1}{2}(a + b + |a - b|),$$
$$a \curlywedge b := \tfrac{1}{2}(a + b - |a - b|).$$

It is easy to see that $a \curlyvee b$ is an upper bound for a and b, for we have

$$(a \curlyvee b) - a = \tfrac{1}{2}\big(b - a + |a - b|\big) = (a - b)^- \geq 0;$$
$$(a \curlyvee b) - b = \tfrac{1}{2}\big(a - b + |a - b|\big) = (a - b)^+ \geq 0.$$

Similarly, one can prove that $a \curlywedge b$ is a lower bound for a and b. Furthermore, we have the following properties:

- $a \curlywedge b = a \iff a \curlyvee b = b \iff a \leq b$;
- $a + b = (a \curlyvee b) + (a \curlywedge b)$;
- $|a - b| = (a \curlyvee b) - (a \curlywedge b)$;
- $a \curlyvee b = b \curlyvee a$ and $a \curlywedge b = b \curlywedge a$;
- $a \curlyvee b = -(-a \curlywedge -b)$;
- $(a \curlyvee b) + c = (a + c) \curlyvee (b + c)$ and $(a \curlywedge b) + c = (a + c) \curlywedge (b + c)$;
- $a \curlyvee (-a) = |a|$ and $a \curlywedge (-a) = -|a|$;
- $a^+ \curlyvee a^- = |a|$ and $a^+ \curlywedge a^- = 0$;
- $a^+ = a \curlyvee 0$ and $a^- = (-a) \curlyvee 0$.

Each of these facts follows easily from the definition.

Note that the quasi-lattice operations coincide with the lattice operations if $\mathcal{A}$ is commutative. We will show in Chapter 8 that $a \curlyvee b$ remains a *minimal* upper bound for a and b in the non-commutative case, but not necessarily a *least* upper bound. In fact, if $a \curlyvee b$ is a supremum for a and b for every pair of elements $a, b \in \mathcal{A}^{\mathrm{sa}}$, then $\mathcal{A}$ must be commutative (cf. Theorem 8.15).

5.3 $*$-homomorphisms and the quasi-lattice operations

We show that $*$-homomorphisms preserve much of the order structure, including the quasi-lattice operations.

Proposition 5.7. *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras, and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism. Then:*

(a) For $a \in \mathcal{A}$ we have $\varphi(\mathrm{Re}(a)) = \mathrm{Re}(\varphi(a))$ and $\varphi(\mathrm{Im}(a)) = \mathrm{Im}(\varphi(a))$;

(b) For $a \in \mathcal{A}^+$ we have $\varphi(a) \in \mathcal{B}^+$;

(c) For $a \in \mathcal{A}^+$ we have $\varphi(a^{1/2}) = \varphi(a)^{1/2}$;

(d) For $a \in \mathcal{A}^{\mathrm{sa}}$ we have $\varphi(|a|) = |\varphi(a)|$;

(e) For $a, b \in \mathcal{A}^{\mathrm{sa}}$ we have $\varphi(a \curlyvee b) = \varphi(a) \curlyvee \varphi(b)$ and $\varphi(a \curlywedge b) = \varphi(a) \curlywedge \varphi(b)$;

(f) For $a \in \mathcal{A}^+$ we have $\varphi(a^+) = \varphi(a)^+$ and $\varphi(a^-) = \varphi(a)^-$;

(g) For $b \in \varphi[\mathcal{A}]^+$ there exists some $a \in \mathcal{A}^+$ with $\varphi(a) = b$;

(h) If φ is injective, then we have $\varphi(c) \in \mathcal{B}^+$ if and only if $c \in \mathcal{A}^+$.

Proof.

(a) We have $\varphi(\mathrm{Re}(a)) = \varphi(\frac{1}{2}a + \frac{1}{2}a^*) = \frac{1}{2}\varphi(a) + \frac{1}{2}\varphi(a)^* = \mathrm{Re}(\varphi(a))$, and analogously for $\mathrm{Im}(a)$.

(b) Write $a = a^{1/2}a^{1/2}$, then we have $\varphi(a) = \varphi(a^{1/2}a^{1/2}) = \varphi(a^{1/2})\varphi(a^{1/2}) = \varphi(a^{1/2})^2$ and we know that squares of self-adjoint elements are positive.

(c) It follows from part (b) that $\varphi(a^{1/2})$ is a positive square root of $\varphi(a)$. By the uniqueness of positive square roots, we have $\varphi(a^{1/2}) = \varphi(a)^{1/2}$.

(d) We have $\varphi(|a|) = \varphi((a^2)^{1/2}) = \varphi(a^2)^{1/2} = (\varphi(a)^2)^{1/2} = |\varphi(a)|$.

(e) This follows from (d) and linearity.

(f) We have $\varphi(a^+) = \varphi\left(\frac{1}{2}|a| + \frac{1}{2}a\right) = \frac{1}{2}|\varphi(a)| + \frac{1}{2}\varphi(a) = \varphi(a)^+$, and analogously for a^-.

(g) There is some $c \in \mathcal{A}$ such that $\varphi(c) = b$ holds. Now set $a := (\mathrm{Re}(c))^+$, then we have

$$\varphi(a) = \varphi(\mathrm{Re}(c)^+) = \varphi(\mathrm{Re}(c))^+ = (\mathrm{Re}(\varphi(c)))^+ = (\mathrm{Re}(b))^+ = b^+ = b.$$

(h) For $c \in \mathcal{A}^+$ we have $\varphi(c) \geq 0$ by part (b). Conversely, suppose that $c \in \mathcal{A}$ is given such that $\varphi(c) \in \mathcal{B}^+$ holds. By (g) there is some $a \in \mathcal{A}^+$ such that $\varphi(a) = \varphi(c)$ holds. By injectivity we have $a = c$, hence $c \in \mathcal{A}^+$. ■

5.4 Disjointness of positive operators on a Hilbert space

One topic of special interest in the theory of Riesz spaces is disjointness. It is said that two positive elements $a, b \in V^+$ in a (real) Riesz space V are *disjoint* if $a \wedge b = 0$ holds. The quasi-lattice operations allows us to define a similar notion for positive operators on a real or complex Hilbert space. We could also extend this to arbitrary C^*-algebras, but we shall have no use for that.

The following result is an extension of [Top65, Lemma 2].

Proposition 5.8. *Let $\mathcal{H}$ be a real or complex Hilbert space. Then, for positive operators $a, b \in B(\mathcal{H})^+$, the following are equivalent:*

(1) $a \curlywedge b = 0$;

(2) $a \curlyvee b = a + b$;

(3) $|a - b| = a + b$;

(4) *there exists a self-adjoint operator $c \in B(\mathcal{H})$ with $a = c^+$ and $b = c^-$;*

(5) $ab = 0$;

(6) $ba = 0$;

(7) $\overline{\operatorname{ran}}(a) \perp \overline{\operatorname{ran}}(b)$;

(8) $\operatorname{ran}(a) \perp \operatorname{ran}(b)$;

(9) $ab = -ba$.

Proof. The equivalences (1) $\Longleftrightarrow$ (3) and (2) $\Longleftrightarrow$ (3) follow immediately from the definition of the quasi-lattice operations.

(3) $\Longrightarrow$ (4). Choose $c := a - b$, then we have $c^+ = \frac{1}{2}(|c| + c) = a$ as well as $c^- = \frac{1}{2}(|c| - c) = b$.

(4) $\Longrightarrow$ (5). For arbitrary $c \in B(\mathcal{H})^{\text{sa}}$ we have $c^+c^- = 0$.

(5) $\Longrightarrow$ (6). We have $ba = b^*a^* = (ab)^* = 0^* = 0$.

(6) $\Longrightarrow$ (7). For all $x \in \mathcal{H}$ we have $ax \in \ker(b)$, so we find $\operatorname{ran}(a) \subseteq \ker(b)$. Since the latter is closed, we also have $\overline{\operatorname{ran}}(a) \subseteq \ker(b)$. In general we have $\ker(b)^\perp = \overline{\operatorname{ran}}(b)$, so it follows that $\overline{\operatorname{ran}}(a) \perp \overline{\operatorname{ran}}(b)$ holds.

(7) $\Longleftrightarrow$ (8). Trivial.

(8) $\Longrightarrow$ (9). For all $x \in \mathcal{H}$ we have $\|abx\|^2 = \langle abx, abx\rangle = \langle bx, a^2bx\rangle = 0$, since we have $bx \in \operatorname{ran}(b)$ and $a^2bx \in \operatorname{ran}(a)$. It follows that $ab = 0$ holds. Analogously, we have $ba = 0$, so in particular we find $ab = -ba$.

(9) $\Longrightarrow$ (3). We have $ab + ba = 0$, hence

$$(a - b)^2 = a^2 - ab - ba + b^2 = a^2 + ab + ba + b^2 = (a + b)^2.$$

We see that $a + b$ is a positive square root of $(a - b)^2$. Seeing as positive square roots are unique, we have $|a - b| = a + b$. ∎

Definition 5.9. Let $\mathcal{H}$ be a real or complex Hilbert space. We say that two positive operators $a, b \in B(\mathcal{H})^+$ are *disjoint*[2] if they satisfy any (and therefore all) of the conditions from Proposition 5.8.

Note that it follows from property (4) that c^+ and c^- are disjoint for every self-adjoint operator c.

The present notion of disjointness will be used in Corollary 6.10 to prove a characterising property of the quasi-supremum $a \curlyvee b$.

5.5 The quasi-supremum has minimal trace

Let $\mathcal{A}$ be a C^*-algebra (or $B(\mathcal{H})$ for a real Hilbert space $\mathcal{H}$). While we do not know whether or not $a \curlyvee b$ actually defines a supremum for a and b, it does have some special properties that sets it apart from other upper bounds for a and b. For one, it coincides with the lattice supremum if $\mathcal{A}$ is commutative. But even in the non-commutative case, if $\mathcal{B} \subseteq \mathcal{A}$ is a C^*-subalgebra containing a and b, then it also contains $a - b$ and $|a - b|$, and therefore $a \curlyvee b$ and $a \curlywedge b$. Finally, we saw in Proposition 5.7 that $*$-homomorphisms automatically preserve the quasi-lattice operations.

We close this chapter by showing one more special property of the quasi-supremum, further reinforcing the notion that $a \curlyvee b$ really is a natural choice of upper bound for a and b. Specifically, we show that the quasi-supremum $a \curlyvee b$ of two trace-class operators has minimal trace among all upper bounds for a and b. Basic familiarity with the theory of trace-class operators is assumed; see for instance [Mur90, Section 2.4]. (Much more on trace-class operators can be found in [Sch60] or [Rin71].)

In keeping with [Mur90, Section 2.4], let $L^1(\mathcal{H}) \subseteq B(\mathcal{H})$ denote the set of trace-class operators. We recall that $L^1(\mathcal{H})$ is a subspace (even an ideal), and that one has $a \in L^1(\mathcal{H})$ if and only if $|a| \in L^1(\mathcal{H})$. Consequently, if a and b are trace-class, then so is $a \curlyvee b$.

Remark 5.10. Note that an upper bound of two trace-class operators is not necessarily trace-class itself (for instance, a sufficiently large multiple of the identity is always an upper bound for a and b). However, we claim that an upper bound c for two (or even one) trace-class operators is *quasi-trace-class*, in the sense that c^- is trace-class.

To prove this claim, suppose that $a, c \in B(\mathcal{H})^{\text{sa}}$ are given with $a \leq c$ and $a \in L^1(\mathcal{H})$. Let p be the orthogonal projection onto $\overline{\text{ran}}(c^-)$, then we have $pcp = -c^-$. Therefore we find $pap \leq pcp = -c^- \leq 0$, or equivalently, $0 \leq c^- \leq -pap$. Since $L^1(\mathcal{H})$ is an ideal, we have $-pap \in L^1(\mathcal{H})$. Now it is clear that $c^- \in L^1(\mathcal{H})$ holds as well: for any orthonormal basis E of $\mathcal{H}$ we have

$$\|c^-\|_1 \;=\; \sum_{x \in E} \langle c^- x, x\rangle \;\leq\; \sum_{x \in E} \langle -papx, x\rangle \;=\; \|-pap\|_1 \;<\; \infty.$$

Thus, any upper bound of two trace-class operators is quasi-trace-class, and as such has a well-defined trace in $\mathbb{R} \cup \{+\infty\}$.

Before we get to the main result of this section, we need the following satellite result. Its (simple) proof is postponed to Section 6.1, for the result is heavily used in that chapter.

Proposition 5.11. *Let $\mathcal{H}$ be a real or complex Hilbert space. For a positive operator $a \in B(\mathcal{H})^+$ one has*

$$\ker(a) = \ker(a^{1/2}) = \{x \in \mathcal{H} \,:\, \langle ax, x\rangle = 0\}.$$

Proof. See Proposition 6.1 below. ∎

We now come to the main result of this section.

Theorem 5.12. *Let $\mathcal{H}$ be a real or complex Hilbert space. For arbitrary self-adjoint trace-class operators $a, b \in L^1(\mathcal{H})^{\mathrm{sa}}$ the quasi-supremum $a \curlyvee b$ is the unique upper bound of minimal trace for a and b.*

Proof. Recall that we have $(a \curlyvee b) - a = (a-b)^-$ and $(a \curlyvee b) - b = (a-b)^+$, and that these two operators have orthogonal range (cf. Proposition 5.8). As such, it follows that $\overline{\operatorname{ran}}((a-b)^-) \subseteq \operatorname{ran}((a-b)^+)^\perp = \ker((a-b)^+)$ holds. Consequently, the choice of $V := \ker((a-b)^-)$ and $W := \overline{\operatorname{ran}}((a-b)^-)$ gives us a decomposition $\mathcal{H} = V \oplus W$, where V and W are orthogonal closed subspaces satisfying $V \subseteq \ker((a-b)^-)$ and $W \subseteq \ker((a-b)^+)$.

Now choose disjoint index sets I and J and orthonormal bases $\{x_i\}_{i\in I}$ and $\{x_j\}_{j\in J}$ for V and W, respectively. We combine these to form an orthonormal basis $\{x_i\}_{i\in I\cup J}$ of $\mathcal{H}$. We will compute all traces with respect to this basis.

Since we have $V \subseteq \ker((a \curlyvee b) - a)$, it follows from Proposition 5.11 that $v \in V$ implies $\langle ((a \curlyvee b) - a)v, v\rangle = 0$, and therefore

$$\langle av, v\rangle = \langle (a \curlyvee b)v, v\rangle \geq \langle bv, v\rangle. \tag{5.13}$$

Similarly, for all $w \in W$ we have

$$\langle bw, w\rangle = \langle (a \curlyvee b)w, w\rangle \geq \langle aw, w\rangle. \tag{5.14}$$

We summarise these results as follows: since every element of our orthonormal basis $\{x_i\}_{i\in I\cup J}$ belongs to either V or W, for all $i \in I \cup J$ we have

$$\langle (a \curlyvee b)x_i, x_i\rangle = \max\big(\langle ax_i, x_i\rangle, \langle bx_i, x_i\rangle\big).$$

As the trace is independent of the choice of orthonormal basis, we find

$$\operatorname{tr}(a \curlyvee b) = \sum_{i\in I\cup J} \langle (a \curlyvee b)x_i, x_i\rangle = \sum_{i\in I\cup J} \max\big(\langle ax_i, x_i\rangle, \langle bx_i, x_i\rangle\big).$$

Now let $c \in B(\mathcal{H})^{\mathrm{sa}}$ be another upper bound for a and b. If c is not trace-class, then we have $\operatorname{tr}(c) = +\infty > \operatorname{tr}(a \curlyvee b)$, and we are done. Assume (for the remainder of this proof) that c is trace-class. Clearly we have $\langle cx, x\rangle \geq \langle ax, x\rangle$ and $\langle cx, x\rangle \geq \langle bx, x\rangle$ for all $x \in \mathcal{H}$, so we find

$$\operatorname{tr}(c) = \sum_{i\in I\cup J} \langle cx_i, x_i\rangle \geq \sum_{i\in I\cup J} \max\big(\langle ax_i, x_i\rangle, \langle bx_i, x_i\rangle\big) = \operatorname{tr}(a \curlyvee b).$$

This shows that $a \curlyvee b$ has minimal trace. Suppose now that equality holds, then we must have $\langle cx_i, x_i\rangle = \max(\langle ax_i, x_i\rangle, \langle bx_i, x_i\rangle)$ for all $i \in I \cup J$. In fact, by (5.13) and (5.14) we have $\langle ax_i, x_i\rangle \geq \langle bx_i, x_i\rangle$ for all $i \in I$, and similarly $\langle bx_j, x_j\rangle \geq \langle ax_j, x_j\rangle$ for all $j \in J$, so we find

$$\langle cx_i, x_i\rangle = \begin{cases} \langle ax_i, x_i\rangle, & \text{if } i \in I; \\ \langle bx_i, x_i\rangle, & \text{if } i \in J. \end{cases}$$

Since $c - a$ and $c - b$ are positive, it follows from Proposition 5.11 that we have $\{x_i\}_{i\in I} \subseteq \ker(c - a)$ and $\{x_j\}_{j\in J} \subseteq \ker(c - b)$. In particular, for all $i \in I$ we have $cx_i = ax_i$, but also $ax_i = (a \curlyvee b)x_i$ (since $x_i \in V \subseteq \ker((a \curlyvee b) - a)$), hence $cx_i = (a \curlyvee b)x_i$. Analogously, for all $j \in J$ we have $cx_j = (a \curlyvee b)x_j$. We see that c and $a \curlyvee b$ coincide on an orthonormal basis, so they must be equal. ■

We note the following immediate consequence.

Corollary 5.15. *Let $\mathcal{H}$ be a real or complex Hilbert space, and let $a, b \in L^1(\mathcal{H})^{\mathrm{sa}}$ be self-adjoint trace-class operators. Then $a \curlyvee b$ is a minimal upper bound for a and b.*

Proof. Let $c \in B(\mathcal{H})^{\mathrm{sa}}$ be an upper bound for a and b with $c \leq a \curlyvee b$. It follows from Remark 5.10 that c^- is trace-class (because $c \geq a$), but also that c^+ is trace-class (because $c \leq a \curlyvee b \in L^1(\mathcal{H})$), so we see that c itself must be trace-class. For all $x \in \mathcal{H}$ we have $\langle cx, x\rangle \leq \langle (a \curlyvee b)x, x\rangle$, so in particular for any orthonormal basis E of $\mathcal{H}$ we have

$$\mathrm{tr}(c) = \sum_{x\in E} \langle cx, x\rangle \leq \sum_{x\in E} \langle (a \curlyvee b)x, x\rangle = \mathrm{tr}(a \curlyvee b).$$

We see that c is an upper bound for a and b with $\mathrm{tr}(c) \leq \mathrm{tr}(a \curlyvee b)$. It follows from Theorem 5.12 that $c = a \curlyvee b$ must hold. We conclude that $a \curlyvee b$ is a minimal upper bound for a and b. ■

This result will be extended to arbitrary self-adjoint operators $a, b \in B(\mathcal{H})^{\mathrm{sa}}$ in Corollary 6.10.

5.6 End notes

1. (page 59) Consider $P := \mathbb{R}^2$ with the coordinate-wise partial order, that is, $(x_1, y_1) \preceq (x_2, y_2) \iff (x_1 \preceq x_2$ and $y_1 \preceq y_2)$. Furthermore, consider the subset $S := \{(0, -1)\} \cup \{(x, 0) : x \in \mathbb{R}\} \subseteq P$. Then $(0, -1)$ is the unique minimal element of S, but it is not the *least* element of S.

2. (page 64) The reader should be warned that the notion of disjointness in Riesz spaces was generalised to ordered vector spaces in [vGK06]. However, this notion is incompatible with Definition 5.9, due to the anti-lattice theorem, so the two concepts should not be confused!

