Asymmetric graphs with quantum symmetry

Constructing a graph $X$ with $\text{Aut}(X) = 1$ and $\text{Qut}(X) \neq 1$
Quantum groups

Discovered from several points of view:

- Deformations of Lie groups/algebras;
- A superclass of groups that allows Pontryagin duality for non-abelian locally compact groups;
- Quantum symmetry objects of spaces in non-commutative geometry.

We focus on the latter.

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Compact quantum groups

Proposition

Let $G$ be a compact group. Then the group structure of $G$ can be recovered from the "comultiplication" $\Delta : C(G) \to C(G \times G)$ given by $\Delta(f)(g, h) = f(gh)$.

Definition (Woronowicz, 1987)

A compact quantum group is a tuple $(\mathcal{A}, \Delta)$ where $\mathcal{A}$ is a unital $C^*$-algebra and $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a unital $\ast$-homomorphism such that

- $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ ("coassociativity");
- $(\mathcal{A} \otimes 1)\Delta(\mathcal{A})$ and $(1 \otimes \mathcal{A})\Delta(\mathcal{A})$ are dense in $\mathcal{A} \otimes \mathcal{A}$ ("cancellation").

Remark: no "coinverse" is assumed – it is often unbounded.

Proposition

If $\mathcal{A}$ is commutative, then $(\mathcal{A}, \Delta)$ comes from a compact group.
Examples of compact quantum groups

Example (The dual of a discrete group)

Let $G$ be a discrete group. Then the (quantum group) dual of $G$ is the compact quantum group $\hat{G} := (\mathcal{A}, \Delta)$, where

$$\mathcal{A} := C^*(G) = C^*(u_g, g \in G \mid u_g u_h = u_{gh}, u_{g^{-1}} = u_g^*),$$

$$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \quad \Delta(u_g) = u_g \otimes u_g.$$

For abelian $G$, this is just the Pontryagin dual, viewed as quantum group.

Example (Compact matrix quantum group)

Let $\mathcal{A}$ be a unital $C^*$-algebra generated by $(u_{ij})_{i,j \in [n]}$ such that:

- The matrices $(u_{ij})_{i,j}$ and $(u_{ij}^*)_{i,j}$ are invertible in $M_n(\mathcal{A})$;
- $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$ defines a unital $*$-homomorphism $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$.

Then $(\mathcal{A}, \Delta)$ is a compact quantum group.
The quantum automorphism group of a space

Definition (Wang, 1998)

The quantum automorphism group of a (non-)commutative space $X$ is the terminal object in the category of quantum groups coacting on $X$:

$$\exists! \text{ morphism} \quad (\mathcal{A}, \Delta) \xrightarrow{\quad \text{coaction} \quad} \text{Qut}(X) \xleftarrow{\quad \text{coaction} \quad} X$$

If Qut($X$) exists, it is necessarily unique.

Theorem (Wang, 1998)

Qut($\{1, \ldots, n\}$) exists (= the quantum symmetric group $\mathbb{S}^+_n$). It is the compact matrix quantum group generated by $(u_{ij})_{i,j \in [n]}$ with relations

- $u_{ij} = u_{ij}^* = u_{ij}^2$ for all $i, j \in [n]$;
- $\sum_{k=1}^n u_{kj} = \sum_{k=1}^n u_{ik} = 1$ for all $i, j \in [n]$. 
Quantum permutation matrices

**Definition**

A quantum permutation matrix is a finite-dimensional representation $\mathbb{C}(\mathbb{S}_n^+) \to M_d(\mathbb{C})$. Equivalently: it is an $n \times n$ matrix with entries in $M_d(\mathbb{C})$ such that:

- every entry is an orthogonal projection;
- the rows and columns sum to $I_d$.

**Example:**

\[
\begin{pmatrix}
(1 & 0 & 0 & 0) & (0 & 0 & 0 & 0) & (0 & 0 & 0 & 0) & (0 & 0 & 0 & 0) \\
(0 & 0 & 0 & 0) & (0 & 1 & 0 & 0) & (0 & 0 & 0 & 0) & (0 & 0 & 0 & 0) \\
(0 & 1 & 0 & 0) & (0 & 0 & 0 & 0) & (0 & 0 & 0 & 0) & (0 & 0 & 0 & 0) \\
(0 & 0 & 0 & 0) & (0 & 0 & 0 & 0) & (0 & 1 & 0 & 0) & (0 & 0 & 0 & 0) \\
(0 & 0 & 0 & 0) & (0 & 0 & 0 & 0) & (0 & 0 & 0 & 0) & \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\
(0 & 0 & 0 & 0) & (0 & 0 & 0 & 0) & \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\end{pmatrix}
\]
The quantum automorphism group of a graph

Theorem (Bichon, 2003 / Banica, 2005)

Let $X$ be a finite graph. Then $\text{Qut}(X)$ exists. It is the compact matrix quantum group generated by $(u_{ij})_{i,j \in V(X)}$ with relations

- $u_{ij} = u_{ij}^* = u_{ij}^2$ for all $i, j \in V(X)$;
- $\sum_{k \in V(X)} u_{kj} = \sum_{k \in V(X)} u_{ik} = 1$ for all $i, j \in V(X)$;
- $uA_X = A_X u$.

