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Certified Soundness of Simplest Known Formulation of First-Order Logic

Abstract. In 1965, Donald Monk published a paper about an axiomatic system for first-order predicate logic that he described as "the simplest known formulation of ordinary logic". In this paper we show work in progress on certifying soundness of this system in the interactive proof assistant Isabelle. Through this work we demonstrate the usefulness of using proof assistants for validating mathematical results. This work also establishes an outline for future work such as a certified completeness proof of the axiomatic system in Isabelle.

Axioms

(C1) $(A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$
(C2) $A \rightarrow \neg A \rightarrow B$
(C3) $(\neg A \rightarrow A) \rightarrow A$
(C4) $\forall x\,(A \rightarrow B) \rightarrow \forall x\,A \rightarrow \forall x\,B$
(C5[1]) $A \rightarrow \forall x\,A$ — x does not occur in A
(C5[2]) $\neg \forall x\,A \rightarrow \forall x\,\neg \forall x\,A$
(C5[3]) $\forall x, y\,A \rightarrow \forall y, x\,A$
(C6) $\neg \forall x\,\neg(x = y)$
(C7) $x = y \rightarrow x = z \rightarrow y = z$
(C8) $x = y \rightarrow A \rightarrow \forall x\,(x = y \rightarrow A)$ — x is different from y

Rules

(R1) From A and $A \rightarrow B$ infer B
(R2) From A infer $\forall x\,A$

J. Donald Monk: *Substitutionless Predicate Logic with Identity*.
Archiv Für Mathematische Logik Und Grundlagenforschung 7(3-4) 102-121 1965

Normalized formulas

$$\forall x, y, z\,(Plus\,(x, y, z) \leftrightarrow Plus\,(y, x, z)) \equiv \forall x, y, z\,(\forall v_0\,(v_0 = x \rightarrow \forall v_1\,(v_1 = y \rightarrow \forall v_2\,(v_2 = z \rightarrow Plus\,(v_0, v_1, v_2)))) \leftrightarrow \forall v_0\,(v_0 = y \rightarrow \forall v_1\,(v_1 = x \rightarrow \forall v_2\,(v_2 = z \rightarrow Plus\,(v_0, v_1, v_2)))))$$

Isabelle formalization

1. Defining the syntax of normalized first-order logic formulas.

datatype *form = Pre string nat | Eq nat nat | Neg form |*
Imp form form | Uni nat form
abbreviation *(input) Falsity ≡ Uni 0 (Neg (Eq 0 0))*
abbreviation *(input) Truth ≡ Neg Falsity*

2. Defining the semantics of the normalized formulas.

fun *semantics :: (nat ⇒ 'a) ⇒ (string ⇒ 'a list ⇒ bool) ⇒ form ⇒ bool*
where
semantics e g (Eq x y) ⟷ (e x) = (e y) |
semantics e g (Pre p arity) ⟷ g p (map e (vars arity)) |
semantics e g (Neg f) ⟷ ¬semantics e g f |
semantics e g (Imp p q) ⟷ (semantics e g p ⟶ semantics e g q) |
semantics e g (Uni x p) ⟷ (∀t. (semantics (e (x:=t)) g p))

3. Defining the axioms and rules of Monk's axiomatic system and a proof system for it.

definition *c1 :: form ⇒ form ⇒ form ⇒ thm*
where
c1 p q r ≡ Thm (Imp (Imp p q) (Imp (Imp q r) (Imp p r)))

inductive *OK :: form ⇒ bool (⊢ - 0)*

4. Defining and proving soundness of the proof system.

theorem *soundness: ⊢ p ⟹ semantics e g p*

Work-in-progress formalization available online. https://bitbucket.org/isafol/isafol/wiki/Home

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