Sequent Calculus, Dialogues, and Cut-Elimination

Two doubtful, personal insights about dialogues in logic

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Curry-Howard Isomorphism - intuitionistic case

$$\Gamma, \varphi \vdash \varphi \qquad \Gamma, x : \varphi \vdash x : \varphi$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \qquad \frac{\Gamma, x : \varphi \vdash M : \psi}{\Gamma \vdash \lambda x.M : \varphi \to \psi}$$

$$\frac{\Gamma \vdash \varphi \to \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi} \qquad \frac{\Gamma \vdash M : \varphi \to \psi \quad \Gamma \vdash N : \varphi}{\Gamma \vdash MN : \psi}$$

Curry-Howard Isomorphism - intuitionistic case

natural deduction	λ-calculus
proof	term
formula	type
normalization	reduction

But why?

- BHK
  - *A construction of* $\varphi_1 \to \varphi_2$ *is a method (function) transforming every construction of* φ_1 *into a construction of* φ_2*;*
- Realizability
  - p realizes $\varphi_1 \to \varphi_2$ if p is the Gödel number of a partial recursive function f of one argument such that, whenever q realizes φ_1, then $f(q)$ realizes φ_2;
- So
  - Proofs are functions/programs.
  - The Curry-Howard makes this concrete/syntactic.

The classical case

$$\frac{\Gamma, \varphi \to \bot \vdash \bot}{\Gamma \vdash \varphi} \qquad \frac{\Gamma, a : \varphi \to \bot \vdash M : \bot}{\Gamma \vdash \mu a.M : \varphi}$$

$$\frac{\text{natural deduction}}{\text{double-negation elimination}} \qquad \frac{\lambda\text{-calculus}}{\text{control operator}}$$

But why?

- Double-negation elimination is a kind of jump in the proof.
  - Yes, but how more concretely?
- Some kind of jumps in BHK?
  - Hmm, BHK is *intentional.*
- An extensional variant of BHK?
  - But not the λ-terms themselves :-).

Prover-skeptic dialogues - interactive BHK

Prover 1: I claim $((p \to p) \to (q \to r \to q) \to s) \to s$ holds.

Skeptic 1: Really? Suppose I provide you with a construction of $(p \to p) \to (q \to r \to q) \to s$. Can you give me one for s?

Prover 2: I got it by applying your construction to a construction of $p \to p$ and then by applying the result to a construction of $q \to r \to q$.

Skeptic 2: Hmm. . . you are making two new assertions. I doubt the first one the most. Are you sure you have a construction of $p \to p$? Suppose I give you a construction of p, can you give me one back for p?

Prover 3: I can just use the same construction!

Prover-skeptic dialogues - interactive BHK

Prover 1: I claim $((p \to p) \to (q \to r \to q) \to s) \to s$ holds.

Skeptic 1: Really? Suppose I provide you with a construction of $(p \to p) \to (q \to r \to q) \to s$. Can you give me one for s?

Prover 2: I got it by applying your construction to a construction of $p \to p$ and then by applying the result to a construction of $q \to r \to q$.

Skeptic 2': Hmm. . . you are making two new assertions. I doubt the second assertion the most. Are you sure you have a construction of $q \to r \to q$? If I give you constructions of q and r, can you give me one for q?

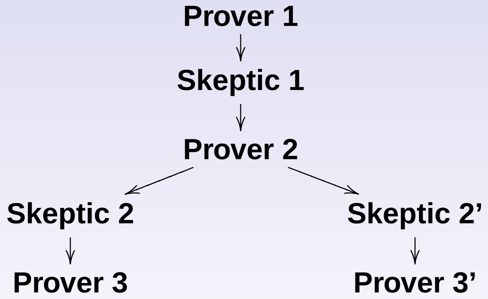
Prover 3': I can use the one you just gave me!

Prover-skeptic dialogues - interactive BHK

- Prover makes an *assertion.*
- Skeptic reacts with a list of *offers* (hypothetical constructions) and a *challenge*.
- Prover must meet the challenge using an offer (just obtained or previous one). Prover may introduce new assertions. Skeptic then reacts to these assertions, etc.
- When prover introduces several new assertions, Skeptic may challenge any one of them—but only one, and only from preceding step.
- Prover must always respond to the latest challenge.
- A dialogue *ends* when the player who is up cannot respond, in which case the other player *wins.*

Prover-skeptic dialogues - interactive BHK

- *Prover strategy:* technique for arguing against the skeptic. When the skeptic has a choice, prover has to anticipate all the different choices and prepare a reaction to each of them.



