Iterative Update of Linearization Model Error for Conductivity Reconstruction from Interior Current Density Information

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Conductivity Reconstruction

Forward problem of Electrical Impedance Tomography (EIT) is defined as

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 & in\ \Omega \\ u = f & on\ \partial\Omega \end{cases} \tag{1}$$

The forward Magnetic Resonance Electrical Impedance Tomography (MREIT) problem employs MRI to detect the magnetic field induced by the interior current. Denoted

$$J = -\sigma \nabla u \quad in\ \Omega \tag{2}$$

as the interior current density, from the Ampere law we have

$$J = \frac{1}{\mu_0} \nabla \times B \quad in\ \Omega \tag{3}$$

The interior current density information (ICDI) J provides a resolution-sufficient number of measurements when numerically solving the problem.

Linearization

Given measurements of ICDI as $J_e \in \mathbb{R}^M$ with

$$J_e := F(\sigma) + e \tag{4}$$

With a background conductivity σ^*, the linearized operator f^{σ^*} is defined as

$$f^{\sigma^*}\sigma = F(\sigma^*) + DF(\sigma^*)(\sigma - \sigma^*) \tag{5}$$

where DF is the Fréchet derivative of F at σ^*. Given model error $m(\sigma) = F(\sigma) - f^{\sigma^*}\sigma$, the forward problem is equivalent to

$$J_e = f^{\sigma^*}\sigma + m(\sigma) + e \tag{6}$$

Bayesian Formulation

$$J_e = f^{\sigma^*}\sigma + m(\sigma) + e,\ \ s.t.\ m(\sigma) = F(\sigma) - f^{\sigma^*}\sigma$$

$$\downarrow$$

$$\pi(\sigma|J_e) \propto \pi(J_e|\sigma)\pi(\sigma)$$

$$\downarrow$$

$$\pi(J_e|\sigma) = \pi_{v|\sigma}(J_e - f^{\sigma^*}\sigma|\sigma),\ \ s.t.\ v = m + e$$

$$\downarrow$$

$$\pi_{v|\sigma}(J_e - f^{\sigma^*}\sigma|\sigma) = \int \pi(m|\sigma)\pi_e(J_e - f^{\sigma^*}\sigma - m)\mathbf{d}m$$

- Nonlinearity of m: Laplace approximation, Gaussian mixture model
- Trackable m: Enhanced Error Model, Iterative Update Error Model
- Sampling methods: MCMC, RTO

Previous works

1. Conventional Error Model:

   Simply ignoring the model error m:

$$\pi(\sigma|J_e) \propto \pi(\sigma)\pi_e(J_e - f^{\sigma^*}\sigma) \tag{7}$$

Previous works

1. Conventional Error Model

2. Enhanced Error Model[1]:

Incorporating the model error under Bayesian framework by

- Assuming m is independent from σ with Gaussian distribution of mean $\bar{m}$ and covariance Γ_m
- Given samples of the prior $\sigma_i^0, i = 1, ..., N$, $\bar{m}$ and Γ_m are approximated upon samples of the model error $M_i = m(\sigma_i^0) = F(\sigma_i^0) - f^{\sigma^*}\sigma_i^0, i = 1, ..., N$ as

$$\bar{m} = \frac{1}{N}\sum_{l=1}^{N} M_i,\ \Gamma_m = \frac{1}{N}\sum_{l=1}^{N}\{(M_i - \bar{m})(M_i - \bar{m})^T\} \quad (8)$$

v is also independent from σ with Gaussian distribution with mean $\bar{v} = \bar{m}$ and covariance $\Gamma_v = \Gamma_m + \Gamma_e$:

$$\pi(\sigma|J_e) \propto \pi(\sigma)\pi_v(J_e - f^{\sigma^*}\sigma) \quad (9)$$

[1] Jari Kaipio and Erkki Somersalo. "Statistical inverse problems: discretization, model reduction and inverse crimes". In: *Journal of computational and applied mathematics* 198.2 (2007), pp. 493–504.

Previous works

1. Conventional Error Model
2. Enhanced Error Model
3. Iterative Updated Error Model[2]:

   Under nonlinear settings:

$$\pi_v(J_e - f^{\sigma^*}\sigma) = \frac{1}{N}\sum_{i=1}^{N} \pi_e(J_e - f^{\sigma^*}\sigma - M_i) \tag{10}$$

$$\pi(\sigma|J_e) \propto \frac{1}{N}\sum_{i=1}^{N} \pi(\sigma)\pi_e(J_e - f^{\sigma^*}\sigma - M_i) \tag{11}$$

   Posterior samples are collected and mapped by m as model error samples for the next iteration:

$$M_i^k = F(\sigma_i^{k-1}) - f^{\sigma^*}\sigma_i^{k-1}, \quad \pi^k(\sigma|J_e) \propto \frac{1}{N}\sum_{i=1}^{N} \pi(\sigma)\pi_e(J_e - f^{\sigma^*}\sigma + M_i^k), \quad k = 1, 2, ... \tag{12}$$

[2] Daniela Calvetti et al. "Iterative updating of model error for Bayesian inversion". In: *Inverse Problems* 34.2 (2018), p. 025008.

