# Fourier method for inverse source problem using correlation of passive measurements

**The 11th Applied Inverse Problems Conference 2023**


**Author**:
Kristoffer Linder-Steinlein

**Supervisors**:
Per Christian Hansen,
Mirza Karamehmedović and
Faouzi Triki

DTU Compute
Department of Applied Mathematics and Computer Science

# Outline

- Introduction
  - Motivation/inspiration
  - The model
- The inverse source problem in 1d
- Numerical experiments
- Conclusion and future work

# Passive sensor imaging

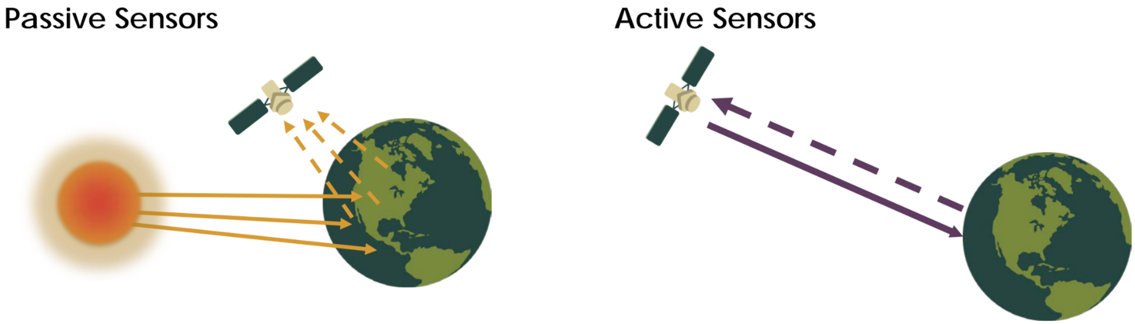



Image credit: NASA

# Passive sensor imaging

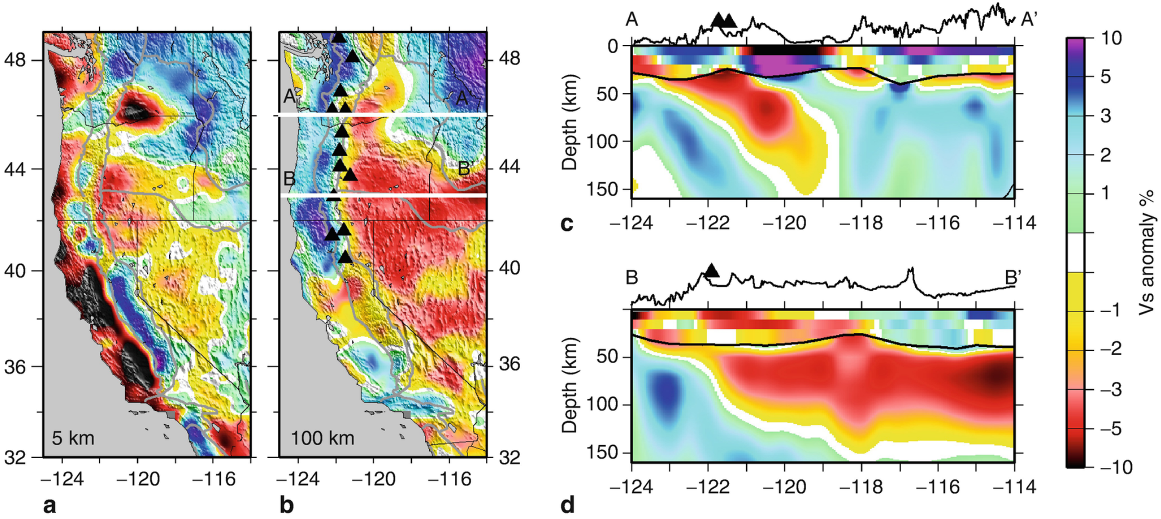



Image credit: [CRS21]

