Edge-Preserving Tomographic Reconstruction with Uncertain View Angles

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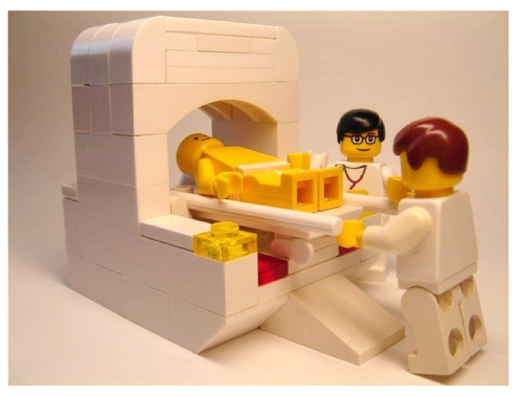
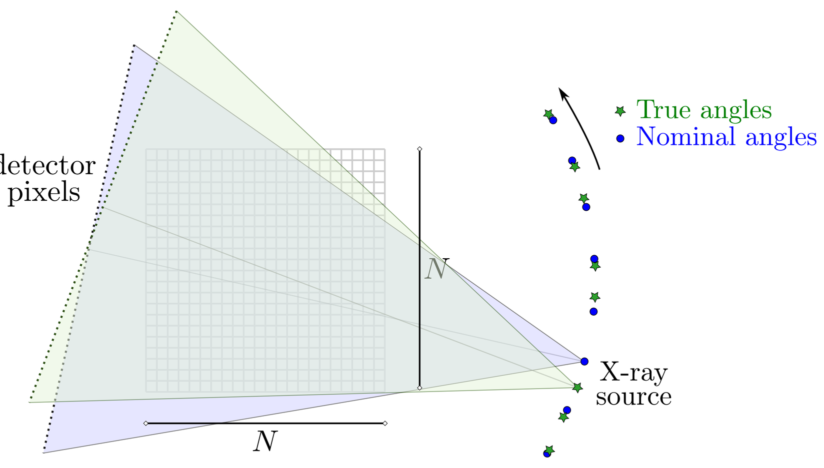
Setting the Stage – True and Nominal View Angles


Computed Tomography (CT)

CT data consist of measurements of the attenuation of X-rays passing through an object.
We reconstruct an image of the linear attenuation coefficient of the object's interior.
For each position of the X-ray source, we measure a set of data referred to as a **view**.

The **true view angles** may differ from the assumed **nominal view angles**:

- The model for the measured data is $\boldsymbol{b} = A_{\text{tru}}\, \boldsymbol{x} + \boldsymbol{e}$, where $\boldsymbol{e}$ is the measurement noise, $\boldsymbol{x}$ represents the image, and A_{tru} is the forward model for the *unknown* true angles.
- A "naive" and bad reconstruction uses the matrix A_{nom} based on the nominal angles.



How to Handle Uncertain View Angles – The Bayesian Framework

We consider the true view angles as unknowns $\boldsymbol{\theta}$, together with the image $\boldsymbol{x}$:

$$\text{find } (\boldsymbol{x}, \boldsymbol{\theta}) \text{ such that } \boldsymbol{b} = A(\boldsymbol{\theta})\, \boldsymbol{x} + \boldsymbol{e} \ .$$

Here, $A(\boldsymbol{\theta})$ denotes the forward model corresponding to the view angles $\boldsymbol{\theta}$.
We apply the Bayesian framework with a likelihood that involves both $\boldsymbol{x}$ and $\boldsymbol{\theta}$:

$$\pi_{\text{pos}}(\boldsymbol{x}, \boldsymbol{\theta}) \propto \pi_{\text{lik}}(\boldsymbol{b} \,|\, \boldsymbol{x}, \boldsymbol{\theta}) \times \pi_{\text{pri}}(\boldsymbol{x}) \times \pi_{\text{pri}}(\boldsymbol{\theta}) \ .$$