RIUEM: Cluster Model Error with GMM

Assumed well-picked T components, the joint distribution of σ and m goes like

$$\pi(\sigma, m) \approx \sum_{i=1}^{T} \omega_i \mathcal{N}\left(\begin{bmatrix}\bar{\sigma}\\ \bar{m}^i\end{bmatrix}, \begin{bmatrix}\Gamma_\sigma & \Gamma^i_{\sigma,m}\\ \Gamma^i_{m,\sigma} & \Gamma^i_m\end{bmatrix}\right), \quad \sum_{i=1}^{T} \omega_i = 1 \tag{13}$$

where $\{\omega_i\}_{i=1}^{T}$ are weights of every components.
Ignoring all correlations between σ and m[3]:

$$\pi(\sigma, m) \approx \sum_{i=1}^{T} \omega_i \mathcal{N}\left(\begin{bmatrix}\bar{\sigma}\\ \bar{m}^i\end{bmatrix}, \begin{bmatrix}\Gamma_\sigma & 0\\ 0 & \Gamma^i_m\end{bmatrix}\right) \tag{14}$$

$$\hat{\pi}(\sigma|J_e) \propto \sum_{i=1}^{T} \omega_i \pi(\sigma) \pi_{v^i}(J_e - f^{\sigma^*}\sigma) \tag{15}$$

where $\pi_{v^i} \sim \mathcal{N}(\bar{m}^i, \Gamma^i_m + \Gamma_e)$.

[3] Junxiong Jia et al. "Recursive linearization method for inverse medium scattering problems with complex mixture Gaussian error learning". In: *Inverse Problems* 35.7 (2019), p. 075003.

RIUEM: Coefficients estimate and model selection in GMM

Given the number of components, we employ the expectation - maximization (EM) algorithm to deduce coefficients of GMM:

1. Expectation Step (E Step): This step defines the expectation of the (penalised[4]) log likelihood w.r.t. currently assumed coefficients.
2. Maximization Step (M Step): This step finds coefficients that maximize the previously defined value, and take them as assumed coefficients for the next iteration.

We incorporate k-means clustering as a preliminary measurement that initialize coefficients. As for the optimal number of components, we experiment with various component numbers, and the best-fit model is selected based on minimizing the (high-dimensional[5]) Bayesian information criterion (BIC).

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[4] Tao Huang, Heng Peng, and Kun Zhang. "Model selection for Gaussian mixture models". In: *Statistica Sinica* (2017), pp. 147–169.

[5] Charles Bouveyron, Stéphane Girard, and Cordelia Schmid. "High-dimensional data clustering". In: *Computational statistics & data analysis* 52.1 (2007), pp. 502–519.

RIUEM: Positive constrains and projection

Given a positive box constraint C, we ascertain if a sample is within the domain. We take no action if it is the case, otherwise we project the sample onto the boundary of the domain w.r.t the minimum distance.

In terms of point estimation of the posterior, we gravitate towards the median.

RIUEM: Algorithm

Algorithm RIterative Update of Model Error with Constraints

Input J_e, $\pi(\sigma)$, $e \sim \mathcal{N}(0, \Gamma_e)$, σ^*, C, number of samples N, maximum iteration L;
Sample $\sigma_i^0 \sim \pi(\sigma)$, then project σ_i^0 as onto C, $i = 1, ..., N$.
for $l = 0, ..., L-1$ **do**
    Calculate $M_i^{l+1} = F(P\sigma_i^l) - f^{\sigma^*} P\sigma_i^l$, $i = 1, ..., N$;
    Get GMM of $\{M_i^{l+1}\}_{i=1,...,N}$. The corresponding weights, means and covariances of t components are ω^t, $\bar{m}^t$ and Γ_m^t, $t = 1, ..., T^{l+1}$;
    Divide the the range $[0, 1]$ into T^{l+1} in proportion to the weight ω^t, $t = 1, ..., T^{l+1}$;
    for $j = 1, ..., N$ **do**
        Pick a random value r within $[0, 1]$;
        Determine the component k corresponding to the segment r falls into;
        Sample $\sigma_j^{l+1} \sim \pi(\sigma)\pi_{v^k}\left(J_e - f^{\sigma^*}\sigma\right)$ where $\pi_{v^k} \sim \mathcal{N}(\bar{m}^k, \Gamma_m^k + \Gamma_e)$, then project σ_i^{l+1} onto C;
    end for
end for