6 Kadison's anti-lattice theorem

In the previous chapter, we defined the so-called quasi-lattice operations in an arbitrary C^*-algebra. If the C^*-algebra is commutative, then it is isometrically $*$-isomorphic to some $C_0(\Omega)$ space, and the quasi-lattice operations coincide with the lattice operations in $C_0(\Omega)$. The next step is to study the class of operator spaces $B(\mathcal{H})$, which are in a sense maximally non-commutative. We will see that the quasi-supremum $a \curlyvee b$ remains a *minimal* upper bound in this setting, but it will not generally be a *least* upper bound. (Recall that an element x of a partially ordered set $(S, \preceq)$ is *minimal* if $s \preceq x$ implies $s = x$ and *least* if $x \preceq s$ holds for all $s \in S$. These two notions are generally different: a minimal element can be incomparable with some of the other elements in S.) Seeing as a *supremum* is defined as a *least* upper bound, we find that $a \curlyvee b$ is not generally a supremum for a and b.

The main result of this chapter is the anti-lattice theorem (Theorem 6.13), due to Richard V. Kadison [Kad51b]. The theorem states that two self-adjoint operators $a, b \in B(\mathcal{H})$ have a supremum if and only if a and b were comparable to begin with (in that case the larger of the two is clearly the supremum).

The proof we give is a bit different from the proofs in the literature. Most proofs rely on results from operator theory to reduce the question to the two-dimensional case (see e.g. [Kad51b], or [LZ71, Theorem 58.4]), but we choose to present a proof which draws more from the geometry of the underlying Hilbert space. It does not rely on the spectral theorem or other advanced tools, and as such qualifies as an "elementary" proof. Furthermore, it is hoped that the geometric approach helps towards a better understanding of the anti-lattice theorem. This approach is taken much further in the next chapter, where we use the results from this chapter to give a geometric classification of a large class of minimal upper bounds for a pair of (incomparable) self-adjoint operators $a, b \in B(\mathcal{H})^{\mathrm{sa}}$.

The proofs in this chapter work equally well if $\mathcal{H}$ is a real Hilbert space, so we prove everything for an arbitrary Hilbert space over the ground field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. A word of warning: we have to be a bit careful with positivity if $\mathcal{H}$ is real: we must check that an operator $a \in B(\mathcal{H})$ is self-adjoint before concluding that a might be positive, as the condition $\langle ax, x\rangle \geq 0$ (for all $x \in \mathcal{H}$) is no longer sufficient in this case. Indeed, matrices such as

$$a_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, a_2 = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$$

satisfy $\langle a_i x, x\rangle \geq 0$ for all $x \in \mathbb{R}^2$, but fail to be positive semidefinite (since they are not symmetric).

6.1 Kernel and range of positive operators

Proposition 6.1. *For a positive operator $a \in B(\mathcal{H})$ one has*

$$\ker(a) = \ker(a^{1/2}) = \{x \in \mathcal{H} : \langle ax, x\rangle = 0\}.$$

Proof. First of all, for $x \in \ker(a^{1/2})$ we have $ax = a^{1/2}a^{1/2}x = a^{1/2}0 = 0$, hence $x \in \ker(a)$. Secondly, for $y \in \ker(a)$ we have $\langle ay, y\rangle = \langle 0, y\rangle = 0$. Finally, let $z \in \mathcal{H}$ be given such that $\langle az, z\rangle = 0$ holds. Then we have

$$\|a^{1/2}z\|^2 = \langle a^{1/2}z, a^{1/2}z\rangle = \langle az, z\rangle = 0,$$

hence $a^{1/2}z = 0$. ■

Proposition 6.2. *Let $a, b \in B(\mathcal{H})$ be positive operators with $0 \leq a \leq b$. Then one has $\ker(a) \supseteq \ker(b)$ and $\overline{\operatorname{ran}}(a) \subseteq \overline{\operatorname{ran}}(b)$.*

Proof. For $x \in \mathcal{H}$ we have $0 \leq \langle ax, x\rangle \leq \langle bx, x\rangle$, so by Proposition 6.1 we have $\ker(a) \supseteq \ker(b)$. Taking orthogonal complements yields $\overline{\operatorname{ran}}(a) \subseteq \overline{\operatorname{ran}}(b)$. ■

It should be pointed out that the stronger inequality $\operatorname{ran}(a) \subseteq \operatorname{ran}(b)$ does not hold in general. The following example will be of importance in Chapter 7 as well.

Example 6.3. Let $\mathcal{H}$ be infinite-dimensional and let $a \in B(\mathcal{H})$ be positive and injective, but not surjective. (Such operators exist: let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal sequence in $\mathcal{H}$, let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}_{>0}$ converging to zero, and let a be the operator given by $x_n \mapsto \alpha_n x_n$, and $w \mapsto w$ for all $w \perp \{x_n\}_{n=1}^{\infty}$.) Now choose some $y \notin \operatorname{ran}(a)$ and let $p \in B(\mathcal{H})$ be the orthogonal projection onto $\operatorname{span}(y)$. Clearly we have $0 \leq p \leq a+p$ and $y \in \operatorname{ran}(p)$. Suppose that $y \in \operatorname{ran}(a+p)$ were to hold, then we could choose some $z \in \mathcal{H}$ such that $(a+p)z = y$ holds. Note that we must have $z \neq 0$, since y cannot be zero. But now we have $az = y - pz \in \operatorname{span}(y)$, and az is non-zero since a is injective. This contradicts our assumption that $y \notin \operatorname{ran}(a)$ holds. We conclude that $y \notin \operatorname{ran}(a+p)$ holds, so we have $\operatorname{ran}(p) \not\subseteq \operatorname{ran}(a+p)$. All we can say with certainty is that $\operatorname{ran}(p) \subseteq \overline{\operatorname{ran}}(p) \subseteq \overline{\operatorname{ran}}(a+p)$ holds. ⊠

We do however have the following result, which is a special case of Douglas' lemma (cf. [Dou66, Theorem 1]).

Lemma 6.4. *Let $a, b \in B(\mathcal{H})$ be positive operators with $0 \leq a \leq b$. Then one has $\operatorname{ran}(a^{1/2}) \subseteq \operatorname{ran}(b^{1/2})$.*

Proof. By propositions 6.1 and 6.2 we have $\ker(b^{1/2}) \subseteq \ker(a^{1/2})$, so the linear map $c' : \operatorname{ran}(b^{1/2}) \to \operatorname{ran}(a^{1/2})$ given by $b^{1/2}x \mapsto a^{1/2}x$ is well-defined. Note that c' is bounded, since we have

$$\|a^{1/2}x\|^2 = \langle a^{1/2}x, a^{1/2}x\rangle = \langle ax, x\rangle \leq \langle bx, x\rangle = \langle b^{1/2}x, b^{1/2}x\rangle = \|b^{1/2}x\|^2.$$

Therefore c' has a unique continuous extension $\overline{\operatorname{ran}}(b^{1/2}) \to \overline{\operatorname{ran}}(a^{1/2})$. We may further extend it to an operator $c \in B(\mathcal{H})$ with $x \mapsto 0$ for all $x \perp \overline{\operatorname{ran}}(b^{1/2})$.

Now we have $a^{1/2} = cb^{1/2}$. Since $a^{1/2}$ and $b^{1/2}$ are self-adjoint, we may rewrite this as $a^{1/2} = (cb^{1/2})^* = b^{1/2}c^*$. Consequently, if $y = a^{1/2}x \in \operatorname{ran}(a^{1/2})$ is given, then we may write $y = b^{1/2}(c^*x) \in \operatorname{ran}(b^{1/2})$. The result follows. ■

A similar result is that $0 \leq a \leq b$ implies $0 \leq a^{1/2} \leq b^{1/2}$ (cf. [Mur90, Theorem 2.2.6]), but we will not need this. Note that the converse does not hold: in Example 6.3 we have $0 \leq p \leq a+p$, but $\operatorname{ran}(p) \not\subseteq \operatorname{ran}(a+p)$, so it follows from Lemma 6.4 that we cannot have $0 \leq p^2 \leq (a+p)^2$. In fact, in Theorem 8.16 we show that a C^*-algebra $\mathcal{A}$ where the implication $0 \leq a \leq b \implies a^2 \leq b^2$ holds for all $a, b \in \mathcal{A}^+$ must necessarily be commutative.

6.2 The projective range

In keeping with [Mur90, Section 2.4], for $x, y \in \mathcal{H}$ we let $x \otimes y \in B(\mathcal{H})$ denote the operator $z \mapsto \langle z, y \rangle x$.[1] It is readily verified that one has $\|x \otimes y\| = \|x\|\|y\|$ and $(x \otimes y)^* = y \otimes x$. Furthermore, note that $x \otimes x$ is positive, since we have $(x \otimes x)^* = x \otimes x$ and

$$\big\langle (x \otimes x)z, z \big\rangle = \big\langle \langle z, x \rangle x, z \big\rangle = \langle z, x \rangle \langle x, z \rangle = |\langle z, x \rangle|^2 \geq 0. \tag{6.5}$$

The operator $x \otimes x$ is a projection if and only if x is a unit vector, and all projections of rank 1 are of this form. More generally, every rank 1 operator in $B(\mathcal{H})$ can be written as $x \otimes y$ for some $x, y \in \mathcal{H}$ (cf. [Mur90, remarks preceding Theorem 2.4.6]).

Note that the map $\mathcal{H} \times \mathcal{H} \to B(\mathcal{H})$, $(x, y) \mapsto x \otimes y$ is sesquilinear, in that we have $(x_1 + x_2) \otimes y = (x_1 \otimes y) + (x_2 \otimes y)$ and $x \otimes (y_1 + y_2) = (x \otimes y_1) + (x \otimes y_2)$, as well as $\lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\overline{\lambda} y)$. It follows from the latter expression that $x \otimes x$ is a positive multiple of the orthogonal projection onto $\operatorname{span}(x)$, for we have $(\lambda x) \otimes (\lambda x) = |\lambda|^2 (x \otimes x)$.

Definition 6.6. For a positive operator $a \in B(\mathcal{H})$, we define the *projective range*[2] $\operatorname{pran}(a) \subseteq \mathcal{H}$ as follows:

$$\operatorname{pran}(a) := \{x \in \mathcal{H} : \varepsilon(x \otimes x) \leq a \text{ for some } \varepsilon > 0\}.$$

In other words, $\operatorname{pran}(a)$ is the set of all $x \in \mathcal{H}$ so that some positive multiple of the orthogonal projection onto $\operatorname{span}(x)$ can be squeezed in between 0 and a. (Note that it does not make sense to extend this definition to operators which are not positive.)

Theorem 6.7. *For any $a \in B(\mathcal{H})^+$ we have $\operatorname{pran}(a) = \operatorname{ran}(a^{1/2})$. In particular, $\operatorname{pran}(a)$ is a linear subspace and satisfies*

$$\operatorname{ran}(a) \subseteq \operatorname{pran}(a) \subseteq \overline{\operatorname{ran}}(a).$$

Proof. Let $y \in \operatorname{ran}(a^{1/2})$ be given. For $y = 0$ we clearly have $y \in \operatorname{pran}(a)$, so assume $y \neq 0$. Choose some $x \in \mathcal{H}$ such that $a^{1/2}x = y$ holds. By (6.5) and the Cauchy–Schwarz inequality, for all $z \in \mathcal{H}$ we have

$$\begin{aligned} \langle (y \otimes y)z, z \rangle &= \big|\big\langle z, a^{1/2}x \big\rangle\big|^2 \\ &= \big|\big\langle a^{1/2}z, x \big\rangle\big|^2 \\ &\leq \big\langle a^{1/2}z, a^{1/2}z \big\rangle \cdot \langle x, x \rangle \\ &= \langle az, z \rangle \cdot \langle x, x \rangle. \end{aligned}$$

Since we assumed $y \neq 0$, we have $x \notin \ker(a^{1/2})$, so in particular $x \neq 0$. It follows that $\langle x, x \rangle > 0$ holds. Therefore we may define $\varepsilon := \frac{1}{\langle x, x \rangle}$. For all $z \in \mathcal{H}$ we have $\langle \varepsilon(y \otimes y)z, z \rangle \leq \langle az, z \rangle$, by the above, so it follows that $\varepsilon(y \otimes y) \leq a$ holds. This proves the inclusion $\operatorname{ran}(a^{1/2}) \subseteq \operatorname{pran}(a)$.

Conversely, let $y \in \operatorname{pran}(a)$ be given. For $y = 0$ it is clear that $y \in \operatorname{ran}(a^{1/2})$ holds, so assume without loss of generality that y is a unit vector. Furthermore, choose $\varepsilon > 0$ such that $\varepsilon(y \otimes y) \leq a$ holds. Note that the square root of $\varepsilon(y \otimes y)$

is $\sqrt{\varepsilon}(y \otimes y)$, so by Lemma 6.4 we have $\operatorname{span}(y) = \operatorname{ran}(\sqrt{\varepsilon}(y \otimes y)) \subseteq \operatorname{ran}(a^{1/2})$. This proves the reverse inclusion $\operatorname{pran}(a) \subseteq \operatorname{ran}(a^{1/2})$, so we have equality.

Clearly $\operatorname{ran}(a^{1/2})$ is a linear subspace. Furthermore, if $y = ax \in \operatorname{ran}(a)$ is given, then we may write $y = a^{1/2}(a^{1/2}x) \in \operatorname{ran}(a^{1/2})$, proving the inclusion $\operatorname{ran}(a) \subseteq \operatorname{ran}(a^{1/2})$. Finally, by Proposition 6.1 we have $\ker(a^{1/2}) = \ker(a)$, so by taking orthogonal complements we find $\operatorname{ran}(a^{1/2}) \subseteq \overline{\operatorname{ran}}(a^{1/2}) = \overline{\operatorname{ran}}(a)$. ■

We note three consequences of Theorem 6.7. Fist of all, note that we have $\overline{\operatorname{pran}}(a) = \overline{\operatorname{ran}}(a)$ for any positive operator $a \in B(\mathcal{H})$.

Secondly, consider the setting where we have positive operators $0 \leq a \leq b$. While the inclusion $\overline{\operatorname{ran}}(a) \subseteq \overline{\operatorname{ran}}(b)$ from Proposition 6.2 cannot be extended to the stronger inclusion $\operatorname{ran}(a) \subseteq \operatorname{ran}(b)$ (cf. Example 6.3), it can be strengthened to $\operatorname{pran}(a) \subseteq \operatorname{pran}(b)$. (This should be clear from the definition.) Then again, we knew this already from Lemma 6.4.

Thirdly, if $\operatorname{ran}(a)$ is closed, then we know exactly what $\operatorname{pran}(a)$ is! However, in general $\operatorname{pran}(a)$ can be different from both $\operatorname{ran}(a)$ and $\overline{\operatorname{ran}}(a)$, and it is easy to come up with such examples (e.g. choose a positive, compact multiplication operator of infinite rank on ℓ^2). The following example shows a case where the first inclusion from Theorem 6.7 is strict, even though it is not at all clear what the square root is.

Example 6.8. Let $\mathcal{H}$, a, y and p be as in Example 6.3. Recall that we have $y \notin \operatorname{ran}(a + p)$ in this setting. Note that $y \otimes y$ is (a positive multiple of) p, so clearly we do have $y \in \operatorname{pran}(a + p)$. So even though it is not at all clear what $(a + p)^{1/2}$ is, we nonetheless showed that $y \in \operatorname{ran}((a + p)^{1/2})$ holds. ⊠

6.3 Minimal upper bounds

Using the projective range, it is relatively easy to recognise minimal upper bounds for a finite set of self-adjoint operators.

Proposition 6.9. *Let $S \subseteq B(\mathcal{H})^{\mathrm{sa}}$ be a non-empty and finite set of self-adjoint operators, and suppose that $c \in B(\mathcal{H})^{\mathrm{sa}}$ is an upper bound of S. Then c is a minimal upper bound of S if and only if*

$$\bigcap_{s \in S} \operatorname{pran}(c - s) \;=\; \{0\}.$$

Proof. Suppose first that there is some non-zero $x \in \bigcap_{s \in S} \operatorname{pran}(c - s)$. Choose $\varepsilon > 0$ sufficiently small such that $\varepsilon(x \otimes x) \leq c - s$ holds for all $s \in S$ (here we use that S is finite). Now we have $s \leq c - \varepsilon(x \otimes x)$ for all $s \in S$, so we see that $c - \varepsilon(x \otimes x)$ is an upper bound for S as well. Since we have $\varepsilon > 0$ and $x \neq 0$, we find $c - \varepsilon(x \otimes x) < c$. In other words: c is not a minimal upper bound.

Conversely, assume that c is not a minimal upper bound, then there is some upper bound d of S with $d < c$. For all $s \in S$ we have $0 < c - d \leq c - s$, hence $\operatorname{ran}(c - d) \subseteq \operatorname{pran}(c - d) \subseteq \operatorname{pran}(c - s)$. It follows that

$$\operatorname{ran}(c - d) \subseteq \bigcap_{s \in S} \operatorname{pran}(c - s).$$

Since we have $c \neq d$, we see that $\operatorname{ran}(c - d)$ is a non-zero subspace. ■

Corollary 6.10. *If $a, b \in B(\mathcal{H})$ are arbitrary self-adjoint operators, then $a \curlyvee b$ is a minimal upper bound for a and b. Furthermore, it is the unique upper bound c of a and b satisfying* $\operatorname{pran}(c-a) \perp \operatorname{pran}(c-b)$.

Proof. Recall that we have $(a \curlyvee b) - a = (a-b)^-$ and $(a \curlyvee b) - b = (a-b)^+$, and that $a \curlyvee b$ is therefore an upper bound for a and b. It follows from Proposition 5.8 that $(a-b)^+$ and $(a-b)^-$ are disjoint, so we have $\overline{\operatorname{ran}}((a-b)^+) \perp \overline{\operatorname{ran}}((a-b)^-)$. By Theorem 6.7 we also have $\operatorname{pran}((a-b)^+) \perp \operatorname{pran}((a-b)^-)$, so in particular we find

$$\operatorname{pran}((a \curlyvee b) - a) \cap \operatorname{pran}((a \curlyvee b) - b) = \{0\}.$$

It follows from Proposition 6.9 that $a \curlyvee b$ is a minimal upper bound for a and b. Now suppose that c is an arbitrary upper bound of a and b satisfying $\operatorname{pran}(c-a) \perp \operatorname{pran}(c-b)$. Then $c-a$ and $c-b$ are disjoint, so property (3) of Proposition 5.8 gives us $|a-b| = 2c - a - b$. Equivalently:

$$c = \tfrac{1}{2}\big(a + b + |a-b|\big) = a \curlyvee b. \qquad \blacksquare$$

Corollary 6.11 (cf. [Top65, Proposition 3])**.** *Let $\mathcal{H}$ be a real or complex Hilbert space, and let $V \subseteq B(\mathcal{H})^{\mathrm{sa}}$ be a space of self-adjoint operators with the property that $a \in V$ implies $|a| \in V$. If $a, b \in V$ have a supremum c relative to V, then one necessarily has $c = a \curlyvee b$.*

Proof. The assumption assures us that $a \curlyvee b \in V$ holds. By the supremum property of c we have $c \leq a \curlyvee b$. Then, by minimality of $a \curlyvee b$ (Corollary 6.10) we must have $c = a \curlyvee b$. $\blacksquare$

An important special case of the preceding corollary occurs when $\mathcal{H}$ is complex and $V = \mathcal{A}^{\mathrm{sa}}$ is the self-adjoint part of a C^*-subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$. We will use this in Chapter 8 as a step towards Sherman's theorem.

6.4 Kadison's anti-lattice theorem

At this time we are ready to prove the main result of this chapter. We first sketch the proof by giving a simple example, and then show that the general case is no harder than this example.