- The distribution of $\boldsymbol{e}$ is determined by the measurements; in CT it is log-Poisson and we approximate it by a Gaussian. Hence, $\pi_{\text{lik}}(\boldsymbol{b} \,|\, \boldsymbol{x}, \boldsymbol{\theta})$ is a **Gaussian**.
- For $\pi_{\text{pri}}(\boldsymbol{x})$ we use a **Laplace distribution of the differences of neighbour pixels** (enables sharp edges in the image; related to total variation (TV) regularization.
- For $\pi_{\text{pri}}(\boldsymbol{\theta})$ we use the **von Mises distribution** (i.e., a *periodic* normal distribution).

But wait, there's more. We introduce scalar **hyperparameters:** λ in the Gaussian likelihood, δ in the Laplace-difference prior for $\boldsymbol{x}$, and κ in the von Mises prior for $\boldsymbol{\theta}$.
All three have exponential distributions $\pi_{\text{hpri}}(\cdot) = \beta \, \exp(-\beta\, \cdot)$ with $\beta = 10^{-4}$. Thus, the posterior takes the form

$$\pi_{\text{pos}}(\boldsymbol{x}, \boldsymbol{\theta}, \lambda, \delta, \kappa) \propto \pi_{\text{lik}}(\boldsymbol{b} \,|\, \boldsymbol{x}, \boldsymbol{\theta}, \lambda) \times \pi_{\text{pri}}(\boldsymbol{x} \,|\, \delta) \times \pi_{\text{pri}}(\boldsymbol{\theta} \,|\, \kappa) \times \pi_{\text{hpri}}(\lambda)\, \pi_{\text{hpri}}(\delta) \times \pi_{\text{hpri}}(\kappa)$$

How to Sample – A Hybrid Gibbs Sampler

Performing statistical inference of the full posterior $\pi_{\text{pos}}(\boldsymbol{x}, \boldsymbol{\theta}, \lambda, \delta, \kappa)$ is challenging: the number of pixels n is large, the forward model $A(\boldsymbol{\theta})$ is nonlinear in the view angles $\boldsymbol{\theta}$, and the prior $\pi_{\text{pri}}(\boldsymbol{x} \,|\, \delta)$ is nondifferentiable due to the 1-norm. We split the posterior and apply different samplers for each parameter, hence the sampler is hybrid.

$$\pi_1(\boldsymbol{x} \,|\, \boldsymbol{\theta}, \lambda, \delta) \propto \exp\left(-\frac{\lambda}{2} \|A(\boldsymbol{\theta})\, \boldsymbol{x} - \boldsymbol{b}\|_2^2 - \delta \big(\|(\boldsymbol{I} \otimes \boldsymbol{D})\, \boldsymbol{x}\|_1 + \|(\boldsymbol{D} \otimes \boldsymbol{I})\, \boldsymbol{x}\|_1 \big) \right)$$

$$\pi_2(\boldsymbol{\theta} \,|\, \boldsymbol{x}, \lambda, \kappa) \propto \exp\left(-\frac{\lambda}{2} \|A(\boldsymbol{\theta})\, \boldsymbol{x} - \boldsymbol{b}\|_2^2 + \kappa \mathbf{1}^{\mathsf{T}} \cos(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}) \right)$$

$$\pi_3(\lambda \,|\, \boldsymbol{x}, \boldsymbol{\theta}) \propto \lambda^{m/2} \exp\left(-\lambda \left[\frac{1}{2} \|A(\boldsymbol{\theta})\, \boldsymbol{x} - \boldsymbol{b}\|_2^2 + \beta \right] \right)$$