Numerical Results: Problem Settings

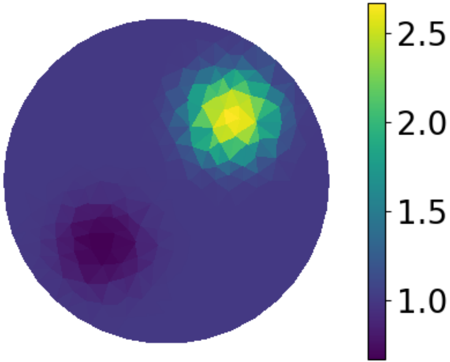


Figure: The conductivity data used in numerical models.

Numerical Results: Problem Settings

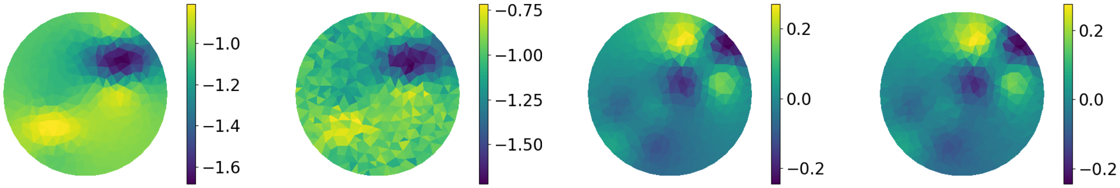


Figure: ICDI and actually measurements with 5% relative noise. Left two: accurate ICDI and measurements on x axis; Right two: accurate ICDI and measurements on y axis.

Numerical Results: Problem Settings

σ^*	1
$\bar{\sigma}$	σ^*
$(\Gamma_\sigma)_{i,j}$	$\gamma \exp\left\{-\frac{\|\|c_i - c_j\|\|_2}{l}\right\}$

We assume value $\gamma = 0.36$ and $l = 0.5$ to ensure that the groundtruth conductivity value on all elements of the mesh are within the 95% credible interval of the prior probability.

Numerical Results: Reference Posterior

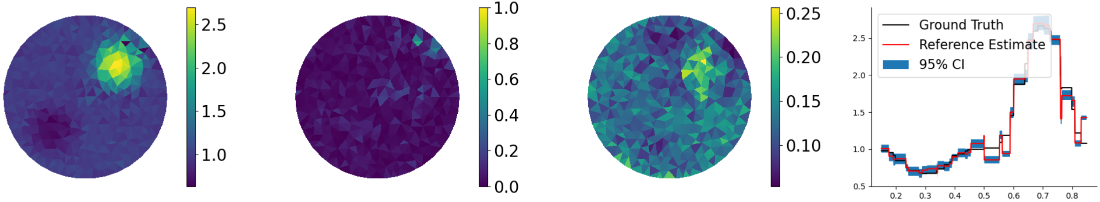


Figure: Properties of the reference posterior. From left to right 1st: the sample median of the reference posterior; 2nd: the coordinate-wise relative error of the sample median comparing to the groundtruth; 3rd: the coordinate-wise width of 95% CI. 4th: values of the groundtruth, the sample median and 95% CI on the line $y = x$.

Numerical Results: Comparisons

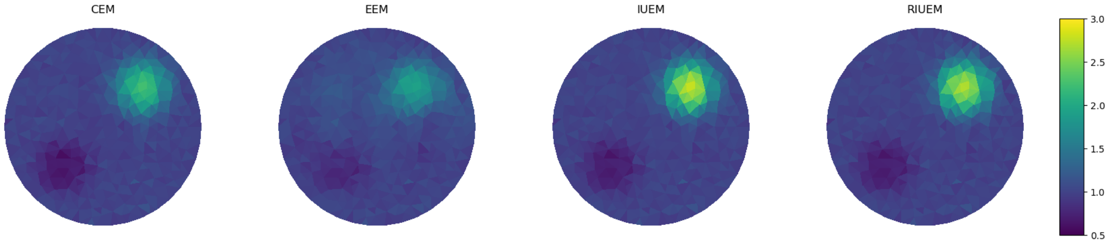


Figure: Posteriors and their comparison with the reference posterior in terms of sample medians.

