Posterior consistency for Bayesian inverse Problems with piecewise constant inclusions

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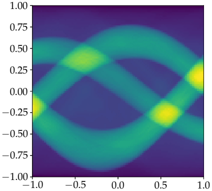
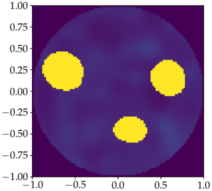
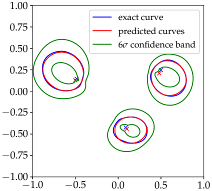
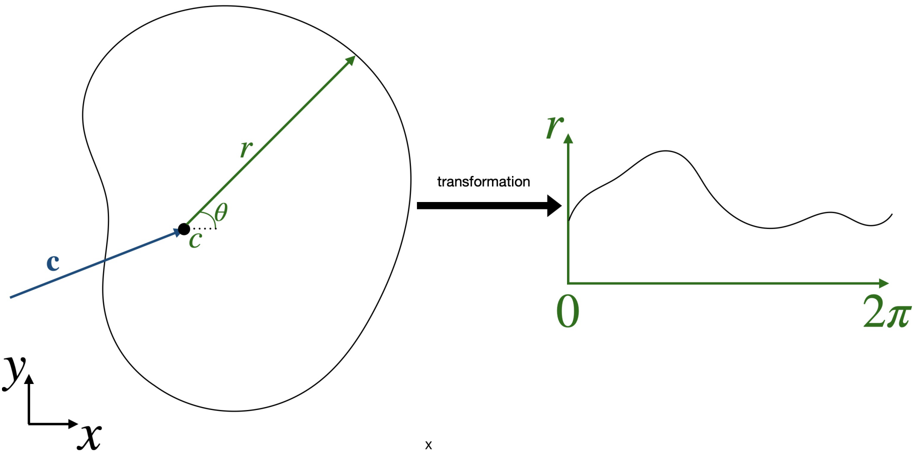
Motivational example: X-ray Computed Tomography





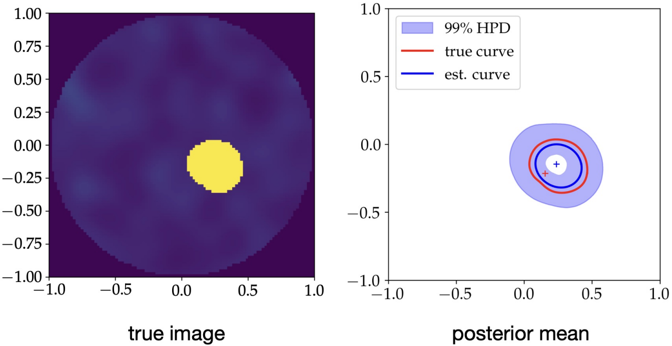


Goal-oriented: extreme model reduction

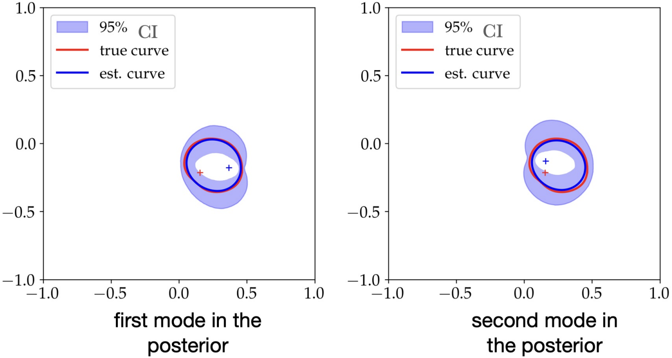


x

Goal-oriented: extreme model reduction



Goal-oriented: extreme model reduction

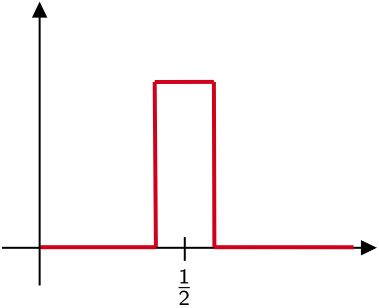


- BMA, Y. Dong, and P. C. Hansen. "Uncertainty quantification of inclusion boundaries in the context of X-ray tomography." SIAM/ASA Journal on Uncertainty Quantification 11.1 (2023): 31-61.
- BMA, N. A. Riis, Y. Dong, P. C. Hansen, (2023). Inferring Features with Uncertain Roughness. arXiv preprint arXiv:2305.04608.