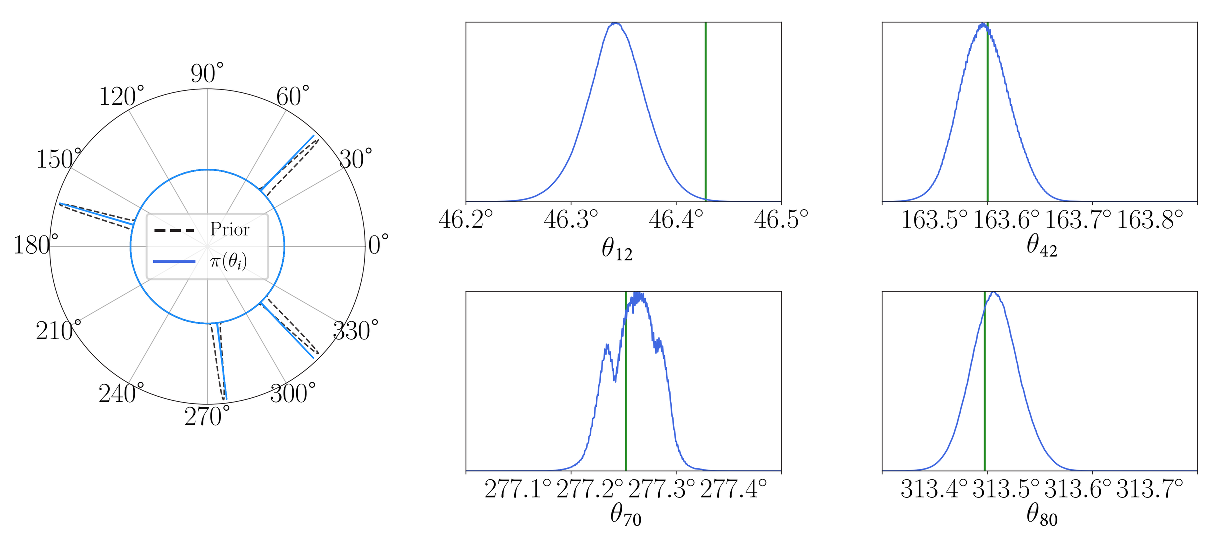
$$\pi_4(\delta \,|\, \boldsymbol{x}) \propto \delta^n \exp\left(-\delta \big[\|(\boldsymbol{I} \otimes \boldsymbol{D})\, \boldsymbol{x}\|_1 + \|(\boldsymbol{D} \otimes \boldsymbol{I})\, \boldsymbol{x}\|_1 + \beta \big] \right)$$

$$\pi_5(\kappa \,|\, \boldsymbol{\theta}) \propto I_0(\kappa)^{-p} \exp\left(-\kappa \left[-\mathbf{1}^{\mathsf{T}} \cos(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}) + \beta \right] \right)$$

$\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{\theta} \in \mathbb{R}^p$, I_0 = 0-order mod. Bessel funct., $\boldsymbol{D} = \text{bidiag}(-1, 1)$, $\bar{\boldsymbol{\theta}}$ = nominal angles.

Initial states $\boldsymbol{x}^{(0)}, \boldsymbol{\theta}^{(0)}, \lambda^{(0)}, \delta^{(0)}, \kappa^{(0)}$
For $j = 1, 2, \ldots, N_{\text{samp}}$
  Sample attenuation coefficients
    $\boldsymbol{x}^{(j)} \sim \pi_1(\cdot \,|\, \boldsymbol{\theta}^{(j-1)}, \lambda^{(j-1)}, \delta^{(j-1)})$
  Sample view angles
    $\boldsymbol{\theta}^{(j)} \sim \pi_2(\cdot \,|\, \boldsymbol{x}^{(j)}, \lambda^{(j-1)}, \kappa^{(j-1)})$
  Sample hyperparameters
    $\lambda^{(j)} \sim \pi_3\left(\cdot \,|\, \boldsymbol{x}^{(j)}, \boldsymbol{\theta}^{(j)}\right), \quad \delta^{(j)} \sim \pi_4\left(\cdot \,|\, \boldsymbol{x}^{(j)}\right), \quad \kappa^{(j)} \sim \pi_5\left(\cdot \,|\, \boldsymbol{\theta}^{(j)}\right)$
End

Results – View Angles



Left: von Mises prior with the respective densities for selected angles in $\boldsymbol{\theta}$.
Right: some component densities and true angles shown as vertical green lines.