Numerical Results: Comparisons

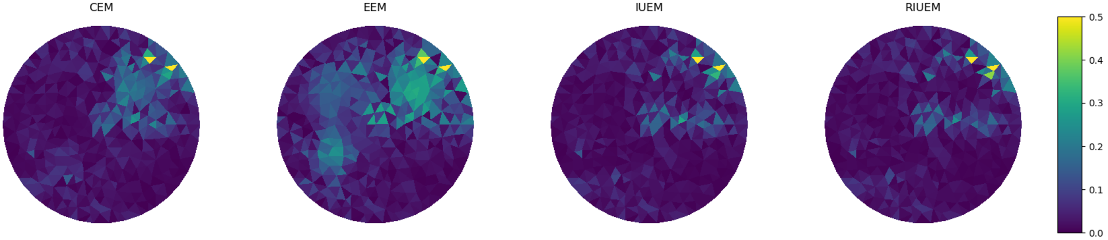


Figure: Posteriors and their comparison with the reference posterior in terms of relative errors w.r.t. medians.

Numerical Results: Comparisons

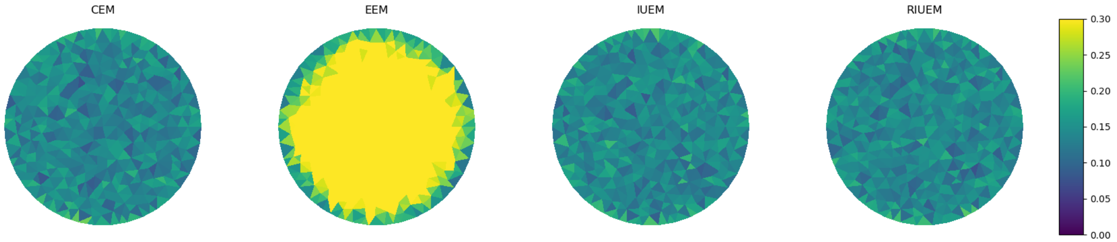


Figure: Posteriors and their comparison with the reference posterior in terms of coordinate-wise widths of 95% CI.

Numerical Results: Comparisons

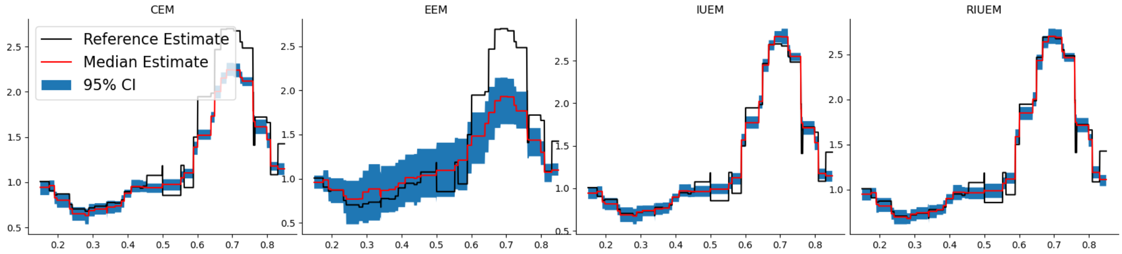
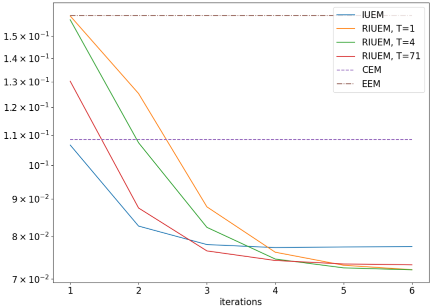


Figure: Posteriors and their comparison with the reference posterior in terms of values of the groundtruth, corresponding sample medians and 95% CIs on the line $y = x$.

Numerical Results: Comparisons



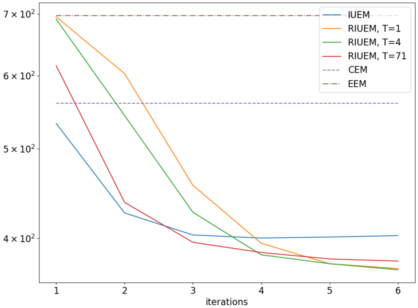


Figure: Relative errors and KL divergence. Left: relative errors of sample estimates comparing to the reference posterior estimate w.r.t. update iterations for both IUEM and RIUEM. Right: approximated KL divergence of posteriors and the reference posterior w.r.t. update iterations for both IUEM and RIUEM.

Contribution and Future Work

Contribution:

- Numerically demonstrate the superior of a refined model error update scheme
- Discuss the influence of a well-picked Gaussian mixture model
- Implement a constraint to prevent the influence of unreasonable samples outside the domain of our interest

Future work:

- Correct the approximated operator for better reconstruction
- Discuss the influence of the discarded correlation
- Implement our method on 3D reconstruction with partial data

Thanks for you attention!