Example:

$A_X = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}$, $u = \begin{pmatrix}
(1 & 0 & 0 & 0 & 0) \\
(0 & 0 & 1 & 0 & 0) \\
(0 & 0 & 1 & 0 & 0) \\
(0 & 0 & 1 & 0 & 0) \\
(0 & 0 & 1 & 0 & 0) \\
\end{pmatrix}$.
Connection with the graph isomorphism game

**Def. (Atserias, Mančinska, Roberson, Šámal, Severini, Varvitsiotis, 2019)**

Given graphs $W$ and $Z$, the $(W, Z)$-isomorphism game is the synchronous non-local game with input and output set $V(W) \sqcup V(Z)$ and winning predicate given by the following conditions:

- $x_A \in V(W) \iff y_A \in V(Z)$ and $x_B \in V(W) \iff y_B \in V(Z)$;
- $x_A = x_B \iff y_A = y_B$ and $y_A = x_B \iff y_B = x_A$;
- $x_A \sim x_B \iff y_A \sim y_B$ and $y_A \sim x_B \iff y_B \sim x_A$.

For $t \in \{\text{loc}, q, qt, qa, qc, ns\}$, we write $W \cong_t Z$ if and only if the $(W, Z)$-isomorphism game admits a perfect $t$-strategy.

**Theorem (Lupini, Mančinska, Roberson, 2020)**

*For connected graphs $W$ and $Z$, one has $W \cong_{qc} Z$ if and only if $\text{Qut}(W \sqcup Z)$ has a quantum orbit that intersects both $W$ and $Z$.***
Quantum vs. classical symmetry

Question (various authors)

What is the maximum disparity between $\text{Aut}(X)$ and $\text{Qut}(X)$?

Is it possible to have:

- $\text{Aut}(X)$ intransitive and $\text{Qut}(X)$ transitive?
  Answer: yes! \cite{Lupini, Mančinska, Roberson, 2020}

- $\text{Aut}(X)$ trivial and $\text{Qut}(X)$ non-trivial?
  Answer: yes! \cite{vDdB, Roberson, Schmidt, 2023}

- $\text{Aut}(X)$ trivial and $\text{Qut}(X)$ transitive?
  Answer: ???

Goal for today:

Theorem (vDdB, Roberson, Schmidt, 2023)

There exists a graph $X$ such that $\text{Aut}(X) = 1$ and $\text{Qut}(X) \neq 1$. 
Proof outline

binary linear constraint system $Mx = b$

homogeneous solution group $\Gamma_0(M)$

coloured graph $\hat{X}(M)$ with $\text{Qut}_c(\hat{X}(M)) \cong \hat{\Gamma}_0(M)$

uncoloured graph $X(M)$ with $\text{Qut}(X(M)) \cong \hat{\Gamma}_0(M)$
(Quantum) automorphisms of coloured graphs

**Definition**

A *coloured graph* is a (finite, simple) graph $X$ together with a vertex colouring $c : V(X) \to C$.

**Definition (Roberson, Schmidt, 2022)**

Let $X$ be a coloured graph. The *quantum automorphism group* $\text{Qut}_c(X)$ of $X$ is the compact matrix quantum group generated by $(u_{ij})_{i,j \in V(X)}$ with relations

- $u_{ij} = u_{ij}^* = u_{ij}^2$ for all $i, j \in V(X)$;
- $\sum_{k \in V(X)} u_{kj} = \sum_{k \in V(X)} u_{ik} = 1$ for all $i, j \in V(X)$;
- $uA_X = A_X u$
- $u_{ij} = 0$ for all $i, j \in V(X)$ with $c(i) \neq c(j)$.
Examples

\[ \text{Qut}(K_n) = S_n^+ \]

\[ \text{Qut}(P) = \text{Aut}(P) = S_5 \]

[Schmidt, 2018]

\[ \text{Qut}_c(X) = \text{Aut}(X) = C_2 \]

\[ \text{Qut}_c(T) = S_2 \ast S_2 \]
The solution group

Definition (Cleve, Liu, Slofstra, 2017)

Let $M \in \mathbb{F}_2^{m \times n}$ and $b \in \mathbb{F}_2^m$. The solution group $\Gamma(M, b)$ for the linear constraint system $Mx = b$ is the finitely presented group generated by $x_1, \ldots, x_n, \gamma$ with relations

- $x_1^2 = \cdots = x_n^2 = \gamma^2 = 1$;
- $[x_j, \gamma] = 1$ for all $j \in [n]$;
- $[x_j, x_k] = 1$ if $\exists i$ such that $M_{ij}, M_{ik} \neq 0$;
- $\prod_{j=1}^n x_j^{M_{ij}} = \gamma^{b_i}$ for all $i \in [m]$;
- $\gamma = 1$ if $b = 0$.