Agenda

1. Consistency for Bayesian Inverse Problems
2. Introduce consistency
3. The case of star-shaped inclusions
4. Computational results

A Bayesian approach for consistent reconstruction of inclusions
BM Afkham, K Knudsen, AK Rasmussen, T Tarvainen
arXiv:2308.13673, 2023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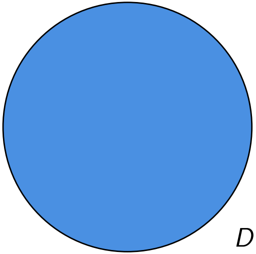
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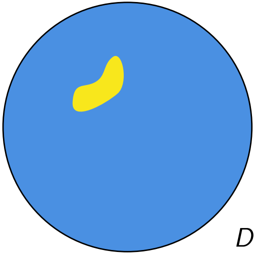

D

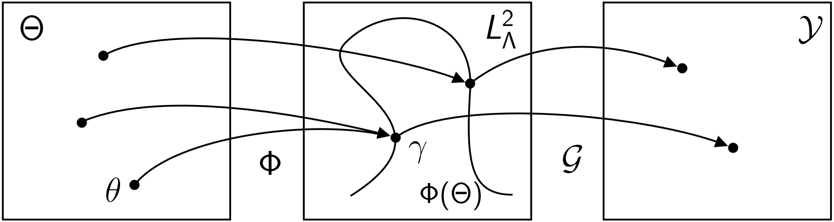
Parametrized inverse problem

Observe γ indirectly

$$Y = \mathcal{G}(\gamma) + \varepsilon\xi, \qquad (\mathcal{Y}, \|\cdot\|) \text{ Hilbert}$$

with white noise ξ. Distribution of Y is P.

$\mathcal{G} : L^2 \to \mathcal{Y}$ is non-linear.



Could be handled in parameter domain Θ, but we are more interested in understanding the physical set $\Phi(\Theta)$.

Bayesian framework

L^2_Λ

Θ

Bayesian framework

Prior. $\pi_{\rm pr}(d\gamma) = (\Phi\pi'_{\rm pr})(d\gamma)$.
Likelihood. $L_\varepsilon(\gamma, Y) = \exp(-\frac{1}{2\varepsilon^2}\|\mathcal{G}(\gamma) - Y\|^2)$.
Posterior.

$$\pi_{\rm pos}(d\gamma) = L_\varepsilon(\gamma, Y)\pi_{\rm pr}(d\gamma)$$

L^2_Λ

Θ

Bayesian framework

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Posterior'.

$$\pi'_{\mathrm{pos}}(d\theta) = L'_\varepsilon(\theta, Y)\pi'_{\mathrm{pr}}(d\theta)$$

Θ

Change of variables.

$$\pi_{\mathrm{pos}} = \Phi\pi'_{\mathrm{pos}}$$

Posterior consistency

The solution to the BIP is the posterior $\Pi(\cdot|Y)$.

In the spirit of classical regularization we would like

$$\Pi(\gamma : \|\gamma - \gamma_0\|_{L^2(D)} \leq C\tilde{r}_n | Y) \to 1 \quad \text{in } P_n^{\gamma_0}\text{-probability},$$

with rate r_n as $n \to \infty$.

A consequence (under additional assumptions on the prior)

$$\|E[\gamma|Y] - \gamma_0\|_{L^2(D)} \to 0 \quad \text{in } P_n^{\gamma_0}\text{-probability},$$

with rate r_n as $n \to \infty$.

Main result

Model

$$Y = \mathcal{G}(\gamma) + \varepsilon\xi$$

Theorem

Let $\Pi(\cdot|Y)$ be the sequence of posterior distributions $\gamma_0 \in L^2_\Lambda(D)$, $\mathcal{G}$ and prior distributions $\Pi = \Pi_n$ satisfying condition (mass, support, covering) for some rate r_n. Then, there exists C such that

$$\Pi(B_{\mathcal{G}}(\gamma_0, Cr_n) \cap A_n | Y) \to 1 \quad \text{in } P_n^{\gamma_0}\text{-probability,}$$

with rate $e^{-bnr_n^2}$ for some $0 < b <$ as $n \to \infty$.