Example 6.12. Let $\mathcal{H}$ be the two-dimensional Hilbert space $\mathbb{F}^2$ with the standard inner product, and let $a, b \in B(\mathcal{H})^{\mathrm{sa}}$ be the projections onto the first and second coordinate, respectively. Now we have

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a \curlyvee b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We know that $a \curlyvee b$ is a minimal upper bound, but there might be other upper bounds which are incomparable with $a \curlyvee b$. Indeed, let $d \in B(\mathcal{H})$ be given by

$$d := \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix},$$

then we have

$$d - a = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}, \quad d - b = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}, \quad d - (a \curlyvee b) = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}.$$

Note that a real symmetric 2×2 matrix is positive if and only if both its trace and its determinant are non-negative. Hence it is clear that $d - a$ and $d - b$ are positive, but neither $d - (a \curlyvee b)$ nor $(a \curlyvee b) - d$ is positive. In other words, d is an upper bound for a and b which is incomparable with $a \curlyvee b$. It follows from Corollary 6.11 that a and b do not have a supremum. ⊠

Theorem 6.13 (Kadison's anti-lattice theorem)**.** *Let $\mathcal{H}$ be a real or complex Hilbert space. Then $B(\mathcal{H})^{\mathrm{sa}}$ is an anti-lattice.*

Proof. Let $a, b \in B(\mathcal{H})^{\mathrm{sa}}$ be incomparable. We show that a and b do not have a supremum. In light of Corollary 6.11, it suffices to show that $a \curlyvee b$ is not the supremum of a and b. It is already a minimal upper bound, so the task at hand is to exhibit another upper bound for a and b which is incomparable with $a \curlyvee b$.

For convenience, let us write $c := a \curlyvee b$. The assumption that a and b are incomparable tells us that both $c - a$ and $c - b$ are non-zero, so we may choose unit vectors $x \in \operatorname{pran}(c - a)$ and $y \in \operatorname{pran}(c - b)$, and choose $\varepsilon > 0$ so that we have $\varepsilon(x \otimes x) \leq c - a$ as well as $\varepsilon(y \otimes y) \leq c - b$. It follows from Corollary 6.10 that we have $x \perp y$.

Now let $d \in B(\mathcal{H})$ be the operator

$$\begin{aligned} d &:= x \otimes x \ + \ \sqrt{2}(x \otimes y) \ + \ \sqrt{2}(y \otimes x) \ + \ y \otimes y \\ &= \frac{1 + \sqrt{2}}{2}(x + y) \otimes (x + y) \ + \ \frac{1 - \sqrt{2}}{2}(x - y) \otimes (x - y) \\ &= (\sqrt{2}x + y) \otimes (\sqrt{2}x + y) \ - \ x \otimes x \\ &= (x + \sqrt{2}y) \otimes (x + \sqrt{2}y) \ - \ y \otimes y. \end{aligned}$$

Analogously to Example 6.12, we will show that $c + \varepsilon d$ is an upper bound of a and b which is incomparable with c.

In order to show that $c + \varepsilon d$ is incomparable with c, it suffices to show that neither d nor $-d$ is positive. Since x and y are orthogonal unit vectors, we have

$$\langle x + y, x - y \rangle = \langle x, x \rangle - \langle x, y \rangle + \langle y, x \rangle - \langle y, y \rangle = 1 - 0 + 0 - 1 = 0,$$

so we see that $x + y$ and $x - y$ are orthogonal as well. The second expression for d shows, therefore, that $x + y$ and $x - y$ are eigenvectors of d. The corresponding eigenvalues are

$$\frac{1 + \sqrt{2}}{2} \cdot \|x + y\|^2 = 1 + \sqrt{2}, \quad \text{and} \quad \frac{1 - \sqrt{2}}{2} \cdot \|x - y\|^2 = 1 - \sqrt{2}.$$

We see that d has a positive and a negative eigenvalue, which proves our claim that neither d nor $-d$ is positive.

In order to see that $c + \varepsilon d$ is an upper bound for a and b, note that we have

$$\begin{aligned} c - a + \varepsilon d &\geq \varepsilon(x \otimes x) + \varepsilon d = \varepsilon\big((\sqrt{2}x + y) \otimes (\sqrt{2}x + y)\big) \geq 0; \\ c - b + \varepsilon d &\geq \varepsilon(y \otimes y) + \varepsilon d = \varepsilon\big((x + \sqrt{2}y) \otimes (x + \sqrt{2}y)\big) \geq 0. \end{aligned}$$

In conclusion: not every upper bound for a and b is comparable with $c = a \curlyvee b$, so the latter is not the supremum of a and b. It follows from Corollary 6.11 that a and b do not have a supremum at all. ∎

6.5 End notes

1. (page 69) The notation $x \otimes y$ is related to tensor products of Hilbert spaces. (Beware: the tensor product of Hilbert spaces does not satisfy the same universal property as the tensor product in abstract algebra!) There is a natural isomorphism $\mathcal{H} \otimes \mathcal{H}^* \cong L^2(H)$ from the tensor product of $\mathcal{H}$ with its dual $\mathcal{H}^*$ to the Hilbert space $L^2(H)$ consisting of all so-called *Hilbert–Schmidt operators* $\mathcal{H} \to \mathcal{H}$. Identifying y with the linear functional $z \mapsto \langle z, y \rangle$ it determines, the pure tensor $x \otimes y \in \mathcal{H} \otimes \mathcal{H}^*$ corresponds with the Hilbert–Schmidt operator $z \mapsto \langle z, y \rangle x$. More on tensor products of Hilbert spaces can be found in [KR97a, Section 2.6].

2. (page 69) The term *projective range* was invented by the author. It seems plausible that the concept already exist in some form in the literature, though we are not aware of a common name for it. In the given proof of the equality $\operatorname{pran}(a) = \operatorname{ran}(a^{1/2})$, we only used a small part of Douglas' lemma (since we derived the inclusion $\operatorname{ran}(a^{1/2}) \subseteq \operatorname{pran}(a)$ from a simple application of the Cauchy–Schwarz inequality), but in fact the entire result follows immediately from the full version of Douglas' lemma.

7 Geometric classification of minimal upper bounds in $B(\mathcal{H})$

In the previous chapter we gave a (somewhat) geometric proof of Kadison's anti-lattice theorem, though the main argument turned out to be rather *ad hoc*. Many questions are left unanswered; in particular one might wonder if incomparable self-adjoint operators $a, b \in B(\mathcal{H})^{\mathrm{sa}}$ have more than one minimal upper bound. In this chapter we shall answer the preceding question through a rigorous study of the minimal upper bounds for a and b.

We start out with a geometric interpretation of the two-dimensional (real) case in Section 7.1. While this section does not really add anything in terms of results, it provides us with the necessary geometric intuition we need before digging a little deeper.

In Section 7.2 we study additional properties enjoyed by minimal upper bounds. After that, we select a subclass of the minimal upper bounds (we call these *of complementary type*) for further investigation, and the succeeding sections give a complete geometric characterisation of these upper bounds. While not all minimal upper bounds are of this type, there will be sufficiently many to give another proof of Kadison's anti-lattice theorem. In fact, we prove the following strengthening: if $a, b \in B(\mathcal{H})^{\mathrm{sa}}$ are incomparable, then the set of minimal upper bounds for a and b is unbounded (Theorem 7.23).

The main results of this chapter are Theorem 7.17 (classification of minimal upper bounds of complementary type), and the aforementioned Theorem 7.23 (the set of minimal upper bounds is unbounded).

While many of the proofs in this chapter would have been much simpler in the finite-dimensional case, we chose to prove everything more generally in arbitrary Hilbert spaces. The upshot is that the classification leads to a purely geometric proof of Kadison's anti-lattice theorem in the general case, but the downside is that the entire classification becomes an exercise in dealing with some of the peculiarities of infinite-dimensional Hilbert spaces (and later on, even some incomplete inner product spaces).

Throughout this chapter, $\mathcal{H}$ denotes a Hilbert space over the ground field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. We use the following geometric properties of $\mathcal{H}$.

Fact 7.1. Let $\mathcal{H}$ be a real or complex Hilbert space and let $V, W \subseteq \mathcal{H}$ be subspaces. Then one has $(V + W)^\perp = V^\perp \cap W^\perp$. Furthermore, if V and W are closed, then one also has $(V \cap W)^\perp = \overline{V^\perp + W^\perp}$, but this is not necessarily equal to $V^\perp + W^\perp$. In fact, $V^\perp + W^\perp$ is closed if and only if $V + W$ is closed.

Proof. See Proposition A.1, Proposition A.2, Example A.3, and Corollary A.10 (in Appendix A below). $\square$

7.1 Geometric interpretation of upper bounds

Before we attempt to obtain a geometric classification of minimal upper bounds, it is instructive to paint the geometric picture in the real two-dimensional case. We follow [LZ71, Chapter 5, §58 (closing remarks)]. Let $\mathcal{H} = \mathbb{R}^2$ be the two-dimensional Euclidean space with standard inner product. In keeping with

[LZ71], we define the *indicatrix* of a positive operator $a \in B(\mathcal{H})$ to be the set

$$I_a := \{x \in \mathcal{H} : \langle ax, x \rangle \leq 1\}.$$

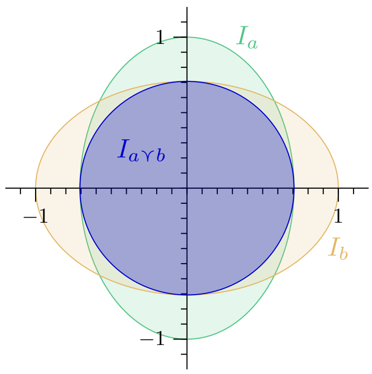
Geometrically, the indicatrix is either an ellipsoidal disk (if a is invertible), the region between two parallel lines (if $\operatorname{rank}(a) = 1$), or all of $\mathbb{R}^2$ (if $a = 0$). Furthermore, for $a, c \geq 0$ we have $a \leq c$ if and only if $I_a \supseteq I_c$.

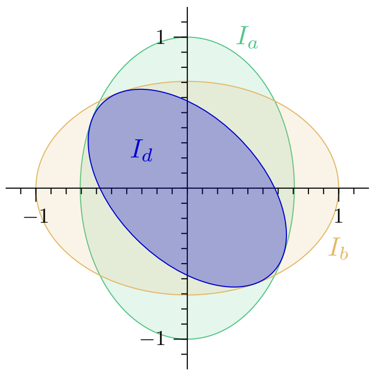
We study the upper bounds for $a, b \in B(\mathcal{H})^{\mathrm{sa}}$ in terms of their indicatrices. Let $\mathbb{1} \in B(\mathcal{H})$ denote the identity, and assume without loss of generality that $a, b \geq \mathbb{1}$ holds (by performing a suitable translation). Then I_a and I_b are ellipsoidal disks contained in the unit disk of $\mathbb{R}^2$. The indicatrix of an upper bound $c \geq a, b$ is an ellipsoidal disk contained in both I_a and I_b. If a and b are incomparable, then I_a and I_b are incomparable (under inclusion), and it follows from Kadison's anti-lattice theorem that there is no *greatest* indicatrix $I_c \subseteq I_a \cap I_b$. We can however find a variety of *maximal* indicatrices $I_c \subseteq I_a \cap I_b$, as the following example shows.

Example 7.2. As in the preceding paragraph, let $\mathcal{H} = \mathbb{R}^2$ be two-dimensional Euclidean space. Further, let $a, b, d \in B(\mathcal{H})$ be the matrices from Example 6.12, except all translated by $\mathbb{1}$, so that we have

$$a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad a \curlyvee b = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad d = \begin{pmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}.$$

We draw the indicatrices of $a \curlyvee b$ and d on top of I_a and I_b.





It is clear from the picture that $a \curlyvee b$ and d are incomparable upper bounds for a and b (as we proved in Example 6.12). Furthermore, we already proved that $a \curlyvee b$ is always a minimal upper bound; the corresponding geometric statement is that there is no indicatrix I_c satisfying $I_{a \curlyvee b} \subsetneq I_c \subseteq I_a \cap I_b$. The image moreover suggests that d is a minimal upper bound as well, and this turns out to be true. (We do not prove this, but in light of Proposition 6.9 it is clear how to verify this: show that $\operatorname{ran}(d - a) \cap \operatorname{ran}(d - b) = \{0\}$ holds.) ⊠

The disks $I_{a \curlyvee b}$ and I_d in the previous example are maximal: they cannot be extended to a larger ellipsoidal disk in $I_a \cap I_b$. We might wonder which indicatrices $I_c \subseteq I_a \cap I_b$ have the same property. We know how to *recognise* minimal upper bounds (Proposition 6.9), but so far we do not know how to *construct* them. That will be the goal for this chapter.

For c to be a minimal upper bound, it seems that the boundary of I_c must touch each of the boundaries of I_a and I_b. But it seems unlikely that we can just specify a point where ∂I_c touches ∂I_a and a point where ∂I_c touches ∂I_b, and then construct an upper bound c with these properties. Furthermore, in the higher-dimensional case the indicatrices might in principle touch in a curve or (hyper)surface instead of a point. And how do we even think about ellipsoidal geometry in an infinite-dimensional Hilbert space?

We translate these questions back to linear algebra. To that end, we make the following observation: in the finite-dimensional case, the condition for an upper bound c to be minimal, namely that $\operatorname{ran}(c-a)\cap\operatorname{ran}(c-b)=\{0\}$ should hold, is equivalent to the condition $\ker(c-a)+\ker(c-b)=\mathcal{H}$, simply by taking orthogonal complements (cf. Fact 7.1). Furthermore, since c is an upper bound for a and b, it follows from Proposition 6.1 that $\ker(c-a)$ is precisely the subspace where ∂I_a and ∂I_c touch! So our geometric questions about touching ellipses are easily turned into questions about kernels and ranges.

The remainder of our studies will be carried out in the setting of linear algebra and functional analysis, so we will not be doing much more geometry. Nevertheless, the geometric intuition obtained in this section should be kept in the back of our minds at all times, as it motivates many of the results to come.

7.2 Kernels and ranges of minimal upper bounds

We proceed with our study of minimal upper bounds of operators $a,b\in B(\mathcal{H})^{\mathrm{sa}}$. This is done in the general setting, so $\mathcal{H}$ is an arbitrary real or complex Hilbert space (not necessarily finite-dimensional). We will shortly see that matters are a little more complicated here, but we will find ways around this.

Recall from Proposition 6.9 that a minimal upper bound c for a and b satisfies $\operatorname{pran}(c-a)\cap\operatorname{pran}(c-b)=\{0\}$. In light of Theorem 6.7, it follows that $\operatorname{ran}(c-a)\cap\operatorname{ran}(c-b)=\{0\}$ holds, but we cannot say with certainty whether or not $\overline{\operatorname{ran}}(c-a)\cap\overline{\operatorname{ran}}(c-b)=\{0\}$ holds. The following example shows that the latter equality might fail in general.

Example 7.3. Let $\mathcal{H}$ be infinite-dimensional, and let $a\in B(\mathcal{H})$ be a positive operator such that $\operatorname{pran}(a)\neq\overline{\operatorname{ran}}(a)$ holds. Choose some $z\in\overline{\operatorname{ran}}(a)\setminus\operatorname{pran}(a)$, and let $p:=z\otimes z$ be (a positive multiple of) the orthogonal projection onto $\operatorname{span}(z)$. Now we have $\operatorname{ran}(p)=\operatorname{pran}(p)=\overline{\operatorname{ran}}(p)=\operatorname{span}(z)$, so it is clear that $a+p$ is a minimal upper bound for a and p (we have $\operatorname{pran}(p)\cap\operatorname{pran}(a)=\{0\}$). However, at the same time we have $\{0\}\subsetneq\operatorname{span}(z)\subseteq\overline{\operatorname{ran}}(p)\cap\overline{\operatorname{ran}}(a)$. ⊠

The property $\operatorname{ran}(c-a)\cap\operatorname{ran}(c-b)=\{0\}$ has the following consequence.

Proposition 7.4. *Let $a,b,c\in B(\mathcal{H})^{\mathrm{sa}}$ be self-adjoint operators such that c is a minimal upper bound for a and b. If $x\in\mathcal{H}$ is such that $ax=bx$ holds, then cx is also equal to this common value. Consequently, one has*

$$\ker(a-b)=\ker(c-a)\cap\ker(c-b).$$

Proof. If $ax=bx$ holds, then we have

$$(c-a)x\;=\;(c-b)x\;\in\;\operatorname{ran}(c-a)\cap\operatorname{ran}(c-b)\;=\;\{0\},$$

hence $cx=ax=bx$. This proves the inclusion $\ker(a-b)\subseteq\ker(c-a)\cap\ker(c-b)$. The reverse inclusion is trivial: if $cx=ax$ and $cx=bx$, then $ax=bx$. ■

Corollary 7.5. *Let $a, b, c \in B(\mathcal{H})^{\mathrm{sa}}$ be self-adjoint operators such that c is a minimal upper bound for a and b. Then one has*

$$\operatorname{ran}(a-b) \subseteq \operatorname{ran}(c-a)+\operatorname{ran}(c-b) \subseteq \overline{\operatorname{ran}}(c-a)+\overline{\operatorname{ran}}(c-b) \subseteq \overline{\operatorname{ran}}(a-b).$$

Proof. If $y \in \operatorname{ran}(a-b)$ is given, then we may choose $x \in \mathcal{H}$ so that $y=(a-b)x$ holds, and then we have

$$y = (a-c)x+(c-b)x \in \operatorname{ran}(c-a)+\operatorname{ran}(c-b). \tag{7.6}$$

This proves the inclusion $\operatorname{ran}(a-b) \subseteq \operatorname{ran}(c-a)+\operatorname{ran}(c-b)$.

The second inclusion is clear.

Thirdly, we prove the inclusion $\operatorname{ran}(c-a)+\operatorname{ran}(c-b) \subseteq \overline{\operatorname{ran}}(a-b)$. It follows from Fact 7.1 and Proposition 7.4 that

$$\begin{aligned}\big(\operatorname{ran}(c-a)+\operatorname{ran}(c-b)\big)^{\perp} &= \operatorname{ran}(c-a)^{\perp} \cap \operatorname{ran}(c-b)^{\perp} \\ &= \ker(c-a) \cap \ker(c-b) \\ &= \ker(a-b).\end{aligned}$$

Taking orthogonal complements, we find

$$\overline{\operatorname{ran}(c-a)+\operatorname{ran}(c-b)} = \overline{\operatorname{ran}}(a-b),$$

which proves the inclusion $\operatorname{ran}(c-a)+\operatorname{ran}(c-b) \subseteq \overline{\operatorname{ran}}(a-b)$.

For the third and final inclusion, note that $\overline{\operatorname{ran}}(a-b)$ is a closed subspace containing $\operatorname{ran}(c-a)$, so it also contains $\overline{\operatorname{ran}}(c-a)$. Analogously, it contains $\overline{\operatorname{ran}}(c-b)$, so it follows that $\overline{\operatorname{ran}}(c-a)+\overline{\operatorname{ran}}(c-b) \subseteq \overline{\operatorname{ran}}(a-b)$ holds. ■

Note that it follows from these results that the study of minimal upper bounds of a and b on $\mathcal{H}$ is no harder than their study on the support of $a-b$, that is, on $\overline{\operatorname{ran}}(a-b)$. It follows, for instance, that all minimal upper bounds of two finite rank operators $a, b \in B(\mathcal{H})^{\mathrm{sa}}$ are finite rank as well, so their properties can be understood completely in the setting where $\mathcal{H}$ is finite-dimensional.

The following example shows that the final inclusion from Corollary 7.5 might be strict.

Example 7.7. Choose a Hilbert space $\mathcal{H}$ and closed subspaces $V, W \subseteq \mathcal{H}$ with $V \cap W = \{0\}$ such that $V+W$ is not closed (a class of subspaces of this type is given in Example A.3). Let $p, q \in B(\mathcal{H})$ be the orthogonal projections onto V and W, respectively. Clearly $p+q$ is an upper bound for p and q, and it is minimal because we have $\overline{\operatorname{ran}}(q) \cap \overline{\operatorname{ran}}(p) = W \cap V = \{0\}$. However, by assumption the sum $\overline{\operatorname{ran}}(q)+\overline{\operatorname{ran}}(p)$ is not closed. ⊠

7.3 Minimal upper bounds of complementary type

In light of the preceding results (Example 7.3, Corollary 7.5, and Example 7.7), we make the following definition.

Definition 7.8. We say that a minimal upper bound c for $a, b \in B(\mathcal{H})^{\mathrm{sa}}$ is *of complementary type*[1] if we have

$$\begin{aligned}\overline{\operatorname{ran}}(c-a) \cap \overline{\operatorname{ran}}(c-b) &= \{0\}; \\ \overline{\operatorname{ran}}(c-a)+\overline{\operatorname{ran}}(c-b) &= \overline{\operatorname{ran}}(a-b).\end{aligned}$$

In other words, $\overline{\operatorname{ran}}(c-a)$ and $\overline{\operatorname{ran}}(c-b)$ are *complementary subspaces* of $\overline{\operatorname{ran}}(a-b)$; see Appendix A.