Remark

The abelianization $(\Gamma(M, b))_{ab}$ is the solution space of the augmented linear system $[M \ b] x = 0$ (if $b \neq 0$) or the system $Mx = 0$ (if $b = 0$).
The construction

**Definition (vDdB, Roberson, Schmidt, 2023)**

Let $M \in \mathbb{F}_2^{m \times n}$ and $b \in \mathbb{F}_2^m$. Let $\hat{X}(M, b)$ be the bipartite graph on the vertex sets $V$ and $Q$, where

$$V = [n] \times \mathbb{F}_2$$

$$Q = \bigsqcup_{i=1}^{m} \left\{ \alpha \in \mathbb{F}_2^{\text{supp}(M_i)} \mid \sum_{j=1}^{n} M_{ij} \alpha_j = b_j \right\}$$

Every vertex $(i, \alpha) \in Q$ is connected to the vertices $(j, \alpha_j) \in V$.

Colour vertices for different variables/equations in different colours.

**Example:**

$$\hat{X}\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) =$$

\[
\begin{array}{ccc}
\text{0?0} & \text{1?1} & \\
\text{0} & \text{1} & \\
\text{0} & \text{1} & \\
\text{0} & \text{1} & \\
\text{0} & \text{1} & \\
\text{0} & \text{1} & \\
\text{0} & \text{1} & \\
\text{0} & \text{1} & \\
\end{array}
\]

\[
\begin{array}{ccc}
x_1 + x_3 = 0 & & \\
x_1 + x_2 + x_3 = 1 & & \\
\end{array}
\]
The quantum automorphism group of $\widehat{X}_0(M)$

Example from before:

$$\widehat{X}\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \left\{\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right\}$$

$$x_1 + x_3 = 0 \quad \text{and} \quad x_1 + x_2 + x_3 = 1$$

Theorem (vDdB, Roberson, Schmidt, 2023)

For all $M \in \mathbb{F}_2^{m \times n}$ and $b \in \mathbb{F}_2^m$, we have

$$\text{Aut}_c(\widehat{X}(M, b)) \cong (\Gamma_0(M))_{ab}$$

$$\text{Qut}_c(\widehat{X}(M, b)) \cong \widehat{\Gamma_0(M)}$$
Decolouring procedure

**Theorem (vDdB, Roberson, Schmidt, 2023)**

For every coloured graph $\hat{X}$, there is an uncoloured graph $X$ such that $\text{Qut}_c(\hat{X}) \cong \text{Qut}(X)$.

Example:
Choosing the right solution group

For every homogeneous solution group \( \Gamma_0(M) \), we have

\[
\text{Aut}(X(M)) \cong (\Gamma_0(M))_{ab} \\
\text{Qut}(X(M)) \cong \hat{\Gamma}_0(M)
\]

Want to find: linear constraint system \( Mx = b \) such that \( \Gamma_0(M) \) is a non-trivial perfect group:

\[
(\Gamma_0(M))_{ab} = \{1\}; \quad \Gamma_0(M) \neq \{1\}.
\]

But \((\Gamma_0(M))_{ab} \cong \ker(M)\). Therefore:

**Goal**

Find a matrix \( M \in \mathbb{F}_2^{m \times n} \) such that \( \ker(M) = \{0\} \) and \( \Gamma_0(M) \neq \{1\} \).
Choosing the right solution group (ctd.)

**Goal**

Find a matrix \( M \in \mathbb{F}_2^{m \times n} \) such that \( \ker(M) = \{0\} \) and \( \Gamma_0(M) \neq \{1\} \).

**Definition**

For a group \( H \), let \( M_H \) be the system of all 3-element “solution group relations” in \( H \); that is:

- The variables \( x_h \) correspond to elements \( h \in H \) of order 2;
- The equations are of the form \( x_{h_1} + x_{h_2} + x_{h_3} = 0 \) for all triples \( h_1, h_2, h_3 \in H \) of order 2 with
  \[
  [h_1, h_2] = [h_1, h_3] = [h_2, h_3] = 1 \quad \text{and} \quad h_1 h_2 h_3 = 1.
  \]

Note: \( H \) satisfies the relations of \( \Gamma_0(M_H) \), so there is a homomorphism \( \phi: \Gamma_0(M_H) \to H \) with \( \phi(x_h) = h \) for all \( h \in H \) of order 2.

**Theorem (vDdB, Roberson, Schmidt, 2023)**

For \( n \geq 7 \), we have \( \Gamma_0(M_{A_n}) \neq \{1\} \) and \( (\Gamma_0(M_{A_n}))_{ab} \cong \ker(M_{A_n}) \cong \{1\} \).
Recap

binary linear constraint system $Mx = b$

homogeneous solution group $\Gamma_0(M) \neq 1$ with $(\Gamma_0(M))_{ab} = 1$

coloured graph $\tilde{X}$ with $Qut_c(\tilde{X}) \cong \Gamma_0(M)$ and $Aut_c(\tilde{X}) = 1$

uncoloured graph $X$ with $Qut(X) \cong \Gamma_0(M)$ and $Aut(X) = 1$

...$

Profit!

Open problem

*Is there a graph $Y$ with $Aut(Y) = 1$ and $Qut(Y)$ transitive?*