Theorem explained

1. The first step reduces convergence of $\{\Pi(\tilde{B}_n|Y)\}_{n=1}^{\infty}$ from sets

   $$\tilde{B}_n = \{\gamma \in L^2_\Lambda(D) : \|\gamma - \gamma_0\|_{L^2(D)} \leq C\tilde{r}_n\}$$

   to sets

   $$B_n = \{\gamma \in L^2_\Lambda(D) : \|\mathcal{G}(\gamma) - \mathcal{G}(\gamma_0)\| \leq Cr_n, \gamma \in A_n\}.$$

   This involves (conditional) inverse stability estimates

   $$\|\gamma_1 - \gamma_2\|_{L^2(D)} \leq \|\mathcal{G}(\gamma_1) - \mathcal{G}(\gamma_2)\|^\nu,$$

   for all $\gamma_1, \gamma_2 \in A_n$ and some $\nu > 0$.

2. The second step involves showing that $\Pi(B_n|Y)$ converges to 1 in $P_n^{\gamma_0}$-probability as $n \to \infty$. This is posterior consistency on the 'forward level'.

Quantitative Photoacoustic Tomography

Light transport in a scattering medium according to

$$-\nabla \cdot \mu \nabla u + \gamma u = 0, \text{ in } D,$$
$$u = g, \text{ on } \partial D$$

μ is the optical diffusion (known)
γ is the absorption (unknown)

Forward map:

$$\mathcal{G} : \gamma \mapsto H := \gamma u, \quad \mathcal{G} : L^2_\Lambda(D) \to L^2(D),$$

Phantom

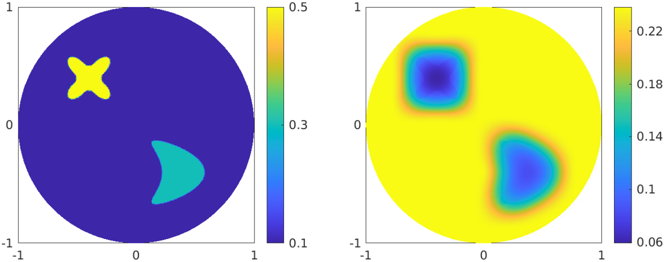


Figure: Simulated absorption γ_0 (left image) and diffusion μ (right image) distributions.

The data

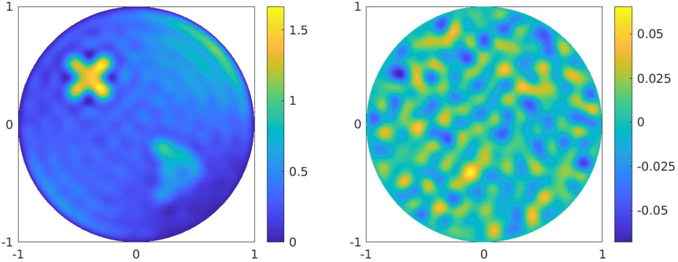
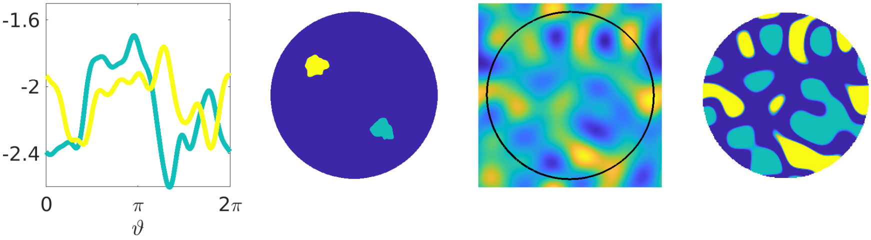


Figure: Projection of absorbed optical energy density H corresponding to phantom (left image) and of the white noise expansion (right image) projected onto the span of $\{e_k\}_{k=1}^{N_d}$.

Sampling from prior



Results

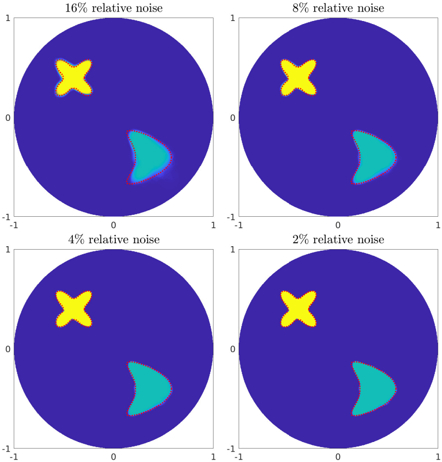


Figure: Posterior mean estimates of the absorption parameter using the star-shaped set parametrization in different noise regimes. The dotted red line indicates the location of γ_0.

references

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