In light of Corollary 7.5, the second condition is equivalent to the assertion that $\overline{\operatorname{ran}}(c-a)+\overline{\operatorname{ran}}(c-b)$ is closed.

Note that we do not require $\operatorname{ran}(c-a)$ and $\operatorname{ran}(c-b)$ to be closed; this would unnecessarily restrict us in the classification ahead.

We saw in examples 7.3 and 7.7 that both equalities from Definition 7.8 might fail. Despite these counterexamples, the following results show that a large class of minimal upper bounds are of complementary type.

Proposition 7.9. *The quasi-supremum $a \curlyvee b$ is of complementary type.*

Proof. For convenience, write $c := a \curlyvee b$. Recall that we have $c-a = (a-b)^-$ and $c-b = (a-b)^+$. By Proposition 5.8, we have $\overline{\operatorname{ran}}((a-b)^-) \perp \overline{\operatorname{ran}}((a-b)^+)$. It follows immediately that $\overline{\operatorname{ran}}(c-a) \cap \overline{\operatorname{ran}}(c-b) = \{0\}$ holds, and that $\overline{\operatorname{ran}}(c-a)+\overline{\operatorname{ran}}(c-b)$ is closed (cf. Corollary A.8). It follows from Corollary 7.5 that we have $\overline{\operatorname{ran}}(c-a)+\overline{\operatorname{ran}}(c-b) = \overline{\operatorname{ran}}(a-b)$. ■

Proposition 7.10. *Suppose that $\operatorname{ran}(a-b)$ is closed, and that c is a minimal upper bound for a and b. Then $\operatorname{ran}(c-a)$ and $\operatorname{ran}(c-b)$ are closed as well, and c is of complementary type.*

Proof. Since $\operatorname{ran}(a-b)$ is closed, we have equality throughout in Corollary 7.5, so we find $\operatorname{ran}(c-a)+\operatorname{ran}(c-b) = \operatorname{ran}(a-b)$. Recall furthermore that we have $\operatorname{ran}(c-a) \cap \operatorname{ran}(c-b) = \{0\}$. As such, every $y \in \operatorname{ran}(a-b)$ can be written uniquely as $y = y_a + y_b$ with $y_a \in \operatorname{ran}(c-a)$ and $y_b \in \operatorname{ran}(c-b)$. Consider the map $\pi_a : \operatorname{ran}(a-b) \to \operatorname{ran}(a-b)$ given by $y \mapsto y_a$. Note that (7.6) gives us an explicit formula for π_a now: the map is given by $(a-b)x \mapsto (a-c)x$. We prove that π_a is continuous.

Since $a-b$ is a self-adjoint operator with closed range, it restricts to an invertible operator $d := (a-b)|_{\operatorname{ran}(a-b)} : \operatorname{ran}(a-b) \to \operatorname{ran}(a-b)$. (Here we use some form of the open mapping theorem.) Now π_a is obtained as the composition $(a-c) \circ d^{-1}$, so in particular it is continuous, as promised.[2]

Note that we have $\ker(\pi_a) = \operatorname{ran}(c-b)$, so it follows from the continuity of π_a that $\operatorname{ran}(c-b)$ is closed. An analogous argument shows that $\operatorname{ran}(c-a)$ is closed as well. Since we already established $\operatorname{ran}(c-a)+\operatorname{ran}(c-b) = \operatorname{ran}(a-b)$ and $\operatorname{ran}(c-a) \cap \operatorname{ran}(c-b) = \{0\}$, it follows that c is of complementary type. ■

We note some of the applications of Proposition 7.10. If $\mathcal{H}$ is finite-dimensional, then all minimal upper bounds are of complementary type. More generally, if $\mathcal{H}$ is arbitrary but a and b have finite rank, then all minimal upper bounds of a and b are of complementary type (and have finite rank as well). Furthermore, if $a-b$ is invertible, then all minimal upper bounds for a and b are of complementary type as well.

In a way, the existence of contrived counterexamples like the one presented in Example 7.3 (where we had $\overline{\operatorname{ran}}(c-a) \cap \overline{\operatorname{ran}}(c-b) \neq \{0\}$) stems from the fact that the projective range of a positive operator sits somewhere between the range and its closure. Since the projective range is the correct tool in Proposition 6.9, it is not surprising that minimal upper bounds are much more well-behaved if the ranges in question are closed.

7.4 From minimal upper bounds to subspace triples

In this section we show that a minimal upper bound of complementary type gives rise to a special kind of decomposition of the underlying Hilbert space $\mathcal{H}$. These decompositions play an important role in the construction of minimal upper bounds in Section 7.5 below.

Recall that a *sesquilinear form* on a Hilbert space $\mathcal{H}$ is a map $g : \mathcal{H}^2 \to \mathbb{F}$ which is linear in the first coordinate and conjugate-linear in the second. We say that g is *bounded* if there is a positive constant γ such that $|g(x,y)| \le \gamma \|x\| \|y\|$ holds for all $x, y \in \mathcal{H}$, and g is *Hermitian* if we have $g(x,y) = g(y,x)^*$ for all $x, y \in \mathcal{H}$. (In the case $\mathbb{F} = \mathbb{C}$ the latter is equivalent with $g(x,x) \in \mathbb{R}$ for all $x \in \mathcal{H}$. Clearly this is no longer true in the case $\mathbb{F} = \mathbb{R}$.)

For $a \in B(\mathcal{H})$, the function $g_a(x,y) := \langle ax, y\rangle = \langle x, a^* y\rangle$ defines a bounded sesquilinear form. Conversely, for every bounded sesquilinear form g there exists a unique operator $a \in B(\mathcal{H})$ such that $g = g_a$ holds (cf. [Mur90, Theorem 2.3.6]). Furthermore, g_a is Hermitian if and only if a is Hermitian.

If g is a sesquilinear form and $S, T \subseteq \mathcal{H}$ are subsets, then we say that S and T are *orthogonal with respect to* g if $g(s,t) = 0$ holds for all $s \in S$, $t \in T$.

For the remainder of this chapter, we understand the term *Hermitian form* to mean "bounded Hermitian sesquilinear form".

The Hermitian form of our interest is the form corresponding with $a - b$, that is, the form given by $g(x,y) := \langle (a-b)x, y\rangle = \langle x, (a-b)y\rangle$. The following simple observations form the core of our classification programme.

Proposition 7.11. *Let c be an upper bound for $a, b \in B(\mathcal{H})^{\mathrm{sa}}$. Then:*

(a) For all $x \in \ker(c-a) \setminus \ker(a-b)$ one has $\langle ax, x\rangle > \langle bx, x\rangle$;

(b) For all $x \in \ker(c-b) \setminus \ker(a-b)$ one has $\langle ax, x\rangle < \langle bx, x\rangle$;

(c) The subspaces $\ker(c-a)$ and $\ker(c-b)$ are orthogonal with respect to the Hermitian form $g(x,y) = \langle (a-b)x, y\rangle = \langle x, (a-b)y\rangle$;

(d) We have $\ker(c-a) + \ker(c-b) = \mathcal{H}$ if and only if c is minimal and of complementary type.

Proof.

(a) For $x \in \ker(c-a) \setminus \ker(a-b)$ we have $cx = ax \ne bx$, hence $x \notin \ker(c-b)$. Since $c - a$ and $c - b$ are positive, it follows from Proposition 6.1 that we have $\langle (c-a)x, x\rangle = 0$ and $\langle (c-b)x, x\rangle > 0$. In other words, we have $\langle ax, x\rangle = \langle cx, x\rangle > \langle bx, x\rangle$.

(b) Analogous.

(c) For $x \in \ker(c-a)$ and $y \in \ker(c-b)$ we have $ax = cx$ and $by = cy$, hence

$$g(x,y) \;=\; \langle (a-b)x, y\rangle \;=\; \langle (c-b)x, y\rangle \;=\; \langle x, (c-b)y\rangle \;=\; \langle x, 0\rangle \;=\; 0.$$

(d) By Fact 7.1, we have that $\ker(c-a) + \ker(c-b)$ is dense in $\mathcal{H}$ if and only if $\overline{\operatorname{ran}}(c-a) \cap \overline{\operatorname{ran}}(c-b) = \{0\}$ holds. Moreover, $\ker(c-a) + \ker(c-b)$ is closed if and only if $\overline{\operatorname{ran}}(c-a) + \overline{\operatorname{ran}}(c-b)$ is closed, and by Corollary 7.5 this is true if and only if $\overline{\operatorname{ran}}(c-a) + \overline{\operatorname{ran}}(c-b) = \overline{\operatorname{ran}}(a-b)$ holds. ∎

It should be noted that Hermitian forms are not quite as well-behaved as the usual inner product. In particular, $\ker(c-a)$ and $\ker(c-b)$ are orthogonal with respect to g, but might nonetheless have non-trivial intersection! This is because the *kernel* (in this case, both the left and the right kernel) of g is equal to $\ker(a-b)$, so in fact $\ker(a-b)$ is orthogonal to all of $\mathcal{H}$! (For much more on Hermitian forms, including notions like left and right kernels, the interested reader is referred to [Lan02, Chapter XV].)

Using the preceding proposition, we show that a minimal upper bound of complementary type gives rise to a decomposition of $\mathcal{H}$ as the (internal) direct sum of a subspace where a and b are equal, a subspace where "a is larger", and a subspace where "b is larger". This is made precise in the following corollary.

Corollary 7.12. *Let c be a minimal upper bound of complementary type for $a, b \in B(\mathcal{H})^{\mathrm{sa}}$. Then the spaces $U := \ker(a-b)$, $V := \ker(c-a) \cap \ker(a-b)^\perp$ and $W := \ker(c-b) \cap \ker(a-b)^\perp$ satisfy the following properties:*

(a) U, V and W are closed linear subspaces of $\mathcal{H}$;

(b) We have $V \cap W = \{0\}$ and $V + W = U^\perp$, and therefore $\mathcal{H} = U \oplus V \oplus W$;

(c) U, V and W are pairwise orthogonal with respect to the Hermitian form $g(x,y) = \langle (a-b)x, y\rangle = \langle x, (a-b)y\rangle$;

(d) For all $x \in U$ we have $ax = bx$, and therefore $\langle ax, x\rangle = \langle bx, x\rangle$;

(e) For all $x \in V \setminus \{0\}$ we have $\langle ax, x\rangle > \langle bx, x\rangle$;

(f) For all $x \in W \setminus \{0\}$ we have $\langle ax, x\rangle < \langle bx, x\rangle$.

Before proceeding with the proof of Corollary 7.12, we pause to discuss the meaning of the expression $\mathcal{H} = U \oplus V \oplus W$, for the notation is ambiguous. We use it to mean that U, V and W are closed subspaces such that every $x \in \mathcal{H}$ can be written uniquely as $x = u + v + w$ with $u \in U$, $v \in V$, $w \in W$ in such a way that the projections $u + v + w \mapsto u$ (or v, or w) are continuous. In this case we say that $\mathcal{H}$ is equal to the *internal direct sum* of U, V and W. This is equivalent to the assertion that $\mathcal{H}$ is linearly isomorphic with the *external direct sum* of the subspaces U, V and W, each of which is a Hilbert space in its own right. With a little work, one can show that this isomorphism is an invertible bounded linear map. (It is not necessarily a Hilbert space isomorphism − this happens if and only if U, V and W are orthogonal.) We do not prove these claims; the interested reader is referred to [Con07, Section III.13].

It should be pointed out that even in the finite-dimensional case (where the topological requirements are automatically met), the statement $\mathcal{H} = U \oplus V \oplus W$ is stronger than $U + V + W = \mathcal{H}$ and $U \cap V = U \cap W = V \cap W = \{0\}$. Simply take three different one-dimensional subspaces of a two-dimensional space to see why. In order for $\mathcal{H} = U \oplus V \oplus W$ to hold, we must have $\mathcal{H} = U + V + W$, but we require a stronger disjointness criterion: any one (and therefore all) of the following equivalent conditions must be met: $V \cap W = U \cap (V + W) = \{0\}$, or $U \cap W = V \cap (U + W) = \{0\}$, or $U \cap V = W \cap (U + V) = \{0\}$. This can be generalised to finite internal direct sums; see [Lan02, Chapter I, §7].

Of course, the situation is relatively easy in Corollary 7.12(b): we have $V \cap W = \{0\}$ and $V + W = U^\perp$, so the condition $V \cap W = U \cap (V + W) = \{0\}$ is evidently met in this case.

Proof of Corollary 7.12.

(a) Trivial.

(b) By Proposition 7.4 we have $\ker(c-a)\cap\ker(c-b)=\ker(a-b)$, hence

$$\begin{aligned} V\cap W &= \ker(c-a)\cap\ker(c-b)\cap\ker(a-b)^{\perp} \\ &= \ker(a-b)\cap\ker(a-b)^{\perp} \\ &= \{0\}. \end{aligned}$$

Next, let $p:\mathcal{H}\to\mathcal{H}$ denote the orthogonal projection onto $\ker(a-b)^{\perp}$, and let ρ denote its restriction to a map $\mathcal{H}\to\ker(a-b)^{\perp}$. Since $\ker(a-b)$ is a subspace of $\ker(c-a)$ and $\ker(c-b)$, it follows from Proposition A.7(b) that we have $V=\rho(\ker(c-a))$ and $W=\rho(\ker(c-b))$. Therefore we have

$$\begin{aligned} V+W &= \rho(\ker(c-a))+\rho(\ker(c-b)) \\ &= \rho\big(\ker(c-a)+\ker(c-b)\big) \\ &= \rho(\mathcal{H}) \\ &= \ker(a-b)^{\perp}, \end{aligned}$$

so we find $V+W=U^{\perp}$. Now the coordinate projections $U^{\perp}\to V$ and $U^{\perp}\to W$ are automatically continuous (cf. Theorem A.4), so we have $U^{\perp}=V\oplus W$, and therefore $\mathcal{H}=U\oplus U^{\perp}=U\oplus V\oplus W$.

(c) It follows from Proposition 7.11(c) that V and W are orthogonal with respect to g. Furthermore, we note that U is the kernel of g, so it is orthogonal to everything (with respect to g).

(d) Trivial.

(e) We have $V\setminus\{0\}\subseteq\ker(c-a)\setminus\ker(a-b)$, so the result follows immediately from Proposition 7.11(a).

(f) Analogous. ■

We close this section with the following satellite lemmas.

Lemma 7.13. *Let c be a minimal upper bound of complementary type for $a,b\in B(\mathcal{H})^{\mathrm{sa}}$, and let $U,V,W\subseteq\mathcal{H}$ be as in Corollary 7.12. Then one has $\ker(c-a)=U+V$ and $\ker(c-b)=U+W$.*

Proof. This follows immediately from parts (b) and (c) of Proposition A.7. ■

Lemma 7.14. *Let c be a minimal upper bound of complementary type for $a,b\in B(\mathcal{H})^{\mathrm{sa}}$, and let $U,V,W\subseteq\mathcal{H}$ be as in Corollary 7.12. Then one has $V=\{0\}$ if and only if $a\leq b$, and similarly $W=\{0\}$ if and only if $a\geq b$.*

Proof. If $V=\{0\}$ holds, then we have $W=U^{\perp}$, hence $\ker(c-b)=U+W=\mathcal{H}$. It follows that $b=c\geq a$ holds.

Conversely, if we have $b\geq a$, then we must necessarily have $b=c$ (it is the only minimal upper bound), hence $V=\ker(b-a)\cap\ker(b-a)^{\perp}=\{0\}$.

This proves that we have $V=\{0\}$ if and only if $a\leq b$. Analogously, we have $W=\{0\}$ if and only if $a\geq b$. ■

7.5 From subspace triples to minimal upper bounds

In the previous section, we took a minimal upper bound c of complementary type, and assigned to it a triple (U, V, W) of subspaces with the properties described in Corollary 7.12. Observe that all of these properties of U, V and W are formulated purely in terms of a and b, with no mention of c. As such, we might wonder which triples (U, V, W) satisfying these properties can actually be obtained for some appropriate choice of c. In this section it will be shown that every such triple can be obtained, and that the assignment $c \mapsto (U, V, W)$ from Corollary 7.12 defines a bijective correspondence between minimal upper bounds of complementary type and triples (U, V, W) satisfying the conditions of Corollary 7.12.

The first step is to show that the assignment $c \mapsto (U, V, W)$ is injective.

Proposition 7.15. *Let c, c' be minimal upper bounds of complementary type for $a, b \in B(\mathcal{H})^{\mathrm{sa}}$, and let $(U, V, W), (U', V', W')$ be the corresponding triples of subspaces (as in Corollary 7.12). If $(U, V, W) = (U', V', W')$, then $c = c'$.*

Proof. Let $x \in \mathcal{H}$ be given, and write $x = u + v + w$ with $u \in U$, $v \in V$ and $w \in V$. Note that we have $U, V \subseteq \ker(c - a)$ and $W \subseteq \ker(c - b)$, by construction, so we find $cx = cu + cv + cw = au + av + bw$. Analogously, we also have $c'x = au + av + bw$, so we find $cx = c'x$ for all $x \in \mathcal{H}$. ■

In short, because we have $\ker(c-a)+\ker(c-b) = \mathcal{H}$, these two kernels provide enough information to reconstruct c uniquely from a and b. It is clear from this argument that there is only one possible candidate for c once we are given the triple (U, V, W). Our goal is to show that this candidate always defines a minimal upper bound of complementary type for a and b.

Theorem 7.16. *Let $a, b \in B(\mathcal{H})^{\mathrm{sa}}$ be given self-adjoint operators on $\mathcal{H}$, and let $U, V, W \subseteq \mathcal{H}$ be subspaces satisfying the properties of Corollary 7.12. Then the map $c : \mathcal{H} \to \mathcal{H}$ given by $u + v + w \mapsto au + av + bw$ is linear, bounded, self-adjoint, and a minimal upper bound of complementary type for a and b. Furthermore, if (U', V', W') denotes the triple of subspaces obtained from c via Corollary 7.12, then we have $(U, V, W) = (U', V', W')$.*

Proof. Let $p \in B(\mathcal{H})$ be the orthogonal projection onto $U^\perp$, and let $q' \in B(U^\perp)$ denote the idempotent $w + v \mapsto w$ corresponding to the complementary pair $W, V \subseteq U^\perp$ (in the sense of Theorem A.4). Extend q' to an operator $q \in B(\mathcal{H})$ with $q(u) = 0$ for all $u \in U$. Now the decomposition map $\mathcal{H} \to U \oplus V \oplus W$ is given by $x \mapsto (1 - p)x \oplus (p - q)x \oplus qx$. Consequently, c can be written as

$$c \;=\; a(1 - p) + a(p - q) + bq \;=\; a(1 - q) + bq,$$

from which it is clear that c is a bounded linear operator.

In order to show that c is self-adjoint, note that U and V, and therefore $U + V$, are orthogonal to W with respect to the Hermitian form g. For all $x, y \in \mathcal{H}$ we have $qx \in W$ and $(1 - q)y \in U + V$ hence

$$\big\langle (1 - q^*)(b - a)qx, y\big\rangle \;=\; \big\langle (b - a)qx, (1 - q)y\big\rangle \;=\; -g\big(qx, (1 - q)y\big) \;=\; 0.$$

As this holds for all $x, y \in \mathcal{H}$, we find $(1 - q^*)(b - a)q = 0$. By taking the adjoint, we also find $q^*(b - a)(1 - q) = 0$, which we will need later.

Now we may write

$$\begin{aligned} c &= a + (b-a)q \\ &= a + (b-a)q - (1-q^*)(b-a)q \\ &= a + q^*(b-a)q, \end{aligned}$$

from which it is clear that c is self-adjoint.