# Space-time stationary and "delta"-correlated noise, homogeneous medium.

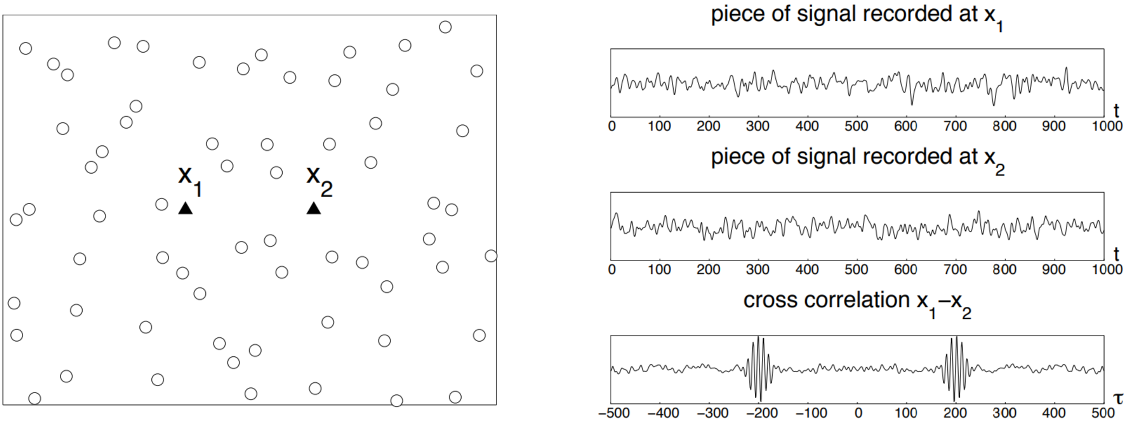



Image credit: Josselin Garnier and George Papanicolao [GP09]

# The data at hand

## Cross correlation

Consider an array of $N$ passive sensors at position $\{x_j\}_{j=1}^{N}$, wave fields $u(t, x_j)$ recorded in the time interval $[0, T]$, $T > 0$, then we write the cross correlation matrix for time lag $\tau$ as:

$$C_T(\tau, x_j, x_l) = \frac{1}{T}\int_0^T u(t, x_j)u(t+\tau, x_l)dt, \quad j, l = \{1, \cdots, N\}, \ \tau \in \mathbb{R} \tag{1}$$

For space-time stationary and "delta"-correlated noise sources, we get [GP16]:

## Green's function proportionality

Denoting by $G$ the Green's function of the scalar wave equation, it is possible to arrive at the following relation:

$$\frac{\partial}{\partial \tau} C_T(\tau, x_1, x_2) \approx -\left[G(-\tau, x_1, x_2) - G(\tau, x_1, x_2)\right] \tag{2}$$

The travel time can be inferred from (2) by considering the singular support of the right-hand side.

# The setup under consideration

## Underlying wave equation

We assume the measured data at sensor positions $\{x_j\}_{j=1}^2$ originates from the following wave equation

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - u_{xx} &= 0, \quad x \in \mathbb{R}, \quad t \in (0,T) \\ u(0,x) = 0, \quad \partial_t u(0,x) &= f(x), \quad x \in \mathbb{R} \end{aligned} \tag{3}$$

where $c > 0$ is the wave speed and $f$ the initial speed.

## Solution space

We seek $f$ in the solution set $\mathcal{Q}_L$ defined as

$$\mathcal{Q}_L := \{f | f \in C(\mathbb{R}), \ \operatorname{supp} f \subseteq [0,1], |f| < L\}$$

for some constant $L > 0$.

# Cross correlation as wave equation

**Definition (The inverse source problem)**

Let $f \in Q_L$, fix $x_1 = \tau = 0$, then the data, at the given values for any combination of $c, T \in \boldsymbol{R}^+$, is defined as originating from:

$$d(x) := \partial_\tau C_T(0,0,x) + c\partial_x C_T(0,0,x) = \frac{1}{2cT}\int_0^T \int_{-ct}^{ct} f(s)ds f(x+ct)dt \quad (4)$$

and the inverse source problem becomes

**Given the data, $d(x)$, seen in (4) recover the initial speed, $f \in Q_L$, of the wave equation (3).**

- Introduction
  - Motivation/inspiration
  - The model
- **The inverse source problem in 1d**
- Numerical experiments
- Conclusion and future work

# Manipulating the data equation

We can express the data as follows:

$$d(x) = \frac{1}{2cT}\int_0^T [F(ct) - F(-ct)]\, F'(x+ct)dt, \quad F(q) = \int_0^q f(s)ds$$

as $c > 0, T > 0$ with the compact support of $f$ in $[0, 1]$ the term $F(-ct)$ vanishes, reducing the above to:

$$d(x) = \frac{1}{2cT}\int_0^T F(ct)F'(x+ct)dt = \frac{1}{2c^2T}\int_0^{cT} F(q)F'(x+q)dq, \quad q = ct$$

A direct application of integration by parts shows the data is equivalent to

$$d(x) = \frac{F(cT)}{2c^2T}F(x+cT) - \frac{1}{2c^2T}\int_0^{cT} F'(q)F(x+q)dq$$

# Fourier domain and mean estimation

## Data in the Fourier domain

$$\mathcal{F}[d'(x)](\omega) = \frac{F(cT)}{2c^2T} e^{2\pi i cT\omega} \hat{f}(\omega) - \frac{1}{2c^2T} |\hat{f}(\omega)|^2, \quad cT \geq 1,$$

where $\hat{f}$ is the Fourier transform of $f$.
*Remark:* It should be noted that $F(cT) = M_x[f]$ is the mean of $f$ by the definition of $F$ and the assumption of the support of $f$. From the above relation it is apparent $F(cT)$ is important to know a priori and hence something that is recovered.