- Prover strategy is *winning* if all its dialogues won by prover.
- Winning prover strategy iff formula provable.
- Winning prover strategies correspond to long normal forms.

The classical case - catch-22 tricks

Prover 1: I assert that $((p \to q) \to p) \to p$ holds.

Skeptic 1: I don't believe you. Suppose that I give you a construction of $(p \to q) \to p$, can you give me a construction of p?

Prover 2: Yes, I have it. I got it by inserting a construction of $p \to q$ into the construction I just got from you!

Skeptic 2: You're lying, suppose I give you a construction of p, can you give me a construction of q?

Prover 3: Ah, your construction of p is actually what you requested in your first move, then we were done already there.

The classical case - catch-22 tricks

Prover 1: I assert that $q \vee \neg q$ holds.

Skeptic 1: Yeah, right. Can you give me a construction of one of the two?

Prover 2: Yes, I have it. It is $\neg q$, i.e. $q \to \bot$.

Skeptic 2: You're lying, suppose I give you a construction of q, can you give me a construction of $\bot$? Ha—got you!

Prover 3: Ah, this construction of q was what you requested in your first step. I should have chosen q, not $\neg q$. So we were done already there.

First insight

- First insight:
  - BHK can be “reformulated” as dialogues.
  - Classical dialogues arise by letting the prover respond to previous challenges.
  - I.e. by adding *jumps* in the arguments.
  - That’s “why” classical proofs correspond to control operators.
- About these dialogues:
  - They (intutionistic variant) appear in proofs that inhabitation is PSPACE complete.
  - They are related to other dialogues in the literature.
  - They are ad-hoc.
- Maybe the Curry-Howard isomorphism has a third angle, with dialogues.

Lorenzen dialogues

- Dialogue: sequence of alternating proponent/opponent steps.
- Proponent begins.
- Opponent refers to the immediately preceding move.
- Proponent refers to any preceding opponent move (classical).
- Proponent may only assert variables asserted by opponent.
- One can attack asserted formulas or defend against a matching attack.

Form.	Attacks
$\sigma \to \tau$	**A** : σ
$\sigma \wedge \tau$	$\mathbf{A}_{\mathrm{L}}$: ø , $\mathbf{A}_{\mathrm{R}}$: ø
$\sigma \vee \tau$	**A** : ø
$\neg\sigma$	**A** : σ

Form.	Attack	Defenses
$\sigma \to \tau$	**A**	**D** : τ
$\sigma \wedge \tau$	$\mathbf{A}_{\mathrm{L}}$	**D** : σ
$\sigma \wedge \tau$	$\mathbf{A}_{\mathrm{R}}$	**D** : τ
$\sigma \vee \tau$	**A**	$\mathbf{D}_{\mathrm{L}}$: σ , $\mathbf{D}_{\mathrm{R}}$: τ

Lorenzen dialogues

A winning proponent strategy for $((p \to p) \to q) \to q$:

$$1\ \mathrm{P}\ \mathbf{D}\ ((p \to p) \to q) \to q\quad 0$$

$$\downarrow$$

$$2\ O\ \mathbf{A}\ (p \to p) \to q\quad 1$$

$$\downarrow$$

$$3\ \mathrm{P}\ \mathbf{A}\ p \to p\quad 2$$

$$\swarrow \qquad \searrow$$

$$4a\ O\ \mathbf{D}\ q\quad 3 \qquad\qquad 4b\ O\ \mathbf{A}\ p\quad 3$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$5a\ \mathrm{P}\ \mathbf{D}\ q\quad 2 \qquad\qquad 5b\ \mathrm{P}\ \mathbf{D}\ p\quad 4b$$

- Winning prover strategy iff provable, usually via Beth-tableux.
- Close to Sequent Calculus, inside out compared to previous dialogues, like ND and SC are inside-out.