Thirdly, we show that c is an upper bound for a and b. For all $x \in \mathcal{H}$ we have $qx \in W$, so by assumption (f) we have

$$\big\langle (c-a)x, x\big\rangle = \big\langle q^*(b-a)qx, x\big\rangle = \big\langle (b-a)qx, qx\big\rangle \geq 0.$$

In fact, the inequality is strict whenever $qx \neq 0$ holds, and we will need this later on. For now, the argument already shows that $c \geq a$ holds. While the inequality $c \geq b$ follows by symmetry (simultaneously interchanging a with b and V with W), we choose to write this out in the present setting, since this will give us another expression for c which we need later. Note that we have

$$\begin{aligned} c &= b + (a-b)(1-q) \\ &= b + (a-b)(1-q) - q^*(a-b)(1-q) \\ &= b + (1-q^*)(a-b)(1-q). \end{aligned}$$

By assumption (d) we have $U \subseteq \ker(a-b)$. Since p is the orthogonal projection onto $U^\perp \supseteq \overline{\operatorname{ran}}(a-b)$, we have $a - b = p(a-b) = (a-b)p = p(a-b)p$. It follows that c can be written as

$$c = b + (p-q^*)(a-b)(p-q).$$

Now it is clear that $c \geq b$ holds: for all $x \in \mathcal{H}$ we have $(p-q)x \in V$, hence

$$\big\langle (c-b)x, x\big\rangle = \big\langle (a-b)(p-q)x, (p-q)x\big\rangle \geq 0,$$

and the inequality is strict whenever $(p-q)x \neq 0$ holds. (Note: here we used that p is always self-adjoint, unlike q.) We conclude that c is an upper bound for a and b, as promised.

Using our knowledge of when the preceding inequalities are strict, it follows from Proposition 6.1 that we have

$$\ker(c-a) = \big\{x \in \mathcal{H} : \langle (b-a)qx, qx\rangle = 0\big\} = \ker(q) = U + V;$$
$$\ker(c-b) = \big\{x \in \mathcal{H} : \langle (b-a)(p-q)x, (p-q)x\rangle = 0\big\} = \ker(p-q) = U + W.$$

In particular, we have $\ker(c-a) + \ker(c-b) = \mathcal{H}$, so c is minimal and of complementary type (by Proposition 7.11(d)).

Since expressions of the form $x = u + v + w$ with $u \in U$, $v \in V$ and $w \in W$ are unique, it is clear that $(U+V) \cap (U+W) = U$ holds. Now it follows from Proposition 7.4 that we have $\ker(a-b) = \ker(c-a) \cap \ker(c-b) = U$. Furthermore, since U and V are orthogonal, we have $V = (V+U) \cap U^\perp$ (cf. Proposition A.7), that is: $V = \ker(c-a) \cap \ker(a-b)^\perp$. Analogously we have $W = \ker(c-b) \cap \ker(a-b)^\perp$, so we see that (U, V, W) coincides with the triple of subspaces obtained from c via Corollary 7.12. ■

In summary, we have the following:

Theorem 7.17. *The constructions from Corollary 7.12 and Theorem 7.16 define a bijective correspondence between the minimal upper bounds of complementary type for a and b, and the triples (U, V, W) of subspaces satisfying the conditions from Corollary 7.12.* ∎

7.6 Finding additional subspace triples

We know from Proposition 7.9 that a minimal upper bound of complementary type exists, namely the quasi-supremum $a \curlyvee b$. The corresponding subspace decomposition is $(\ker(a-b), \overline{\operatorname{ran}}((a-b)^+), \overline{\operatorname{ran}}((a-b)^-))$. For our second proof of the anti-lattice theorem, we show that this is not the only one (unless a and b are comparable).

In order to find additional triples (U, V, W) satisfying the conditions of Corollary 7.12, it helps to understand the behaviour of Hermitian forms. While they share some properties with the usual inner product, we will see that their behaviour can at times be counterintuitive. We start with the following proposition, which is nothing out of the ordinary.

Proposition 7.18. *Let $g(x, y) = \langle (a-b)x, y\rangle = \langle x, (a-b)y\rangle$ be the Hermitian form from before. For a non-empty subset $X \subseteq \mathcal{H}$, the orthogonal complement of X with respect to g is given by $X^{\perp_g} = ((a-b)X)^\perp = (a-b)^{-1}(X^\perp)$. In particular, $X^{\perp_g}$ is a closed subspace.*

Proof. By definition we have

$$\begin{aligned} X^{\perp_g} &= \big\{ y \in \mathcal{H} \,:\, \langle (a-b)x, y\rangle = 0 \text{ for all } x \in X \big\} \\ &= \big\{ y \in \mathcal{H} \,:\, \langle z, y\rangle = 0 \text{ for all } z \in (a-b)X \big\} \\ &= ((a-b)X)^\perp; \end{aligned}$$

$$\begin{aligned} X^{\perp_g} &= \big\{ y \in \mathcal{H} \,:\, \langle x, (a-b)y\rangle = 0 \text{ for all } x \in X \big\} \\ &= \big\{ y \in \mathcal{H} \,:\, (a-b)y \in X^\perp \big\} \\ &= (a-b)^{-1}(X^\perp). \end{aligned}$$

It follows from either of these expressions that $X^{\perp_g}$ is a closed subspace. ∎

The following example shows some of the stranger things that can happen.

Example 7.19. Consider an injective positive operator $a \in B(\mathcal{H})$ with dense range such that $\operatorname{pran}(b) \neq \mathcal{H}$ holds. Choose some vector $z \notin \operatorname{pran}(a)$, and let $p := z \otimes z$ be (a positive multiple of) the orthogonal projection onto $\operatorname{span}(z)$. By the remarks in Example 7.3, in this setting we have that $a + p$ is a minimal upper bound for a and p. Furthermore, in light of Proposition 7.11(c), the subspaces $V := \ker(p) = \{z\}^\perp$ and $W := \ker(a) = \{0\}$ are orthogonal with respect to $g(x, y) = \langle (a-p)x, y\rangle$.

We prove that $V^{\perp_g} = \{0\}$ holds. In light of Proposition 7.18, we have $V^{\perp_g} = (a-p)^{-1}\operatorname{span}(z)$. Thus, for $x \in V^{\perp_g}$ we have $(a-p)x \in \operatorname{span}(z)$.

Now we also have $ax \in \operatorname{span}(z)$ (since $px \in \operatorname{span}(z)$), hence $ax = 0$ (because $z \notin \operatorname{pran}(a) \supseteq \operatorname{ran}(a)$), and therefore $x = 0$ (since a is injective), proving our claim that $V^{\perp_g} = \{0\}$ holds.

In this example we have $V + V^{\perp_g} \neq \mathcal{H}$. Note furthermore that we have $W^{\perp_g} = \mathcal{H}$, so additionally we find $(V^{\perp_g})^{\perp_g} = W^{\perp_g} = \mathcal{H} \neq V$. ⊠

At the very least, the preceding example shows that we need to construct V and W simultaneously. We cannot simply construct one of the two, say V, and set $W := V^{\perp_g}$. (Not only would our construction of V have to make sure that properties (e) and (f) of Corollary 7.12 hold; additional care would have to be taken to show that $V + V^{\perp_g} = U^\perp$ even holds for the chosen construction!) It seems that this approach is doomed.

The approach we take instead is to make slight modifications to an existing solution. If the triple (U, V, W) satisfies the properties of Corollary 7.12, then g defines an inner product on V, and similarly $-g$ defines an inner product on W. While inner product spaces are not quite as well-behaved as Hilbert spaces (e.g. they do not always have an orthonormal basis), we do have the following tool at our disposal.

Proposition 7.20. *Let $(X, \langle\,\cdot\,,\,\cdot\,\rangle)$ be an inner product space, and let $Y \subseteq X$ be a finite-dimensional subspace. Then one has $Y \cap Y^\perp = \{0\}$ and $Y + Y^\perp = X$.*

Proof. The equality $Y \cap Y^\perp = \{0\}$ is clear, since $\langle x, x\rangle = 0$ implies $x = 0$.

Note that Y is complete, since it is finite-dimensional, so we see that Y is a Hilbert space. Choose some orthonormal basis $b_1, \ldots, b_n$ for Y. Let $x \in X$ be given and set $x' := x - \sum_{k=1}^n \langle x, b_k\rangle b_k$. A straightforward computation shows that $x' \perp \{b_1, \ldots, b_n\}$ holds, so we have $x' \in Y^\perp$ and $x \in Y + Y^\perp$. ∎

In particular, by passing to the inner product spaces (V, g) and $(W, -g)$, we find the following corollary.

Corollary 7.21. *Let $U, V, W \subseteq \mathcal{H}$ be subspaces satisfying the properties of Corollary 7.12. If $Y \subseteq V$ and $Z \subseteq W$ are finite-dimensional subspaces, then one has $V = Y \oplus (V \cap Y^{\perp_g})$ and $W = Z \oplus (W \cap Z^{\perp_g})$.*

Proof. Note that $V \cap Y^{\perp_g}$ is precisely the orthogonal complement of Y in the inner product space (V, g), so the result follows from propositions 7.20 and 7.18. (We are working in a Hilbert space now, so the notation $\oplus$ is reserved for closed subspaces.) For $(W, -g)$ the argument is analogous, where we note that the orthogonal complement with respect to $-g$ is equal to the orthogonal complement with respect to g. ∎

This leads to the following construction.

Construction 7.22. Let a and b be incomparable, and let (U, V, W) be a triple satisfying the properties of Corollary 7.12. Then by Lemma 7.14 we have $V \neq \{0\}$ and $W \neq \{0\}$, so we may choose one-dimensional subspaces $Y \subseteq V$, $Z \subseteq W$. By Corollary 7.21 we have $U^\perp = Y \oplus Z \oplus (V \cap Y^{\perp_g}) \oplus (W \cap Z^{\perp_g})$, and the four summands are pairwise orthogonal with respect to g. We shall make a modification in $Y \oplus Z$ and leave $(V \cap Y^{\perp_g}) \oplus (W \cap Z^{\perp_g})$ invariant.

Since g and $-g$ define inner products on V and W, respectively, we may choose unit vectors $y \in Y$ and $z \in Z$ with respect to these inner products. In other words, we normalise the generators of Y and Z so that we have $g(y,y) = 1$ and $g(z,z) = -1$. Now choose some arbitrary $\alpha \in (0, \frac{1}{2})$, and define

$$Y' := \operatorname{span}\big((1-\alpha)y + \alpha z\big), \qquad \text{and} \qquad Z' := \operatorname{span}\big(\alpha y + (1-\alpha)z\big).$$

Clearly we have $Y' \cap Z' = \{0\}$ and $Y' + Z' = Y + Z$, so we find a second decomposition $U^\perp = Y' \oplus Z' \oplus (V \cap Y^{\perp_g}) \oplus (W \cap Z^{\perp_g})$. We claim that (U, V', W') is another triple satisfying the properties of Corollary 7.12, where V' and W' are given by

$$V' := Y' + (V \cap Y^{\perp_g}), \qquad \text{and} \qquad W' := Z' + (W \cap Z^{\perp_g}).$$

To see that V' and W' are closed, recall that the sum of a closed subspace and a finite-dimensional subspace is again closed (cf. [Rud91, Theorem 1.42]). Furthermore, U is unchanged and we have $U^\perp = V' \oplus W'$, so we see that properties (a), (b) and (d) are satisfied.

In order to see that V' and W' are orthogonal with respect to g, write $y' = (1-\alpha)y + \alpha z \in Y'$ and $z' = \alpha y + (1-\alpha)z \in Z'$, and note that we have

$$\begin{aligned} g(y',z') &= (1-\alpha)\alpha \cdot g(y,y) + (1-\alpha)^2 g(y,z) + \alpha^2 g(z,y) + \alpha(1-\alpha) \cdot g(z,z) \\ &= (1-\alpha)\alpha \cdot 1 \;+\; 0 \;+\; 0 \;+\; \alpha(1-\alpha) \cdot -1 \\ &= 0. \end{aligned}$$

As such we have $Y' \perp_g Z'$. Moreover, the three summands in the decomposition $U^\perp = (Y \oplus Z) \oplus (V \cap Y^{\perp_g}) \oplus (W \cap Z^{\perp_g}) = (Y' \oplus Z') \oplus (V \cap Y^{\perp_g}) \oplus (W \cap Z^{\perp_g})$ were orthogonal to begin with, so it follows that the four summands in the decomposition $U^\perp = Y' \oplus Z' \oplus (V \cap Y^{\perp_g}) \oplus (W \cap Z^{\perp_g})$ are pairwise orthogonal with respect to g. In particular we have $V' \perp_g W'$, and property (c) is satisfied.

In order to see that g is positive definite on V' and negative definite on W', note that we have

$$\begin{aligned} g(y',y') &= (1-\alpha)^2 \cdot g(y,y) + (1-\alpha)\alpha g(y,z) + \alpha(1-\alpha) g(z,y) + \alpha^2 \cdot g(z,z) \\ &= (1-\alpha)^2 \cdot 1 \;+\; 0 \;+\; 0 \;+\; \alpha^2 \cdot -1 \\ &= 1 - 2\alpha; \end{aligned}$$

$$\begin{aligned} g(z',z') &= \alpha^2 \cdot g(y,y) + \alpha(1-\alpha) g(y,z) + (1-\alpha)\alpha g(z,y) + (1-\alpha)^2 \cdot g(z,z) \\ &= \alpha^2 \cdot 1 \;+\; 0 \;+\; 0 \;+\; (1-\alpha)^2 \cdot -1 \\ &= 2\alpha - 1. \end{aligned}$$

Since we assumed $\alpha < \frac{1}{2}$, we find that g is positive definite on Y' and negative definite on Z'. Now we note that V' is defined as the sum of two g-orthogonal subspaces and that g is positive definite on each of the two summands. From this it follows that g is positive definite on all of V'. Analogously, g is negative definite on W', so we see that properties (e) and (f) are satisfied.

Finally, note that the obtained decomposition (U, V', W') is different from the original, since we have $y' \in V'$ but $y' \notin V$. (After all, $y' \in V$ would imply $z \in V \cap W = \{0\}$, but z is non-zero by assumption.) In fact, we note that, by the same argument, different choices of $\alpha \in (0, \frac{1}{2})$ give rise to different decompositions (U, V', W'). $\boxtimes$

At this time we can give our "geometric" proof of the anti-lattice theorem.

Second proof of the anti-lattice theorem. Let $a, b \in B(\mathcal{H})^{\mathrm{sa}}$ be incomparable. There is at least one triple (U, V, W) satisfying the properties of Corollary 7.12 (namely the triple corresponding to $a \curlyvee b$), so it follows from Construction 7.22 that there are infinitely many such triples (corresponding to different choices of $\alpha \in (0, \frac{1}{2})$). Therefore a and b admit infinitely many minimal upper bounds of complementary type. On the other hand, if a and b were to have a supremum, then there would be exactly one minimal upper bound. ■

In fact, Construction 7.22 gives us the following strengthening of the anti-lattice theorem.

Theorem 7.23. *Let $a, b \in B(\mathcal{H})^{\mathrm{sa}}$ be incomparable, then the set of all minimal upper bounds of complementary type for a and b is unbounded.*

Proof. Let $U, V, W \subseteq \mathcal{H}$ be subspaces satisfying the properties of Corollary 7.12. Furthermore, let $\alpha \in (0, \frac{1}{2})$ and $Y, Z, Y', Z' \subseteq \mathcal{H}$ and $y, z, y', z' \in \mathcal{H}$ be as in Construction 7.22. We will vary the value of α at a later point. (The spaces Y' and Z' and the elements $y' \in Y'$ and $z' \in Z'$ depend on α, so these will be understood to change accordingly.)

Now let $q \in B(\mathcal{H})^{\mathrm{sa}}$ be the idempotent $u + v' + w' \mapsto w'$ for the triple (U, V', W'), as in the proof of Theorem 7.16. As established in the proof of said theorem, for all $x \in \mathcal{H}$ we have $\langle (c' - a)x, x\rangle = \langle (b - a)qx, qx\rangle = -g(qx, qx)$, where $c' \in B(\mathcal{H})$ denotes the minimal upper bound corresponding to the new triple of subspaces (U, V', W').

We write our "old" vector $z \in Z$ from the construction in terms of the "new" vectors $y' \in Y'$ and $z' \in Z'$. A direct computation shows that we have

$$z = \frac{1}{1 - 2\alpha}\big(-\alpha y' + (1 - \alpha)z'\big).$$

In particular, it follows that $qz = \frac{1-\alpha}{1-2\alpha}z'$ holds, so we have

$$\begin{aligned}
\big\langle (c' - a)z, z\big\rangle &= -g\left(\frac{1-\alpha}{1-2\alpha}z', \frac{1-\alpha}{1-2\alpha}z'\right) \\
&= \left(\frac{1-\alpha}{1-2\alpha}\right)^2 \cdot -g(z', z') \\
&= \left(\frac{1-\alpha}{1-2\alpha}\right)^2 \cdot (1 - 2\alpha) \\
&= \frac{(1-\alpha)^2}{1-2\alpha}.
\end{aligned}$$

Letting α increase to $\frac{1}{2}$, we find that $\langle (c' - a)z, z\rangle$ goes to infinity. Consequently, $\|c' - a\|$ goes to infinity, so the minimal upper bounds obtained from this construction are unbounded in norm. ■

Example 7.24. To illustrate the conclusion of Theorem 7.23, we once again consider the setting from Example 7.2:

$$a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad a - b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The standard subspace triple for this example is

$$\begin{aligned}(U, V, W) &= \big(\ker(a-b), \overline{\operatorname{ran}}((a-b)^+), \overline{\operatorname{ran}}((a-b)^-)\big)\\ &= \big(\{0\}, \operatorname{span}\{(1,0)\}, \operatorname{span}\{(0,1)\}\big).\end{aligned}$$

Let us carry out Construction 7.22 with respect to this triple. Then we see that the one-dimensional subspaces $Y \subseteq V$ and $Z \subseteq W$ from Construction 7.22 are unique. For the normalised generators of Y and Z, let us choose $y = (1,0)$ and $z = (0,1)$. (Indeed, we have $\langle (a-b)y, y\rangle = 1$ and $\langle (a-b)z, z\rangle = -1$, so these vector are normalised with respect to g.) Choose some $\alpha \in (0, \frac{1}{2})$, then the construction gives us

$$y' = \begin{pmatrix} 1-\alpha \\ \alpha \end{pmatrix}, \qquad z' = \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix}.$$

In this setting, the new subspace decomposition is given simply by

$$(U, V', W') = \big(\{0\}, \operatorname{span}\{y'\}, \operatorname{span}(z')\big).$$

The corresponding minimal upper bound for a and b is given by mapping y' to ay' and z' to bz'. Concretely, this is given by the matrix

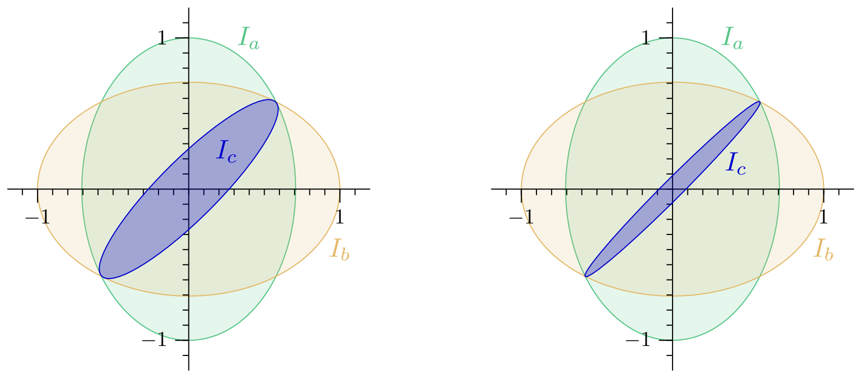
$$\begin{aligned} c &= \begin{pmatrix} 2-2\alpha & \alpha \\ \alpha & 2-2\alpha \end{pmatrix} \cdot \begin{pmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{pmatrix}^{-1} \\ &= \frac{1}{1-2\alpha} \begin{pmatrix} \alpha^2 - 4\alpha + 2 & \alpha^2 - \alpha \\ \alpha^2 - \alpha & \alpha^2 - 4\alpha + 2 \end{pmatrix}. \end{aligned}$$

Letting α increase to $\frac{1}{2}$, this “converges” (or rather, diverges) to

$$\frac{1}{+\infty} \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix},$$

so indeed we see that the entries of the minimal upper bounds of a and b are unbounded.