## Lemma (Mean of the source from data)

*Assume access to the data $d(x)$, and the sensor signal $u_0(t) = u(t, 0)$, then the mean $M_x[f]$ can be calculated as:*

$$M_x[f] = \frac{2c\,\mathrm{Leb}(\operatorname{supp} d) M_x[d]}{M_t[u_0]}$$

## Zero mean, non-uniqueness. and phase-retrieval

Now, assume $M_x[f] = 0$ and introduce $\breve{f}(x) = f(-x)$, then

$$2c^2Td'(x) = -\left(f * \breve{f}\right)(-x)$$

with $*$ the usual convolution operator. Applying the Fourier transform we find:

$$2i\omega c^2 T\hat{d}(\omega) = -\hat{f}(-\omega)\hat{\breve{f}}(-\omega) = -\hat{f}(\omega)\hat{f}(-\omega)$$

since $f$ is real-valued, we have $\hat{f}(-\omega) = \overline{\hat{f}(\omega)}$, the complex conjugate, which yields the phase retrieval problem

$$-2i\omega c^2 T\hat{d}(\omega) = \left|\hat{f}(\omega)\right|^2$$

**Phase retrival**

Hence, in the case where the mean of the source function is zero, we are faced with the following problem

$$\text{Reconstruct } f \text{ from } \left|\hat{f}\right|$$

# Zero mean, non-uniqueness. and phase-retrieval

**Theorem (Akutowicz [Aku57])**

*Let* $f, g \in Q_L$, *and denote by* $\hat{f}$ *and* $\hat{g}$ *the Fourier transform. Then* $\hat{f}$ *and* $\hat{g}$ *admits a holomorphic extensions,* $F$ *and* $G$ *to all of* $\mathbb{C}$. *If* $|\hat{f}| = |\hat{g}|$ *then* $\{z_n\} \cup \{\bar{z}_n\} = \{z'_n\} \cup \{\bar{z}'_n\}$ *where* $\{z_n\} \cup \{\bar{z}_n\}$ *and* $\{z'_n\} \cup \{\bar{z}'_n\}$ *are the sets of zeros of* $F$ *and* $G$ *respectively. This implies*

$$\hat{g}(z) = \hat{h}(z)e^{i\theta z}\gamma, \quad \theta \in \mathbb{R},\ \gamma \in \mathbb{C},\ |\gamma| = 1$$

*for some function* $h \in \mathcal{E}'(\mathbb{R})$ *completely determined by* $\{z_n\} \cup \{\bar{z}_n\}$.

**Lemma**

*Let* $f \in Q_L$, $M_x[f] = 0$, *and assume* $cT \geq 1$ *the non-linear inverse source problem based on the data seen in* (4) *is ill-posed. Furthermore, it is a phase retrieval problem.*

# Non-zero reconstruction theorem

**Theorem**

Let $f \in \mathcal{Q}_L(\mathbb{R})$, $M_x[f] \neq 0$, assume $cT \geq 1$ put $\mathcal{G}(\omega) = 2c^2 T \mathcal{F}[d'(x)](\omega)$ and $\alpha(\omega) = E_{cT}(\omega)$ then the non-linear inverse source problem based on the data seen in (4) can be reconstructed from:

$$\Re \hat{f} = \frac{\pm\sqrt{\kappa^4 - 4\kappa^2 \Re\mathcal{G} - 4(\Im\mathcal{G})^2}|\Re\alpha| + \Re\alpha\kappa^2 + 2\Im\alpha\Im\mathcal{G}}{2\kappa}$$

$$\Im \hat{f} = \frac{-\kappa\Im\alpha\Re\hat{f} - \Im\mathcal{G}}{\kappa\Re\alpha}$$

where $\kappa = F(cT)$. Furthermore, we obtain the following stability estimate

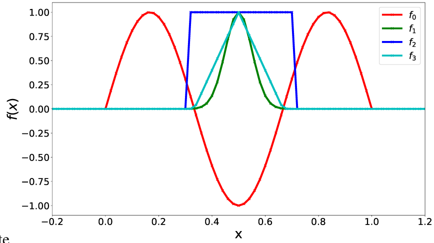
$$\|f_1 - f_2\|_{L^2(\mathbb{R})} \leq \kappa^{-1} C(L) \left(2\kappa^2 + 1\right) |\mathcal{G}_1 - \mathcal{G}_2|^{1/2}_{L^\infty(\mathbb{R})}$$