Some Details of the Algorithm

π_1: non-differentiable due to $\|\cdot\|_1$ and nonlinear in $\boldsymbol{\theta}$. Use *Laplace's approximation*, i.e., a Gaussian $\pi_G = \mathcal{N}(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{H}^{-1})$ with $\boldsymbol{H}(\boldsymbol{x}^{(j-1)}) \approx$ Hessian of $-\log \pi_1$ and

$$\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{x}) = \lambda\, \boldsymbol{H}^{-1}(\boldsymbol{x})\, A(\boldsymbol{\theta})^{\mathsf{T}} \boldsymbol{b} = \text{MAP estimator of } \pi_1,$$

Much easier to work with a Gaussian but we miss the heavy tails of π_1. We use 10 CGLS iterations to compute the LS solution that gives the sample $\boldsymbol{x}^{(j)}$.

π_2: samples from π_2 are drawn sequentially by componentwise Metropolis:

$$\theta_1^{[k+1]} \sim \pi_2\left(\boldsymbol{\theta} \,|\, \boldsymbol{x}, \lambda, \kappa, \left[\theta_2^{[k]}, \theta_3^{[k]}, \ldots, \theta_p^{[k]}\right] \right),$$

$$\theta_2^{[k+1]} \sim \pi_2\left(\boldsymbol{\theta} \,|\, \boldsymbol{x}, \lambda, \kappa, \left[\theta_1^{[k+1]}, \theta_3^{[k]}, \ldots, \theta_p^{[k]}\right] \right),$$