In terms of our geometric view of Example 7.2, the result is that the maximal ellipsoidal disks in $I_a \cap I_b$ can become arbitrarily thin, as illustrated in the figure below.



$\alpha = 0.49$ $\alpha = 0.499$

The boundary of the indicatrix I_c of c touches ∂I_a and ∂I_b in some multiple of y' and z', respectively. These directions get arbitrarily close together as α increases to $\frac{1}{2}$. As a result, the indicatrix I_c becomes arbitrarily thin, reflecting the fact that one the eigenvalues of c becomes arbitrarily large. ⊠

7.7 Constructions in the finite-dimensional case

In the infinite-dimensional case it was not so clear how to construct additional subspace triples, except by making tiny adjustments to an existing triple. The situation is different if $\mathcal{H}$ is finite-dimensional. The results are well understood in this setting, and form part of the standard theory of symmetric bilinear forms (which can easily be extended to results on Hermitian forms for the complex case). The theory can be found in many textbooks in linear algebra. We paraphrase results from [Lan02, Chapter XV, §1–§5].[3] (Note: in the literature it is customary to make a distinction between *Hermitian sesquilinear forms* on a complex vector space and *symmetric bilinear forms* on a real vector space, but we have put these two together under the name *Hermitian forms*. For our purposes they are the same.)

In our subspace decompositions we always have $U = \ker(a - b)$, so this part is uninteresting. Therefore, let us assume without loss of generality that $U = \{0\}$ holds, so that the operator $a - b$ is invertible and the Hermitian form g is non-degenerate. Here we have the following tools at our disposal.

Proposition 7.25. *Let $\mathcal{H}$ be a finite-dimensional real or complex vector space, and let $g : \mathcal{H}^2 \to \mathbb{F}$ be a Hermitian form. Then:*

(a) If $\{b_1, \ldots, b_n\}$ is a g-orthogonal basis of $\mathcal{H}$, then we have $g(b_i, b_i) \neq 0$ for all $i \in \{1, \ldots, n\}$ if and only if g is non-degenerate on $\mathcal{H}$.

Assume furthermore that g is non-degenerate. Then:

(b) If $V \subseteq \mathcal{H}$ is a subspace, then one has $\dim(V) + \dim(V^{\perp_g}) = \dim(\mathcal{H})$ and $(V^{\perp_g})^{\perp_g} = V$. As a consequence, the following are equivalent:

(1) g is non-degenerate on V;

(2) g is non-degenerate on $V^{\perp_g}$;

(3) $V + V^{\perp_g} = \mathcal{H}$.

(c) If $\dim(\mathcal{H}) \geq 1$ holds, then there exist vectors $x \in \mathcal{H}$ with $g(x, x) \neq 0$.

(d) Every finite g-orthogonal set $\{b_1, \ldots, b_k\}$ satisfying $g(b_i, b_i) \neq 0$ for all $i \in \{1, \ldots, k\}$ can be extended to a g-orthogonal basis of $\mathcal{H}$.

(e) $\mathcal{H}$ admits a g-orthogonal decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ such that g is positive definite on $\mathcal{H}^+$ and negative definite on $\mathcal{H}^-$. The dimension of $\mathcal{H}^+$ (or $\mathcal{H}^-$) is the same in all such decompositions.

Complete proofs of these claims can be found in [Lan02, Chapter XV, §1–§5]. We sketch the most important techniques.

Proof (outline).

(a) If $g(x, b_j) = 0$ holds for all $j \in \{1, \ldots, n\}$, then we have $x \perp_g \mathcal{H}$. Thus, if g is non-degenerate, then we must have $g(b_i, b_i) \neq 0$, for otherwise we would have $b_i \perp_g \mathcal{H}$.

Conversely, assume that g is degenerate, and choose some non-zero vector $x \perp_g \mathcal{H}$. Write $x = \lambda_1 b_1 + \ldots + \lambda_n b_n$, and choose some $i \in \{1, \ldots, n\}$ such that $\lambda_i \neq 0$ holds. Then we have $g(x, b_i) = \lambda_i g(b_i, b_i)$, by the g-orthogonality of $\{b_1, \ldots, b_n\}$. But we also have $g(x, b_i) = 0$, since we assumed $x \perp_g \mathcal{H}$. We conclude that $g(b_i, b_i) = 0$ must hold.

(b) We have $V^{\perp_g} = (a-b)^{-1}(V^{\perp})$, by Proposition 7.18, and $a-b$ is invertible by assumption. Hence it is clear that $\dim(V)+\dim(V^{\perp_g}) = \dim(\mathcal{H})$ holds. Furthermore, by Proposition 7.18 and invertibility of $a - b$ we have

$$(V^{\perp_g})^{\perp_g} = ((a-b)V^{\perp_g})^{\perp} = ((a-b)(a-b)^{-1}(V^{\perp}))^{\perp} = V^{\perp\perp} = V.$$

The g-orthogonal complement of V within V is given by $V \cap V^{\perp_g}$, so it is clear that g is non-degenerate on V if and only if $V \cap V^{\perp_g} = \{0\}$ holds. By a dimension argument, this happens if and only if $V + V^{\perp_g} = \mathcal{H}$ holds. This proves the equivalence (1) $\Longleftrightarrow$ (3). Since we have $(V^{\perp_g})^{\perp_g} = V$, the equivalence (3) $\Longleftrightarrow$ (2) follows immediately.

(c) This follows from the real and complex polarisation identities:

$$4g(x,y) = \begin{cases} g(x+y, x+y) - g(x-y, x-y) & \text{if } \mathbb{F} = \mathbb{R}; \\ \displaystyle\sum_{k=0}^{3} i^k g(x + i^k y, x + i^k y) & \text{if } \mathbb{F} = \mathbb{C}. \end{cases}$$

Consequently, if $g(x, x) = 0$ holds for all $x \in \mathcal{H}$, then we have $g(x, y) = 0$ for all $x, y \in \mathcal{H}$, so we must have $\mathcal{H} = \{0\}$ (since g is assumed to be non-degenerate).

(d) First we prove that $\{b_1, \ldots, b_k\}$ is linearly independent. To that end, let $x = \lambda_1 b_1 + \cdots + \lambda_k b_k$ be a linear combination, not all coefficients equal to zero. Fix some $i \in \{1, \ldots, k\}$ with $\lambda_i \neq 0$. Then, by g-orthogonality of $\{b_1, \ldots, b_k\}$, we have $g(x, b_i) = \lambda_i g(b_i, b_i) \neq 0$. In particular, it follows that $x \neq 0$ holds, proving that $\{b_1, \ldots, b_k\}$ is linearly independent.

Now consider $V := \operatorname{span}(b_1, \ldots, b_k)$. Then $\{b_1, \ldots, b_k\}$ is a g-orthogonal basis for V. Since we have $g(b_i, b_i) \neq 0$ for all $i \in \{1, \ldots, k\}$, it follows from part (a) that g is non-degenerate on V. Consequently, by part (b) we have $V + V^{\perp_g} = \mathcal{H}$, and g is non-degenerate on $V^{\perp_g}$. Choose some $b_{k+1} \in V^{\perp_g}$ with $g(b_{k+1}, b_{k+1}) \neq 0$, using part (c), and proceed by induction.

(e) It follows from parts (c) and (d) that we can choose a g-orthogonal basis $\{b_1, \ldots, b_n\}$ of $\mathcal{H}$. Then, by part (a), for all i we have $g(b_i, b_i) \neq 0$. Let $\mathcal{H}^+$ be the span of all b_i with $g(b_i, b_i) > 0$ and $\mathcal{H}^-$ the span of all b_j with $g(b_j, b_j) < 0$. From the g-orthogonality of $\{b_1, \ldots, b_n\}$ it is clear that g is positive definite on $\mathcal{H}^+$ and negative definite on $\mathcal{H}^-$.

The invariance of $\dim(\mathcal{H}^+)$ and $\dim(\mathcal{H}^-)$ is an equivalent form of what is known as *Sylvester's law of inertia*. (Cf. [Lan02, Theorem XV.4.1].) ■

In particular, parts (d) and (e) tell us how to construct many more subspace decompositions (and therefore minimal upper bounds) in this setting: simply choose a vector $x \in \mathcal{H}$ with $g(x,x) \neq 0$, extend it to an orthogonal basis for $\mathcal{H}$, and construct $\mathcal{H}^+$ and $\mathcal{H}^-$ as in the proof of Proposition 7.25(e). In terms of our geometric interpretation from Section 7.1, this means that we can choose a point which lies on the boundary of ∂I_a and in the interior of I_b (or vice versa) and construct a minimal upper bound $c \geq a, b$ such that ∂I_c touches the respective boundary in the chosen point. (More generally, any g-orthogonal set of such touching points can be specified.)

To summarise: the results from this chapter show that the finite-dimensional version of Kadison's anti-lattice theorem is equivalent to the following statement.

Fact 7.26. Let $\mathcal{H}$ be a finite-dimensional real or complex vector space, and let $g : \mathcal{H}^2 \to \mathbb{F}$ be a non-degenerate Hermitian form. Then the g-orthogonal decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ from Proposition 7.25(e) (with the property that g is positive definite on $\mathcal{H}^+$ and negative definite on $\mathcal{H}^-$) is unique if and only if g is either positive definite or negative definite.

(Actually, we only showed that this fact implies the anti-lattice theorem, and not the other way around. For the reverse implication we still have to prove the following: if there is a unique minimal upper bound c for a and b, then c is the supremum of a and b. We prove this using a compactness argument. Let c be the unique minimal upper bound and let d be an arbitrary upper bound for a and b. Since $\mathcal{H}$ is finite-dimensional, the intersection of order intervals $[a,d] \cap [b,d]$ is compact, so there exists an element $c' \in [a,d] \cap [b,d]$ of minimal trace. Since the trace defines a strictly positive linear map, it is clear that c' is a minimal upper bound. By uniqueness we must have $c = c'$, so we find $c \leq d$. We conclude that c is the supremum of a and b.)

7.8 End notes

1. (page 78) This terminology (*minimal upper bound of complementary type*) was invented by the author.

2. (page 79) There is a slight abuse of notation here. While both d and π_a are maps $\operatorname{ran}(a-b) \to \operatorname{ran}(a-b)$, the operator $a-c$ is still a map $\mathcal{H} \to \mathcal{H}$. Strictly speaking we should note that $\operatorname{ran}(a-b)$ is a reducing subspace for $a-c$, and write $\pi_a = e \circ d^{-1}$, where $e := (a-c)|_{\operatorname{ran}(a-b)}$ denotes the restriction of $a-c$ to a map $\operatorname{ran}(a-b) \to \operatorname{ran}(a-b)$. The argument still stands: π_a is a composition of continuous operators, and as such it continuous as well.

3. (page 90) The treatment of symmetric and Hermitian forms in [Lan02] is rather terse. A more detailed discussion can be found in, for instance, [Gre75, Chapter IX]. Then again, any textbook on (advanced) linear algebra is likely to contain the results we claim in Section 7.7 (look for the theory of symmetric bilinear forms.)

8 Sherman's theorem

In this chapter we prove one of the main theorems relating order to algebraic structure: the positive cone in a C^*-algebra $\mathcal{A}$ is a lattice cone if and only if $\mathcal{A}$ is commutative. We proceed via representation theory in order to deduce this from Kadison's anti-lattice theorem.

Throughout this chapter, $\mathcal{H}$ denotes a *complex* Hilbert space.

8.1 $*$-homomorphisms preserve lattice structure

Recall from Section 5.3 that $*$-homomorphisms are positive and preserve the quasi-lattice operations. The following two lemmas show that, under certain circumstances, even suprema and infima are preserved by $*$-homomorphisms.

Proposition 8.1. *Let $\mathcal{A}$ be a C^*-algebra and let $a, b \in \mathcal{A}^{\mathrm{sa}}$ be arbitrary self-adjoint elements. Then $a \curlyvee b$ is a minimal upper bound for a and b.*

Proof. Choose a faithful representation $\varphi : \mathcal{A} \to B(\mathcal{H})$, so that we may identify $\mathcal{A}$ with the C^*-subalgebra $\varphi[\mathcal{A}] \subseteq B(\mathcal{H})$. Since $\varphi(a) \curlyvee \varphi(b) = \varphi(a \curlyvee b)$ is a minimal upper bound for $\varphi(a)$ and $\varphi(b)$ relative to all of $B(\mathcal{H})^{\mathrm{sa}}$, it certainly is a minimal upper bound relative to the C^*-subalgebra $\varphi[\mathcal{A}] \subseteq B(\mathcal{H})$. The result follows since φ defines an isomorphism $\mathcal{A} \cong \varphi[\mathcal{A}]$ of C^*-algebras. ■

Corollary 8.2 (cf. Corollary 6.11)**.** *Let $\mathcal{A}$ be a C^*-algebra. If $a, b \in \mathcal{A}^{\mathrm{sa}}$ have a supremum c, then one necessarily has $c = a \curlyvee b$.*

Proof. By the supremum property of c we have $c \leq a \curlyvee b$. Then, by minimality of $a \curlyvee b$ (Proposition 8.1) we must have $c = a \curlyvee b$. ■

This is complemented by the following lemma.

Lemma 8.3. *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a surjective $*$-homomorphism between the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. If $a, b \in \mathcal{A}^{\mathrm{sa}}$ are given such that $a \curlyvee b$ is the supremum of a and b, then $\varphi(a) \curlyvee \varphi(b)$ is the supremum of $\varphi(a)$ and $\varphi(b)$.*

Proof. Let $c \in \mathcal{A}$ be given such that $\varphi(c)$ is self-adjoint with $\varphi(c) \geq \varphi(a)$ and $\varphi(c) \geq \varphi(b)$. Now define $d := \mathrm{Re}(c)$, so that we have $d \in \mathcal{A}^{\mathrm{sa}}$ and

$$\varphi(d) \;=\; \varphi(\tfrac{1}{2}c + \tfrac{1}{2}c^*) \;=\; \tfrac{1}{2}\varphi(c) + \tfrac{1}{2}\varphi(c)^* \;=\; \varphi(c).$$

Furthermore, define $e := d + (d-a)^- + (d-b)^-$. Then, by Proposition 5.7(f) we have

$$\begin{aligned}\varphi(e) \;&=\; \varphi(d) + \big(\varphi(d) - \varphi(a)\big)^- + \big(\varphi(d) - \varphi(b)\big)^- \\ &=\; \varphi(c) + \big(\varphi(c) - \varphi(a)\big)^- + \big(\varphi(c) - \varphi(b)\big)^- \\ &=\; \varphi(c) + 0 + 0 \;=\; \varphi(c),\end{aligned}$$

as well as

$$\begin{aligned}e - a \;&=\; d - a + (d-a)^- + (d-b)^- \;=\; (d-a)^+ + (d-b)^- \;\geq\; 0; \\ e - b \;&=\; d - b + (d-a)^- + (d-b)^- \;=\; (d-a)^- + (d-b)^+ \;\geq\; 0.\end{aligned}$$

Therefore we have $e \geq a \curlyvee b$. Since φ is positive and preserves the quasi-lattice operations, we find $\varphi(c) = \varphi(e) \geq \varphi(a \curlyvee b) = \varphi(a) \curlyvee \varphi(b)$. ■

8.2 The anti-lattice theorem for strongly dense C^*-subalgebras

In this section, Kadison's anti-lattice theorem is extended to strongly dense C^*-subalgebras of $B(\mathcal{H})$.

Before we proceed, we establish a bit of notation. If $\mathcal{A}$ is unital, $a \in \mathcal{A}$ is self-adjoint, and $f : \mathbb{R} \to \mathbb{C}$ is continuous, then we write $f(a)$ for the element $\varphi(f)$, where $\varphi : C(\sigma(a)) \to \mathcal{A}$ is the continuous functional calculus at a. This can be thought of as a generalisation of polynomial expressions in a. Indeed, the two notions coincide if p happens to be a polynomial. (In general, the functional calculus boils down to using polynomials to uniformly approximate a general continuous function applied to a.)

Remark 8.4. If $\mathcal{A}$ is unital, then the positive square root $a^{1/2}$ of a positive element and the absolute value $|a|$ of a self-adjoint element $a \in \mathcal{A}$ are special cases of the functional calculus. However, the functional calculus requires $\mathcal{A}$ to be unital, while $a^{1/2}$ and $|a|$ also exist in non-unital C^*-algebras. More generally, in non-unital C^*-algebras we have a functional calculus precisely for those continuous functions $f : \mathbb{R} \to \mathbb{C}$ satisfying $f(0) = 0$. However, we will not need that, since we will do most of our work in $B(\mathcal{H})$.

We recall the following topology from the theory of C^*-algebras (and von Neumann algebras).

Definition 8.5 (cf. [Mur90, page 113])**.** For $x \in \mathcal{H}$ we define the seminorm $p_x : B(\mathcal{H}) \to \mathbb{R}_{\geq 0}$ by $a \mapsto \|a(x)\|$. The Hausdorff locally convex topology on $B(\mathcal{H})$ generated by the family of seminorms $\{p_x : x \in \mathcal{H}\}$ is called the *strong operator topology* (SOT). Thus, a net $\{a_\lambda\}_{\lambda\in\Lambda}$ in $B(\mathcal{H})$ converges *strongly* to $a \in B(\mathcal{H})$ if and only if $a(x) = \lim_{\lambda\in\Lambda} a_\lambda(x)$ holds for all $x \in \mathcal{H}$.

Proposition 8.6. *The positive cone $B(\mathcal{H})^+$ is strongly closed.*

Proof. Let $\{a_\lambda\}_{\lambda\in\Lambda}$ be a net in $B(\mathcal{H})^+$ that converges strongly to some $a \in B(\mathcal{H})$. For fixed $x \in \mathcal{H}$ we have $\lim\limits_{\lambda\in\Lambda} a_\lambda x = ax$, hence

$$\langle ax, x\rangle = \left\langle \lim_{\lambda\in\Lambda} a_\lambda x, x \right\rangle = \lim_{\lambda\in\Lambda} \langle a_\lambda x, x\rangle.$$

We see that $\langle ax, x\rangle$ is the limit of a net in $\mathbb{R}_{\geq 0}$. Since $\mathbb{R}_{\geq 0}$ is closed, it follows that $\langle ax, x\rangle \in \mathbb{R}_{\geq 0}$ holds as well. This holds for all $x \in \mathcal{H}$, so we have $a \geq 0$. ■

Proposition 8.7. *The set $B(\mathcal{H})^{\mathrm{sa}}$ of self-adjoint operators is strongly closed.*

Proof. Analogous to Proposition 8.6. ■

We furthermore recall the Kaplansky density theorem. To formulate this, we use on the following notion.

Definition 8.8. A continuous function $f : \mathbb{R} \to \mathbb{C}$ is said to be *strongly continuous* if for every Hilbert space $\mathcal{H}$ and for every net $\{a_\lambda\}_{\lambda\in\Lambda}$ of self-adjoint operators on $\mathcal{H}$ converging strongly to $a \in B(\mathcal{H})$ we also have $\lim_{\lambda\in\Lambda} f(a_\lambda) = a$ in the strong operator topology.