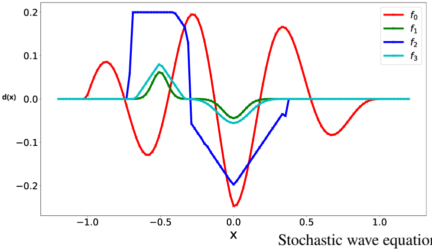
- Introduction
  - Motivation/inspiration
  - The model
- The inverse source problem in 1d
- Numerical experiments
- Conclusion and future work

# Example of numerical sources

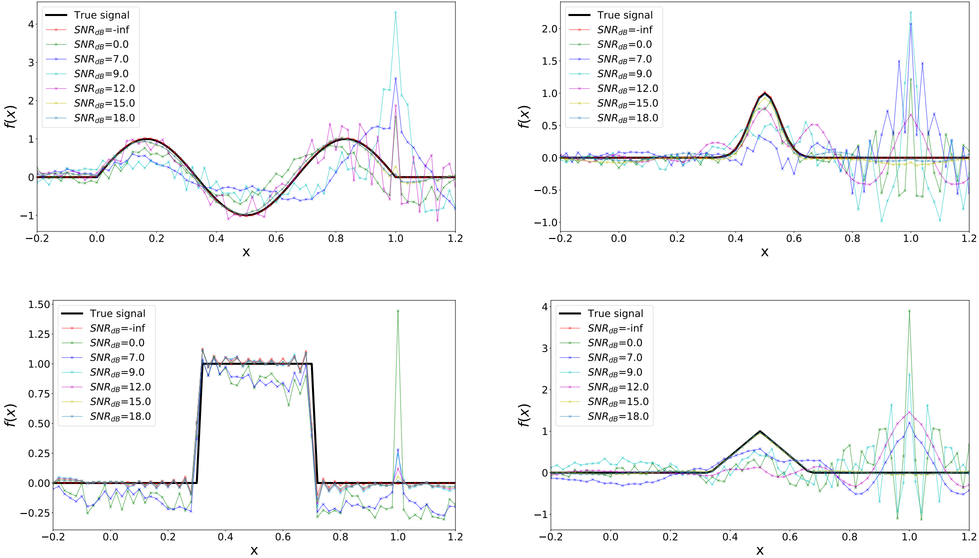
$$f_0(x) = \sin(3\pi x)\mathbb{1}_A, \qquad f_1(x) = e^{-\frac{1}{2}\left(\frac{x-1/2}{1/20}\right)^2}\mathbb{1}_A,$$

$$f_2(x) = \begin{cases} 1, & 0.3 < x < 0.7 \\ 0, & otherwise \end{cases}, \qquad f_3(x) = \begin{cases} 0, & x < 1/3 \\ 6\left(x - 1/3\right), & x \geq 1/3 \wedge x < 1/2 \\ -6\left(x - 2/3\right), & x \geq 1/2 \wedge x < 2/3, \\ 0, & x \geq 2/3 \end{cases}$$



# Reconstructions



- Introduction
  - Motivation/inspiration
  - The model
- The inverse source problem in 1d
- Numerical experiments
- **Conclusion and future work**

# Conclusion and future work

Conclusion:

- Proved non-uniqueness in the case of a source with mean zero, using results from phase retrieval in one dimension.
- Derived a reconstruction method and stability estimate where the source has a non-zero mean in one dimension.
- Numerical proof of concept using simulated data in one dimension.

Future work

- Extend to higher dimensions (in progress)
- Consider the stochastic setting in terms of initial conditions and time fluctuations (in progress)
- Possible real applications, e.g. within seismology and astronomy.

# References


[Aku57] Edwin J. Akutowicz. "On the Determination of the Phase of a Fourier Integral, II". In: *Proceedings of the American Mathematical Society* 8.2 (1957), pp. 234–238.

[BCJ01] N. Bleistein, J. Cohen, and John Jr. *Mathematics of Multidimensional Seismic Imaging, Migration, and Inversion*. Vol. 13. Jan. 2001. ISBN: 978-1-4612-6514-6.