$$\vdots$$

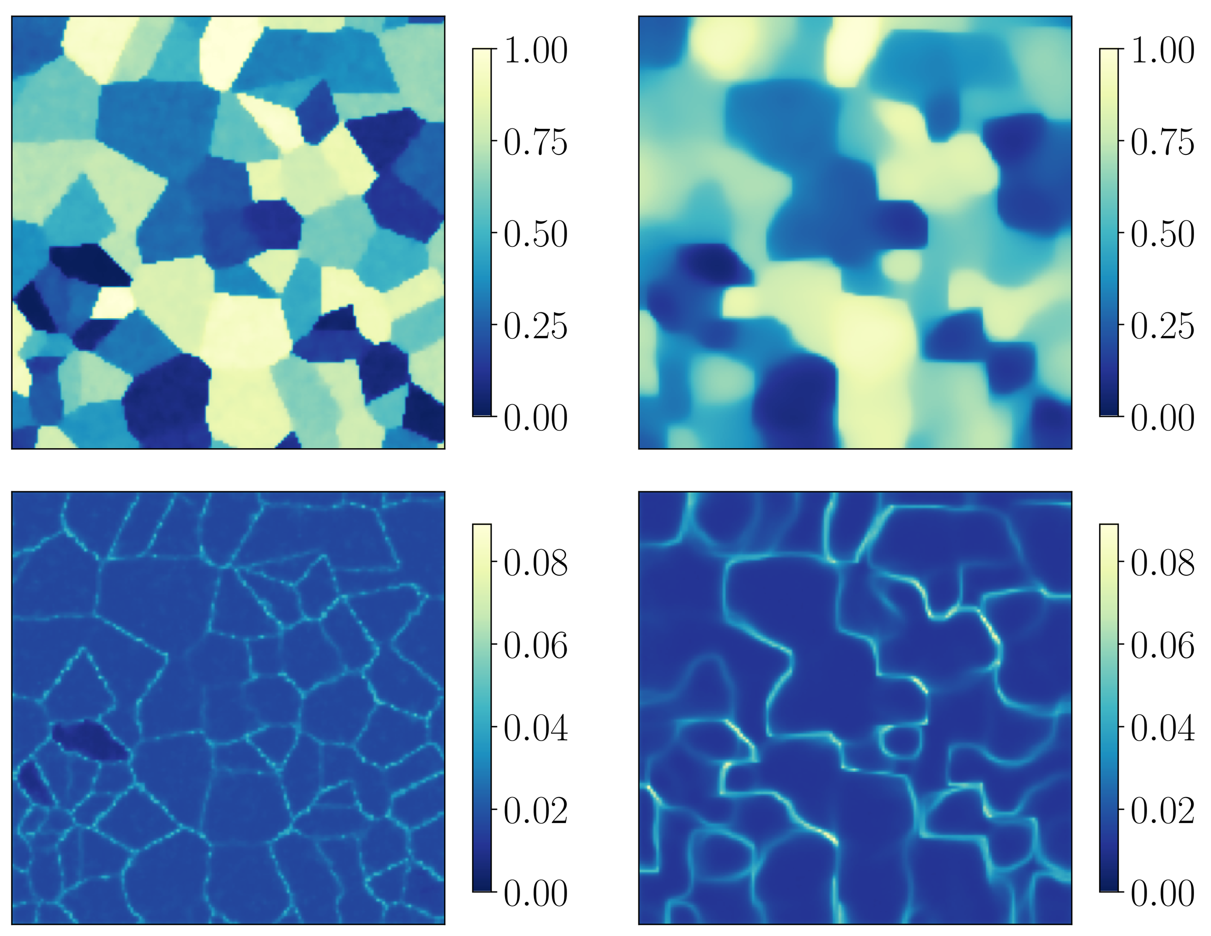
$$\theta_p^{[k+1]} \sim \pi_2\left(\boldsymbol{\theta} \,|\, \boldsymbol{x}, \lambda, \kappa, \left[\theta_1^{[k+1]}, \theta_2^{[k+1]}, \ldots, \theta_{p-1}^{[k+1]}\right] \right).$$

After 20 cycles we obtain $\boldsymbol{\theta}^{(j)} = \boldsymbol{\theta}^{[20]}$.

π_3 and π_4: can be written and approximated, respectively, in closed form.

π_5: sampled with standard random-walk Metropolis.

Results – Metallic Grains Phantom



Left: our method.
Right: using the incorrect nominal angles.
Top: posterior mean
Bottom: st. dev.

Nominal angles give a blurry image with uncertain boundaries.

We compute a sharper image with uncertainty confined to pixels on grain boundaries.

Appendix – Definition of Priors

$$\pi_{\text{pri}}(\boldsymbol{x} \,|\, \delta) = \left(\frac{\delta}{2} \right)^n \exp\big(-\delta (\|(\boldsymbol{I} \otimes \boldsymbol{D})\, \boldsymbol{x}\|_1 + \|(\boldsymbol{D} \otimes \boldsymbol{I})\, \boldsymbol{x}\|_1) \big)$$

$$\pi_{\text{pri}}(\boldsymbol{\theta} \,|\, \kappa) = \left(\frac{1}{2\pi\, I_0(\kappa)} \right)^p \exp\left(\kappa\, \mathbf{1}^{\mathsf{T}} \cos(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}) \right)$$

Reference and Funding

F. Uribe, J. M. Bardsley, Y. Dong, P. C. Hansen, N. Riis, *A hybrid Gibbs sampler for edge-preserving tomographic reconstruction with uncertain view angles*, SIAM/ASA J. Uncertain. Quantific., 10 (2022), pp. 1293–1320, doi: 10.1137/21M1412268.

A part of the project Computational Uncertainty Quantification for Inverse problems (CUQI), funded by Villum Investigator grant no. 25893 from The Villum Foundation.