In the special case where $\{a_\lambda\}_{\lambda\in\Lambda}$ is a sequence, the situation is illustrated in the following figure.

$$\begin{tikzcd}
 & f\big|_{\sigma(a_1)} \in C(\sigma(a_1)) \arrow[r, "\text{functional}", "\text{calculus}"'] & f(a_1) \in B(\mathcal{H}) \\
 & f\big|_{\sigma(a_2)} \in C(\sigma(a_2)) \arrow[r] & f(a_2) \in B(\mathcal{H}) \\
f \in C(\mathbb{R}) \arrow[uur, bend left] \arrow[ur, bend left] \arrow[r] \arrow[ddr, bend right] & f\big|_{\sigma(a_3)} \in C(\sigma(a_3)) \arrow[r] & f(a_3) \in B(\mathcal{H}) \arrow[dd, "\text{SOT}"', "\text{convergence}"'] \\
 & \vdots & \\
 & f\big|_{\sigma(a)} \in C(\sigma(a)) \arrow[r] & f(a) \in B(\mathcal{H})
\end{tikzcd}$$

Theorem 8.9 (Kaplansky)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a bounded continuous function. Then f is strongly continuous.*

Proof. See [Mur90, Theorem 4.3.2]. ■

Corollary 8.10. *Let $f : \mathbb{R} \to \mathbb{C}$ be continuous, but not necessarily bounded, and let $\mathcal{H}$ be a Hilbert space. If $\{a_\lambda\}_{\lambda\in\Lambda}$ is a norm-bounded net of self-adjoint operators converging strongly to a, then $\{f(a_\lambda)\}_{\lambda\in\Lambda}$ converges strongly to $f(a)$.*

Proof. Since the net $\{a_\lambda\}_{\lambda\in\Lambda}$ is norm-bounded, there is some $M \in \mathbb{R}_{>0}$ with

$$\|a\| \leq M \quad \text{and} \quad \|a_\lambda\| \leq M, \quad \text{for all } \lambda \in \Lambda.$$

Therefore we have $\sigma(a) \subseteq [-M, M]$ and $\sigma(a_\lambda) \subseteq [-M, M]$ for all $\lambda \in \Lambda$. Thus, the value of f outside the compact interval $[-M, M]$ is irrelevant. Define $g : \mathbb{R} \to \mathbb{C}$ by

$$g(x) = \begin{cases} f(x), & \text{if } x \in [-M, M]; \\ f(-M), & \text{if } x < -M; \\ f(M), & \text{if } x > M. \end{cases}$$

Now g is a bounded continuous function and we have $g(a) = f(a)$ as well as $g(a_\lambda) = f(a_\lambda)$ for all $\lambda \in \Lambda$. The result now follows from Theorem 8.9. ■

Theorem 8.11 (Kaplansky's density theorem)**.** *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a C^*-subalgebra with strong closure $\mathcal{B}$. Then the following results hold:*

(a) $\mathcal{B}$ is a C^*-subalgebra of $B(\mathcal{H})$,

(b) $\mathcal{A}^{\mathrm{sa}}$ is strongly dense in $\mathcal{B}^{\mathrm{sa}}$,

(c) $\mathrm{ball}(\mathcal{A}^{\mathrm{sa}})$ is strongly dense in $\mathrm{ball}(\mathcal{B}^{\mathrm{sa}})$,

(d) $\mathrm{ball}(\mathcal{A})$ is strongly dense in $\mathrm{ball}(\mathcal{B})$.

(e) $\mathrm{ball}(\mathcal{A}^+)$ is strongly dense in $\mathrm{ball}(\mathcal{B}^+)$,

For (a)–(d), see [Mur90, Theorem 4.3.3]. For (e), see [KR97a, Corollary 5.3.6].

We now have all the necessary tools to get to business. The main result of this section is contained in the following theorem. We use Kaplansky's density theorem in order to extend Kadison's anti-lattice theorem to strongly dense C^*-subalgebras of $B(\mathcal{H})$. Informally, this is because the quasi-lattice operations are strongly continuous on bounded sets, by Corollary 8.10.

Theorem 8.12. *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a C^*-subalgebra with strong closure $\mathcal{B}$. If $a, b \in \mathcal{A}^{\mathrm{sa}}$ have a supremum relative to $\mathcal{A}$, then this is also the supremum for a and b relative to $\mathcal{B}$.*

Proof. Recall from Corollary 8.2 that the supremum of a and b relative to $\mathcal{A}$ can only be $a \curlyvee b$. Thus, our goal is to show that $a \curlyvee b$ remains the supremum relative to the larger C^*-subalgebra $\mathcal{B}$.

Since we have $\mathcal{B}^+ = \mathcal{B} \cap B(\mathcal{H})^+$, it follows from Proposition 8.6 that $\mathcal{B}^+$ is strongly closed. The vector space operations are strongly continuous, so for every $e \in \mathcal{B}^{\mathrm{sa}}$ the translate $\mathcal{B}^+ + e$ is strongly closed as well.

Suppose now that $c \in \mathcal{B}^{\mathrm{sa}}$ is any upper bound for a and b, that is, we have $c \in (\mathcal{B}^+ + a) \cap (\mathcal{B}^+ + b)$. By Theorem 8.11(e) (Kaplansky's density theorem), we can approximate c in the strong operator topology by a norm-bounded net $\{a_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{A}^+ + a$, as well as by a norm-bounded net $\{b_\mu\}_{\mu\in M}$ in $\mathcal{A}^+ + b$. Since the vector space operations are strongly continuous, we see that $\{a_\lambda - b_\mu\}_{(\lambda,\mu)\in\Lambda\times M}$ is a norm-bounded net of self-adjoint operators converging strongly to 0. Consequently, by Corollary 8.10, the net $\{|a_\lambda - b_\mu|\}_{(\lambda,\mu)\in\Lambda\times M}$ converges strongly to 0 as well. Therefore we have

$$\lim_{(\lambda,\mu)\in\Lambda\times M} a_\lambda \curlyvee b_\mu = \lim_{(\lambda,\mu)\in\Lambda\times M} \tfrac{1}{2}\big(a_\lambda + b_\mu + |a_\lambda - b_\mu|\big) = \tfrac{1}{2}(c + c + 0) = c.$$

For all $(\lambda, \mu) \in \Lambda \times M$ we have $a_\lambda \curlyvee b_\mu \in \mathcal{A}^{\mathrm{sa}}$ as well as

$$a_\lambda \curlyvee b_\mu \geq a_\lambda \geq a \qquad \text{and} \qquad a_\lambda \curlyvee b_\mu \geq b_\mu \geq b.$$

It follows that $a_\lambda \curlyvee b_\mu \geq a \curlyvee b$ holds for all $(\lambda, \mu) \in \Lambda \times M$ (because $a \curlyvee b$ is the supremum of a and b relative to $\mathcal{A}$).

We see that c can be approximated in the strong operator topology by a net of elements in $\mathcal{A}^+ + (a \curlyvee b) \subseteq \mathcal{B}^+ + (a \curlyvee b)$. Since $\mathcal{B}^+ + (a \curlyvee b)$ is strongly closed, we have $c \in \mathcal{B}^+ + (a \curlyvee b)$, or equivalently: $c \geq a \curlyvee b$. This proves that $a \curlyvee b$ is the supremum for a and b relative to $\mathcal{B}$ as well. ■

Corollary 8.13. *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a strongly dense C^*-subalgebra. Then $\mathcal{A}^{\mathrm{sa}}$ is an anti-lattice.*

Proof. Suppose that $a, b \in \mathcal{A}^{\mathrm{sa}}$ have a supremum relative to $\mathcal{A}$, then this is also a supremum relative to $B(\mathcal{H})$ by Theorem 8.12. It follows from Kadison's anti-lattice theorem (Theorem 6.13) that a and b must be comparable. ■

8.3 Sherman's theorem

We are now in the position to prove Sherman's theorem, as well as a stronger theorem concerning suprema in arbitrary C^*-algebras. Before we do this, recall that the *universal atomic representation* of a C^*-algebra $\mathcal{A}$ is the direct sum of the GNS representations $\varphi_\tau : \mathcal{A} \to B(\mathcal{H}_\tau)$, where the sum is taken over all *pure* states τ on $\mathcal{A}$. The universal atomic representation is faithful. (See [Tak79, Definition 6.35], or [KR97b, Proposition 10.3.10].)

Recall that faithful representations are bipositive (cf. Proposition 5.7(h)), so for arbitrary $a, b \in \mathcal{A}^{\text{sa}}$ we have $a \leq b$ if and only if $\varphi(a) \leq \varphi(b)$ holds for every irreducible representation $\varphi : \mathcal{A} \to B(\mathcal{H})$. This leads us to the following characterisation of suprema and infima in a C^*-algebra.

Theorem 8.14. *Let $\mathcal{A}$ be a C^*-algebra. Then $a, b \in \mathcal{A}^{\text{sa}}$ have a supremum (or infimum) if and only if $\varphi(a)$ and $\varphi(b)$ are comparable for every irreducible representation $\varphi : \mathcal{A} \to B(\mathcal{H})$.*

Proof. First suppose that a and b have a supremum, and let $\varphi : \mathcal{A} \to B(\mathcal{H})$ be an irreducible representation. By Lemma 8.3, $\varphi(a)$ and $\varphi(b)$ have a supremum relative to the C^*-subalgebra $\varphi[\mathcal{A}] \subseteq B(\mathcal{H})$. But $\varphi[A]$ is strongly dense in $B(\mathcal{H})$, since the representation is irreducible, so it follows from Corollary 8.13 that $\varphi(a)$ and $\varphi(b)$ must be comparable.

Conversely, suppose that $\varphi(a)$ and $\varphi(b)$ are comparable for every irreducible representation. For a fixed irreducible representation $\varphi : \mathcal{A} \to B(\mathcal{H})$, we know that $\varphi(a)$ and $\varphi(b)$ are comparable, so clearly $\varphi(a) \curlyvee \varphi(b)$ is equal to the larger of the two. As such, $\varphi(a) \curlyvee \varphi(b)$ is the supremum of $\varphi(a)$ and $\varphi(b)$ relative to $\varphi[\mathcal{A}]$ (and even to $B(\mathcal{H})$, but we don't need that). Now, for any upper bound $c \geq a, b$ we have $\varphi(c) \geq \varphi(a)$ and $\varphi(c) \geq \varphi(b)$, hence $\varphi(c) \geq \varphi(a) \curlyvee \varphi(b) = \varphi(a \curlyvee b)$. Note that this holds for every irreducible representation $\varphi : \mathcal{A} \to B(\mathcal{H})$. Thus, passing to the universal atomic representation, we find that $c \geq a \curlyvee b$ must hold, showing that $a \curlyvee b$ is the supremum of a and b. ■

The main theorem of this chapter follows easily.

Theorem 8.15 (Sherman)**.** *Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}^{\text{sa}}$ is a lattice if and only if $\mathcal{A}$ is commutative.*

Proof. If $\mathcal{A}$ is commutative, then it is clear from the Gelfand representation that $\mathcal{A}^{\text{sa}}$ is a lattice.

Conversely, suppose that $\mathcal{A}^{\text{sa}}$ is a lattice, and let $\varphi : \mathcal{A} \to B(\mathcal{H})$ be an arbitrary irreducible representation. We show that $\mathcal{H}$ must be one-dimensional. Let $c = \varphi(b) \in \varphi[\mathcal{A}]^{\text{sa}}$ be given, then we may choose $a \in A^{\text{sa}}$ such that $\varphi(a) = c$ holds (set $a := \text{Re}(b)$). By assumption, 0 and a have a supremum in $\mathcal{A}^{\text{sa}}$, so it follows from Theorem 8.14 that 0 and c are comparable in $B(\mathcal{H})$. We see that every element of $\varphi[\mathcal{A}]^{\text{sa}}$ is either positive or negative:

$$\varphi[A]^{\text{sa}} \subseteq B(\mathcal{H})^+ \cup (-B(\mathcal{H})^+).$$

Since $B(\mathcal{H})^+ \cup (-B(\mathcal{H})^+)$ and $B(\mathcal{H})^{\text{sa}}$ are strongly closed (by propositions 8.6 and 8.7), it follows from Theorem 8.11(b) (Kaplansky's density theorem) that

$$B(\mathcal{H})^{\text{sa}} = \overline{\varphi[\mathcal{A}]^{\text{sa}}}^{s} \subseteq B(\mathcal{H})^+ \cup (-B(\mathcal{H})^+).$$

We see that every self-adjoint operator on $\mathcal{H}$ is either positive or negative. It follows that $\mathcal{H}$ is one-dimensional.

We have that every irreducible representation is one-dimensional, so it is clear from the universal atomic representation that $\mathcal{A}$ is commutative. ■

Note that it is easy to come up with examples of C^*-algebras where some pairs of self-adjoint elements have a supremum and others don't. For instance, consider the algebra $\mathcal{A} := B(\mathbb{C}^2) \oplus B(\mathbb{C}^2)$ of 4×4 complex matrices in block diagonal form

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \qquad (a_1, a_2 \in B(\mathbb{C}^2)).$$

Then two self-adjoint operators $a = a_1 \oplus a_2$ and $b = b_1 \oplus b_2$ have a supremum if and only if a_i and b_i are comparable for $i \in \{1, 2\}$. Now, for instance, the operators $I \oplus 0$ and $0 \oplus I$ have a supremum, despite being incomparable.

8.4 Ogasawara's theorem

Using the general strategy taken in this chapter, we can prove various other commutativity theorems with little extra effort. As an example we show how the techniques can be used to prove Ogasawara's theorem.

Theorem 8.16 (Ogasawara)**.** *Let $\mathcal{A}$ be a C^*-algebra with the property that $0 \le a \le b$ implies $0 \le a^2 \le b^2$. Then $\mathcal{A}$ must be commutative.*

Proof. First of all, if $\mathcal{H}$ is a (complex) Hilbert space with $\dim(\mathcal{H}) > 1$, then it is easy to come up with examples of positive operators $a, b \in B(\mathcal{H})^+$ satisfying $0 \le a \le b$ but $a^2 \not\le b^2$. For instance, a straightforward computation shows that the following choice suffices:

$$a := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad b := \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

(These are 2×2 matrices, but they can of course be extended to operators in $B(\mathcal{H})$ by choosing a 2-dimensional subspace $V \subseteq \mathcal{H}$ for a and b to act on, and setting $ax = bx = 0$ for all $x \in V^\perp$.)

Secondly, suppose that $\mathcal{B} \subseteq B(\mathcal{H})$ is a strongly dense C^*-subalgebra. Let $a, b \in B(\mathcal{H})^+$ be such that $0 \le a \le b$ holds but $a^2 \not\le b^2$. By Theorem 8.11(e) (Kaplansky's density theorem) we may choose norm-bounded nets $\{a_\lambda\}_{\lambda \in \Lambda}$ and $\{b_\mu\}_{\mu \in M}$ in $\mathcal{B}^+$ converging strongly to a and $b - a$, respectively. Then, for all $(\lambda, \mu) \in \Lambda \times M$ we have $0 \le a_\lambda \le a_\lambda + b_\mu$, and furthermore we have

$$a = \lim_{\lambda \in \Lambda} a_\lambda, \qquad \text{and} \qquad b = \lim_{(\lambda,\mu) \in \Lambda \times M} a_\lambda + b_\mu.$$

Hence, by Corollary 8.10 we have

$$a^2 = \lim_{\lambda \in \Lambda} a_\lambda^2, \qquad \text{and} \qquad b^2 = \lim_{(\lambda,\mu) \in \Lambda \times M} (a_\lambda + b_\mu)^2,$$

and therefore

$$\lim_{(\lambda,\mu) \in \Lambda \times M} (a_\lambda + b_\mu)^2 - a_\lambda^2 = b^2 - a^2 \notin B(\mathcal{H})^+.$$

Since $B(\mathcal{H})^+$ is strongly closed (by Proposition 8.6), there must be some choice of $(\lambda, \mu) \in \Lambda \times M$ such that $(a_\lambda + b_\mu)^2 - a_\lambda^2 \notin B(\mathcal{H})^+$ holds. In other words, the strongly dense C^*-subalgebra $\mathcal{B} \subseteq B(\mathcal{H})$ also contains positive operators $c, d \in \mathcal{B}^+$ satisfying $0 \le c \le d$ but $c^2 \not\le d^2$ (namely $c := a_\lambda$, $d := a_\lambda + b_\mu$).

Thirdly, suppose for the sake of contradiction that $\mathcal{A}$ admits an irreducible representation $\varphi : \mathcal{A} \to B(\mathcal{H})$ with $\dim(\mathcal{H}) > 1$. Then, by the above, there exist $a, b \in \mathcal{A}$ with $0 \leq \varphi(a) \leq \varphi(b)$ but $\varphi(a)^2 \not\leq \varphi(b)^2$. By Proposition 5.7(g), we may assume without loss of generality that $0 \leq a \leq b$ holds. But then, by assumption, we also have $a^2 \leq b^2$, hence $\varphi(a)^2 \leq \varphi(b)^2$. This is a contradiction, so we conclude that all irreducible representations of $\mathcal{A}$ are one-dimensional.

By looking at the universal atomic representation, we see that $\mathcal{A}$ must be commutative. ■

In the proof of Theorem 8.16, we used that $a \mapsto a^2$ is strongly continuous on bounded sets (as a consequence of Corollary 8.10). It is well-known that multiplication $B(\mathcal{H}) \times B(\mathcal{H}) \to B(\mathcal{H})$ is not strongly continuous if $\mathcal{H}$ is infinite-dimensional; see for instance [Mur90, Exercise 4.3]. However, it can be shown that multiplication is strongly continuous on $S \times B(\mathcal{H})$ whenever $S \subseteq B(\mathcal{H})$ is bounded; see [Mur90, Remark 4.3.1].

Finally, we point out that there is a different way to deduce Ogasawara's theorem (and many other commutativity theorems) directly from Sherman's theorem. This is the approach taken in [Top65].

A Complementary subspaces

In this appendix, we cover some well-known and some not so well-known results regarding complementary subspaces in a real or complex Hilbert space $\mathcal{H}$.

Recall that subspaces V, W of a vector space X are said to be *complementary* if one has $V \cap W = \{0\}$ and $V + W = X$. In the case where X is a Banach space (or more generally a topological vector space), it is customary to require both V and W to be *closed* subspaces (cf. [Rud91, Definition 4.20], and [Con07, Section III.13]). In this setting, a closed subspace $V \subseteq X$ is said to be *complemented* if there exists a closed subspace $W \subseteq X$ such that V and W are complementary. In general it is not true that every closed subspace of a Banach space X is complemented, but this is true if X is a Hilbert space (given a closed subspace $V \subseteq X$, set $W := V^{\perp}$). It should be noted, however, that the complement W is generally far from being unique, even in the finite-dimensional case.

This appendix serves to study complementary pairs in Hilbert spaces, their properties, and how to recognise them. The results and techniques developed here are used extensively towards the end of Chapter 6.

A.1 Orthogonal complements of sums and intersections

We briefly recall how orthogonal complements behave under sums and intersections. While the results presented here are basic exercises in undergraduate level functional analysis, they are sufficiently important to our investigation to warrant treatment in some detail.

Proposition A.1. *Let $\mathcal{H}$ be a real or complex Hilbert space, and let $V, W \subseteq \mathcal{H}$ be subspaces. Then one has $(V + W)^{\perp} = V^{\perp} \cap W^{\perp}$.*

Proof. On the one hand, since we have $V \subseteq V + W$, we find $V^{\perp} \supseteq (V + W)^{\perp}$. Analogously we have $W^{\perp} \supseteq (V + W)^{\perp}$, so we find $(V + W)^{\perp} \subseteq V^{\perp} \cap W^{\perp}$. Conversely, for all $v \in V$, $w \in W$ and $y \in V^{\perp} \cap W^{\perp}$ we have

$$\langle v + w, y\rangle \;=\; \langle v, y\rangle + \langle w, y\rangle \;=\; 0 + 0 \;=\; 0,$$

so we see that $V + W$ and $V^{\perp} \cap W^{\perp}$ are orthogonal. The reverse inclusion $V^{\perp} \cap W^{\perp} \subseteq (V + W)^{\perp}$ follows. ∎

For closed subspaces, we have the following partial converse.

Proposition A.2. *Let $\mathcal{H}$ be a real or complex Hilbert space, and let $V, W \subseteq \mathcal{H}$ be closed subspaces. Then one has $(V \cap W)^{\perp} = \overline{V^{\perp} + W^{\perp}}$.*

Proof. Set $X := V^{\perp}$ and $Y := W^{\perp}$, so that we have $V = \overline{V} = V^{\perp\perp} = X^{\perp}$, and similarly $W = Y^{\perp}$. By Proposition A.1 we have $(X + Y)^{\perp} = X^{\perp} \cap Y^{\perp}$. Taking orthogonal complements yields

$$\overline{V^{\perp} + W^{\perp}} \;=\; \overline{X + Y} \;=\; (X + Y)^{\perp\perp} \;=\; (X^{\perp} \cap Y^{\perp})^{\perp} \;=\; (V \cap W)^{\perp}. \qquad ∎$$

It should be noted that the stronger equality $(V \cap W)^\perp = V^\perp + W^\perp$ fails in general, even though V and W are closed subspaces (which we expect to be rather well-behaved). This is because the sum of two closed subspaces of a Hilbert space is not necessarily closed, as the following example shows.