[GP09] Josselin Garnier and George Papanicolaou. "Passive Sensor Imaging Using Cross Correlations of Noisy Signals in a Scattering Medium". In: *SIAM Journal on Imaging Sciences* 2.2 (2009), pp. 396–437.

[GP16] J. Garnier and G. Papanicolaou. *Passive Imaging with Ambient Noise*. Cambridge Monographs on Applied Mathematics. Cambridge University Press, 2016. ISBN: 9781107135635.

[Fou+18] Jacques Fournier et al. "Correlation-based imaging of fast moving objects using a sparse network of passive receivers". In: (2018), pp. 1618–1622.

[Mos+20] Miguel Moscoso et al. "The Noise Collector for sparse recovery in high dimensions". In: *Proceedings of the National Academy of Sciences* 117.21 (2020), pp. 11226–11232.

[CRS21] Michel Campillo, Philippe Roux, and Nikolai M. Shapiro. "Seismic, Ambient Noise Correlation". In: *Encyclopedia of Solid Earth Geophysics*. Ed. by Harsh K. Gupta. Cham: Springer International Publishing, 2021, pp. 1557–1562. ISBN: 978-3-030-58631-7.

[Gar21] Josselin Garnier. "Passive Communication with Ambient Noise". In: *SIAM Journal on Applied Mathematics* 81.3 (2021), pp. 814–833.

[Tou+21] Rita Touma et al. "A distortion matrix framework for high-resolution passive seismic 3-D imaging: application to the San Jacinto fault zone, California". In: *Geophysical Journal International* 226.2 (Mar. 2021), pp. 780–794.

# Wave equation

## Wave equation

$$\frac{1}{c^2(x)}\frac{\partial^2 u}{\partial t^2} - \Delta_x u = n(t,x), \quad x \in \boldsymbol{R}^n, \quad t \in \boldsymbol{R}^+ \quad *$$

## Green's function

$$\frac{1}{c^2(x)}\frac{\partial^2 G}{\partial t^2} - \Delta_x G = \delta(t)\delta(x-y), \quad x, y \in \boldsymbol{R}^n, \quad t \in \boldsymbol{R}$$

$$G(t,x,y) = 0, \quad \text{for all } t < 0$$

## Solution (Causal)

$$u(t,x) = \int_{-\infty}^{t}\int_{\boldsymbol{R}^n} G(t-s,x,y)n(s,y)\mathrm{d}y\mathrm{d}s$$

Extra

# Space-time stationary and "delta"-correlated noise, homogeneous medium

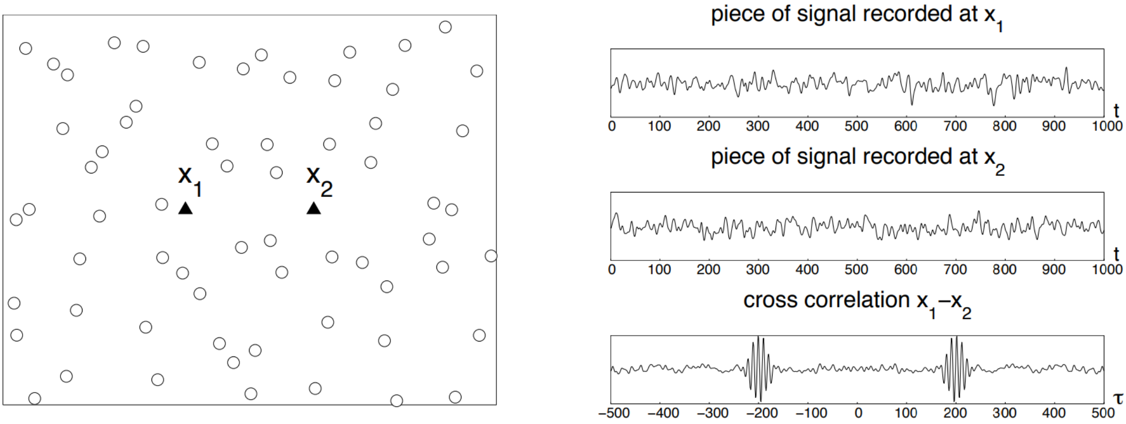



Image credit: [GP09]

# Travel time estimation

**Singular support**

Consider any distribution $f \in \mathcal{D}'(\Omega)$ for $\Omega \subseteq \boldsymbol{R}^n$, then we define singular support as:

$$\text{singsupp}(f) = \{x \in \Omega,\ f\ is\ C^\infty\ near\ x\}^c$$

**Derivative and singular support**

For any differential operator, $D$, and distribution, $f$, that;

$$\text{singsupp}\,(Df) \subseteq \text{singsupp}(f)$$