Lorenzen dialogues

- Dialogues can be viewed as parameterized over:
  - The set Φ of formulas
  - For each φ, the set of attacks on φ;
  - For each φ and attack on φ, the set of defenses of φ.
- φ formula, τ_i the attacks, and Σ_i the defenses against τ_i:

$$\varphi \lhd (\tau_1 \vdash \Sigma_1) \ldots (\tau_n \vdash \Sigma_n)$$

- Examples:
  - $\varphi_1 \wedge \varphi_2 \lhd (\vdash \varphi_1)(\vdash \varphi_2)$.
  - $\varphi_1 \vee \varphi_2 \lhd (\vdash \varphi_1, \varphi_2)$.
  - $\varphi_1 \to \varphi_2 \lhd (\varphi_1 \vdash \varphi_2)$.
  - $\neg\varphi \lhd (\varphi \vdash)$.
  - $p \lhd ()$ and $\emptyset \lhd ()$.

From LK to dialogues

Take standard LK: cut, structural rules, Axiom, plus logical rules. Replace latter with:

$$\frac{\Gamma \vdash \tau_i, \Delta \quad \Gamma, \sigma^i_1 \vdash \Theta \quad \cdots \quad \Gamma, \sigma^i_{m_i} \vdash \Theta}{\Gamma, \varphi \vdash \Delta, \Theta} \; (\mathrm{L}_i)$$

$$\frac{\Gamma, \tau_1 \vdash \Sigma_1, \Delta \quad \cdots \quad \Gamma, \tau_n \vdash \Sigma_n, \Delta}{\Gamma \vdash \varphi, \Delta} \; (\mathrm{R})$$

$$[\varphi \lhd (\tau_1 \vdash \Sigma_1) \ldots (\tau_n \vdash \Sigma_n) \qquad \Sigma_i = \{\sigma^i_1, \ldots, \sigma^i_{m_i}\},\ 1 \leq i \leq n]$$

Yields standard rules plus Cut elimination!

From LK to dialogues

Now replace all rules by:

$$\frac{\Gamma, \sigma_1 \vdash \Sigma_1, \Delta \quad \cdots \quad \Gamma, \sigma_n \vdash \Sigma_n, \Delta \qquad \Gamma, \rho_1 \vdash \Delta \quad \cdots \quad \Gamma, \rho_m \vdash \Delta}{\Gamma \vdash \Delta} \text{(L)}$$

$$[\varphi \in \Gamma,\ \varphi \lhd \cdots (\sigma \vdash \rho_1, \ldots, \rho_m) \cdots,\ \sigma \in (\Phi^{\emptyset} - \Upsilon) \cup \Gamma,\ \sigma \lhd (\sigma_1 \vdash \Sigma_1) \ldots (\sigma_n \vdash \Sigma_n)]$$

$$\frac{\Gamma, \sigma_1 \vdash \Sigma_1, \Delta \quad \cdots \quad \Gamma, \sigma_n \vdash \Sigma_n, \Delta}{\Gamma \vdash \Delta} \text{(R)}$$

$$[\varphi \in \Delta,\ \varphi \lhd (\sigma_1 \vdash \Sigma_1) \ldots (\sigma_n \vdash \Sigma_n),\ \varphi \in (\Phi - \Upsilon) \cup \Gamma\ ]$$

$$\frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma, \varphi \vdash \Theta}{\Gamma \vdash \Delta, \Theta} \text{(Cut)}$$

From LK to dialogues

The rules in LKD formalize winning proponent strategies.

- In $\Gamma \vdash \Delta$ we read:
  - Γ as the assertions that *have* been stated by the opponent and thus may be attacked by the proponent
  - Δ as assertions that *may* be asserted by the proponent, as defenses or as the initial formula.
- the right rule corresponds to a node where the proponent states φ in a defense, and the strategy has a branch for each possible opponent attack on φ.
- The left rule corresponds to a node where the proponent attacks a formula φ stated by the opponent. In this case the strategy has a branch for each possible opponent defense and for each opponent counter-attack.

Second insight

Classical Lorenzen Dialogues and LK are two different presentations of the same thing:

- replace traditional right rules for disjunction with a single rule;
- restrict (Ax) to variables;
- adopt G3-style and use sets to eliminate structural rules;
- build (Ax) into (R);
- replace the concrete connectives with a specification of attacks and defenses;
- avoid redundant occurrences of (R);
- adopt L-R regime.

Cut-elimination does not require the actual connectives.