Example A.3 ([Con07, Exercises II.3.9 & II.3.10])**.** Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and let $a : \mathcal{H} \to \mathcal{H}$ be an injective operator with dense range which nonetheless fails to be surjective. Define $X, Y \subseteq \mathcal{H} \oplus \mathcal{H}$ by setting $X := \operatorname{graph}(a)$ and $Y := \mathcal{H} \oplus \{0\}$. It is routinely verified that X and Y are closed subspaces with $X \cap Y = \{0\}$, and that $X + Y$ is dense in $\mathcal{H} \oplus \mathcal{H}$ with $X + Y \neq \mathcal{H} \oplus \mathcal{H}$. Consequently, if we set $V := X^\perp$ and $W := Y^\perp$, then we have

$$(V \cap W)^\perp = \overline{V^\perp + W^\perp} = \overline{X + Y} = \mathcal{H} \oplus \mathcal{H},$$

whereas $V^\perp + W^\perp = X + Y$ fails to be closed. ⊠

The main result of this appendix is that the closure in Proposition A.2 is superfluous if (and only if) $V + W$ is closed.

A.2 Complementary subspaces and idempotents

We now turn our attention to complementary subspaces. Recall from before that two subspaces V and W of a Hilbert space $\mathcal{H}$ are *complementary* if V and W are closed, $V \cap W = \{0\}$, and $V + W = \mathcal{H}$.

If $V, W \subseteq \mathcal{H}$ are complementary subspaces, then every $x \in \mathcal{H}$ can be uniquely written as $x = x_v + x_w$ with $x_v \in V$ and $x_w \in W$. It is routinely verified that the maps $x \mapsto x_v$ and $x \mapsto x_w$ are linear. It turns out that these maps give rise to an equivalent algebraic characterisation of complementary subspaces. We state and prove the following theorem for Hilbert spaces, but the result can be generalised to Banach spaces ([Con07, Theorem III.13.2]), or even Fréchet spaces ([Rud91, Theorem 5.16]), with identical proof.

Theorem A.4. *Let $\mathcal{H}$ be a real or complex Hilbert space.*

(a) If $V, W \subseteq \mathcal{H}$ are complementary subspaces and $e : \mathcal{H} \to \mathcal{H}$ is defined by $e(v + w) := v$ for all $v \in V$ and all $w \in W$, then e is a continuous linear operator such that $e^2 = e$, $\operatorname{ran}(e) = V$, and $\ker(e) = W$.

(b) If $e \in B(\mathcal{H})$ and $e^2 = e$, then $\operatorname{ran}(e)$ and $\ker(e)$ are complementary subspaces of $\mathcal{H}$.

Proof.

(a) Clearly we have $e^2 = e$, $\operatorname{ran}(e) = V$ and $\ker(e) = W$. We show that e is continuous, using the closed graph theorem. To that end, suppose that $\{(v_n + w_n) \oplus v_n\}_{n=1}^\infty$ is a sequence in $\operatorname{graph}(e) \subseteq \mathcal{H} \oplus \mathcal{H}$ which converges to some $x \oplus y \in \mathcal{H} \oplus \mathcal{H}$. Projecting onto the first and second coordinates, we find

$$\lim_{n\to\infty} v_n + w_n = x \qquad \text{and} \qquad \lim_{n\to\infty} v_n = y.$$

Consequently, we also have

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} (w_n + v_n) - v_n = x - y.$$

Since V and W are closed, we have $y \in V$ and $x - y \in W$, so we find

$$e(x) = e\big(y + x - y\big) = y,$$

showing that $x \oplus y \in \operatorname{graph}(e)$ holds. Indeed, the graph of e is closed, so it follows from the closed graph theorem that e is continuous.

(b) For convenience, let us write $V := \operatorname{ran}(e)$ and $W := \ker(e)$. Note that we have $\ker(1 - e) = V$:

- Let $y \in \operatorname{ran}(e)$ be given. Then we may choose $x \in \mathcal{H}$ with $ex = y$, so that we have $ey = e^2 x = ex = y$, hence $(1 - e)y = 0$.
- For $y \notin \operatorname{ran}(e)$, we have $ey \in \operatorname{ran}(e)$, so in particular $ey \neq y$. In this case we find $(1 - e)y \neq 0$.

Note furthermore that we have $(1 - e)^2 = 1 - 2e + e^2 = 1 - e$, so the preceding proof also shows that we have $W = \operatorname{ran}(1 - e)$.

Since e and $1 - e$ are continuous, clearly the subspaces $V = \ker(1 - e)$ and $W = \ker(e)$ are closed.

For $x \in V \cap W$ we have $ex = 0$ as well as $(1 - e)x = 0$, hence $1x = 0$. This shows that we have $V \cap W = \{0\}$.

Finally, note that any $x \in \mathcal{H}$ can be written as $x = ex + (1 - e)x$, where we have $ex \in \operatorname{ran}(e) = V$ and $(1 - e)x \in \operatorname{ran}(1 - e) = W$. This shows that V and W are complementary subspaces of $\mathcal{H}$. ■

In abstract algebra, an element e of a ring (or algebra) satisfying $e^2 = e$ is called an *idempotent*. In operator theory, the term *projection* is sometimes used, though usually projections are also required to be self-adjoint. It can be shown that an idempotent $e \in B(\mathcal{H})$ is self-adjoint if and only if $\ker(e)$ and $\operatorname{ran}(e)$ are orthogonal; cf. [Con07, Proposition II.3.3].

The preceding theorem shows that complementary pairs (V, W) of subspaces of $\mathcal{H}$ are in bijective correspondence with idempotents of $B(\mathcal{H})$. Furthermore, the proof reveals that interchanging the spaces V and W corresponds with passing from e to $1 - e$, which is again an idempotent.

A.3 Orthogonal complements of complementary pairs

Using the characterisation of complementary subspaces in terms of idempotents (Theorem A.4), it is relatively easy to show that taking orthogonal complements preserves complementary pairs.

Theorem A.5. *Let $\mathcal{H}$ be a real or complex Hilbert space, and let $V, W \subseteq \mathcal{H}$ be complementary subspaces. Then $V^\perp$ and $W^\perp$ are complementary as well.*

Proof. Let $e \in B(\mathcal{H})$ be the idempotent corresponding with the pair (V, W), that is: $e(v + w) = v$. Now consider its adjoint $e^* \in B(\mathcal{H})$. Note that we have $(e^*)^2 = (e^2)^* = e^*$, so e^* is an idempotent as well. Furthermore, we have

$$\begin{aligned} \ker(e^*) &= \operatorname{ran}(e)^\perp = V^\perp; \\ \operatorname{ran}(e^*) &= \ker(1 - e^*) = \operatorname{ran}(1 - e)^\perp = W^\perp. \end{aligned}$$

This shows that $V^\perp$ and $W^\perp$ are complementary. ■

This result can be generalised to closed subspaces $V, W \subseteq \mathcal{H}$ with the property that $V+W$ is closed, by passing to the appropriate quotient space. We will prove this in some detail. In this setting the result is the following.

Theorem A.6. *Let $\mathcal{H}$ be a real or complex Hilbert space, and let $V, W \subseteq \mathcal{H}$ be closed subspaces such that $V+W$ is closed. Then $V^\perp + W^\perp$ is closed as well. Consequently, one has $(V \cap W)^\perp = V^\perp + W^\perp$.*

Before giving the proof of Theorem A.6, we recall part of the theory of quotient spaces of a Hilbert space. Let $\mathcal{H}$ be a Hilbert space, let $X \subseteq \mathcal{H}$ be a closed subspace, and let $\pi : \mathcal{H} \to \mathcal{H}/X$ denote the natural map $v \mapsto v + X$. Seeing as $\mathcal{H}/X$ is the quotient of a normed space by a closed subspace, it becomes a normed space with the quotient norm. Using basic properties of the orthogonal decomposition $\mathcal{H} = X \oplus X^\perp$, it is readily verified that π restricts to an isometric isomorphism $\varphi : X^\perp \to \mathcal{H}/X$. Concretely, this isomorphism is given by $\varphi(v) = v + X$ and $\varphi^{-1}(v + X) = pv$, where $p \in B(\mathcal{H})$ denotes the orthogonal projection onto $X^\perp$. It follows that $\mathcal{H}/X$ is a Hilbert space as well, with inner product given by $\langle v + X, w + X\rangle = \langle pv, pw\rangle$.

While parts of the proof of Theorem A.6 are perhaps easier understood in terms of quotient spaces, we choose to work within the "internal" quotients of the form $X^\perp \cong \mathcal{H}/X$, for this simplifies the notation. The following proposition casts familiar properties of the natural map π in terms of this internal quotient.

Proposition A.7. *Let $\mathcal{H}$ be a real or complex Hilbert space, $X \subseteq \mathcal{H}$ a closed subspace, $p : \mathcal{H} \to \mathcal{H}$ the orthogonal projection onto $X^\perp$, and ρ the restriction of p to a map $\mathcal{H} \to X^\perp$. Then:*

(a) The linear map $\rho : \mathcal{H} \to X^\perp$ is surjective with $\ker(\rho) = X$;

(b) For a subspace $X \subseteq Y \subseteq \mathcal{H}$ we have $\rho(Y) = Y \cap X^\perp$ and $\rho^{-1}(\rho(Y)) = Y$;

(c) For a subspace $Y' \subseteq X^\perp$ we have $\rho^{-1}(Y') = Y' + X$ and $\rho(\rho^{-1}(Y')) = Y'$;

(d) The formulas $Y \mapsto \rho(Y)$ and $Y' \mapsto \rho^{-1}(Y')$ define a bijective correspondence between subspaces $X \subseteq Y \subseteq \mathcal{H}$ and subspaces $Y' \subseteq X^\perp$;

(e) Under the aforementioned correspondence, the subspace Y is closed if and only if its corresponding subspace Y' is closed.

Proof.

(a) Clearly ρ is surjective, as it is defined of the restriction of p to a map $\mathcal{H} \to \operatorname{ran}(p)$. Furthermore, p is precisely the idempotent corresponding to the complementary pair $(X^\perp, X)$, in the sense of Theorem A.4, so we have $\ker(\rho) = \ker(p) = \operatorname{ran}(1 - p) = X$.

(b) For $v \in Y \cap X^\perp$ we have $\rho(v) = v$ (because $v \in X^\perp$), hence $v \in \rho(Y)$ (because $v \in Y$). This proves the inclusion $Y \cap X^\perp \subseteq \rho(Y)$.

Conversely, let $y \in Y$ be given. Write y as its orthogonal decomposition with respect to $(X^\perp, X)$, that is, write $y = v + w$ with $v \in X^\perp$ and $w \in X$. Note that we have $v = \rho(y)$. Furthermore, since we assumed $X \subseteq Y$, we have $\rho(y) = v = y - w \in Y$, so we find $\rho(y) \in Y \cap X^\perp$. This proves the reverse inclusion $\rho(Y) \subseteq Y \cap X^\perp$.

Finally, we have $Y \subseteq \rho^{-1}(\rho(Y))$ by elementary set theory. For the reverse inclusion, let $v \in \rho^{-1}(\rho(Y))$ be given, and choose some $y \in Y$ such that $\rho(v) = \rho(y)$ holds. Then we have $v - y \in \ker(\rho) = X \subseteq Y$, hence $v = (v - y) + y \in Y$. This proves the reverse inclusion $\rho^{-1}(\rho(Y)) \subseteq Y$.

(c) Let $y \in Y'$ be given, then we have $\rho(y) = y$. Consequently, if $v \in \mathcal{H}$ is such that $\rho(v) = y$ holds, then we have $v - y \in \ker(\rho) = X$, hence $v \in Y' + X$. This proves the inclusion $\rho^{-1}(Y') \subseteq Y' + X$.

Conversely, for $v = y+x \in Y'+X$ we have $\rho(v) = \rho(y)+\rho(x) = y+0 \in Y'$, proving the reverse inclusion $Y' + X \subseteq \rho^{-1}(Y')$.

Finally, we have $\rho(\rho^{-1}(Y')) \subseteq Y'$ by elementary set theory, with equality because ρ is surjective.

(d) This is immediate, for we have $\rho^{-1}(\rho(Y)) = Y$ and $\rho(\rho^{-1}(Y')) = Y'$.

(e) If Y is closed subspace containing X, then $\rho(Y) = Y \cap X^\perp$ is closed as well. Conversely, if Y' is closed, then $\rho^{-1}(Y')$ is closed as well, because it is the inverse image of a closed set under a continuous map. ■

It should be noted that the equality $\rho(S) = S \cap X^\perp$ does not hold for arbitrary subsets $S \subseteq \mathcal{H}$, or even arbitrary subspaces (which we no longer require to contain X). Furthermore, ρ is typically not a closed mapping. One can show, however, that ρ is an open mapping, either by invoking the open mapping theorem, or with a much more elementary proof using the fact that the quotient map π is an open mapping; cf. [Rud91, Theorem 1.41(a)].

We deduce the following well-known fact as a corollary of Proposition A.7.

Corollary A.8. *Let $\mathcal{H}$ be a real or complex Hilbert space, and let $V, W \subseteq \mathcal{H}$ be orthogonal closed subspaces. Then $V + W$ is closed as well.*

Proof. Let $p \in B(\mathcal{H})$ denote the orthogonal projection onto $V^\perp$, and let ρ denote its restriction to a map $\mathcal{H} \to V^\perp$. Since V and W are orthogonal, we have $W \subseteq V^\perp$. It follows from parts (c) and (e) of Proposition A.7 that $\rho^{-1}(W) = V + W$ is closed. ■

Now that we know how to deal with quotients, we briefly look at quotients of subspaces. Suppose that $X, Z \subseteq \mathcal{H}$ are closed subspaces with $X \subseteq Z$. In this setting, Z is a Hilbert space in its own right, so the quotient Z/X is naturally isomorphic with the orthogonal complement of X in Z, that is, with $Z \cap X^\perp$. The crucial observation is this: because we have $X \subseteq Z$, we also have $Z^\perp \subseteq X^\perp$, and by the same argument the quotient $X^\perp/Z^\perp$ is naturally isomorphic with $X^\perp \cap (Z^\perp)^\perp = X^\perp \cap Z$: the same space as before! As such, we get a natural isomorphism $Z/X \cong Z \cap X^\perp \cong X^\perp/Z^\perp$, which we use to prove Theorem A.6. Before we do so, we must find a way to describe orthogonal complements in either of these spaces.

Proposition A.9. *Let $\mathcal{H}$ be a real or complex Hilbert space and let $X \subseteq Y \subseteq Z$ be closed subspaces of $\mathcal{H}$. Let $\phi : Z/X \to Z \cap X^\perp$ and $\psi : X^\perp/Z^\perp \to Z \cap X^\perp$ denote the natural isomorphisms. Then we have $\phi(Y/X)^\perp = \psi(Y^\perp/Z^\perp)$.*

Note that the orthogonal complement $\phi(Y/X)^\perp$ is taken inside $Z \cap X^\perp$. In the larger space $\mathcal{H}$ the statement would be $\phi(Y/X)^\perp \cap Z \cap X^\perp = \psi(Y^\perp/Z^\perp)$.

Proof of Proposition A.9. For convenience, let us write $V := \phi(Y/X)$ and $W := \psi(Y^\perp/Z^\perp)$. Furthermore, let $\rho : Z \to Z \cap X^\perp$ and $\sigma : X^\perp \to Z \cap X^\perp$ denote the internal quotient maps (as in Proposition A.7), so that the following diagram commutes.

$$\begin{tikzcd}
& Z \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & & X^\perp \arrow[dl, "\sigma"'] \arrow[dr, "\pi'"] & \\
Z/X \arrow[rr, "\phi"'] & & Z \cap X^\perp & & X^\perp/Z^\perp \arrow[ll, "\psi"]
\end{tikzcd}$$

Now we have $V = \rho(Y)$ and $W = \sigma(Y^\perp)$. By Proposition A.7(b) we have $V = Y \cap X^\perp$ and $W = Y^\perp \cap Z$.

Since we have $X \subseteq Y$, we see that X and $Y^\perp$ are orthogonal, so we have $\overline{Y^\perp + X} = Y^\perp + X$ by Corollary A.8. Hence the orthogonal complement of V inside $Z \cap X^\perp$ is given by

$$\begin{aligned}
V^\perp \cap Z \cap X^\perp &= (Y \cap X^\perp)^\perp \cap X^\perp \cap Z \\
&= \overline{Y^\perp + X} \cap X^\perp \cap Z \\
&= (Y^\perp + X) \cap X^\perp \cap Z \\
&= \rho(Y^\perp + X) \cap Z \\
&= \rho(\rho^{-1}(Y^\perp)) \cap Z \\
&= Y^\perp \cap Z,
\end{aligned}$$

using parts (b) and (c) of Proposition A.7. (In order to see that the equality $\rho(Y^\perp + X) = (Y^\perp + X) \cap X^\perp$ holds, note that we have $X \subseteq Y^\perp + X$.) ■

At this time we are ready to prove the main theorem.

Proof of Theorem A.6. Let $V, W \subseteq \mathcal{H}$ be closed subspaces such that $V + W$ is closed. Set $X := V \cap W$ and $Z := V + W$, and let $\rho : Z \to Z \cap X^\perp$ and $\sigma : X^\perp \to Z \cap X^\perp$ be as in the proof of Proposition A.9. The situation is described by the following commutative diagram, an adaptation of the diagram in the proof of Proposition A.9 to the current situation.

$$\begin{tikzcd}
& V + W \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & & (V \cap W)^\perp \arrow[dl, "\sigma"'] \arrow[dr, "\pi'"] & \\
(V+W)/(V \cap W) \arrow[rr, "\phi"'] & & (V+W) \cap (V \cap W)^\perp & & (V \cap W)^\perp/(V+W)^\perp \arrow[ll, "\psi"]
\end{tikzcd}$$

Now we have $\rho(V) = V \cap (V \cap W)^\perp$ and $\rho(W) = W \cap (V \cap W)^\perp$. Note that these are complementary subspaces of $(V + W) \cap (V \cap W)^\perp$, for we have

$$\begin{aligned}
\rho(V) \cap \rho(W) &= V \cap W \cap (V \cap W)^\perp = \{0\}; \\
\rho(V) + \rho(W) &= \rho(V + W) = (V + W) \cap (V \cap W)^\perp.
\end{aligned}$$

Consequently, it follows from Theorem A.5 that the orthogonal complements of $\rho(V)$ and $\rho(W)$ inside $(V+W)\cap(V\cap W)^\perp$ form a complementary pair as well. We know from (the proof of) Proposition A.9 that these relative orthogonal complements are given by $\rho(V)^\perp = \sigma(V^\perp)$ and $\rho(W)^\perp = \sigma(W^\perp)$. But now we have

$$\begin{aligned} V^\perp + W^\perp &= \sigma^{-1}(\sigma(V^\perp + W^\perp)) \\ &= \sigma^{-1}(\sigma(V^\perp) + \sigma(W^\perp)) \\ &= \sigma^{-1}(Z \cap X^\perp) \\ &= X^\perp \;=\; (V \cap W)^\perp. \end{aligned}$$

(To see that we may apply Proposition A.7(b) in the first step, we must again be careful to check that we have $\ker(\sigma) \subseteq V^\perp + W^\perp$, but this is the case since we have $\ker(\sigma) = (V+W)^\perp = V^\perp \cap W^\perp$.)

In particular, we conclude that $V^\perp + W^\perp$ is closed. ■

Due to the self-dual nature of orthogonal complements, we immediately get the following corollary.

Corollary A.10. *Let $\mathcal{H}$ be a real or complex Hilbert space and let $V, W \subseteq \mathcal{H}$ be closed subspaces. Then $V + W$ is closed if and only if $V^\perp + W^\perp$ is closed. If this is the case, then one has $(V \cap W)^\perp = V^\perp + W^\perp$.*

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