Reduced order model approach for imaging with waves

Josselin Garnier (Ecole polytechnique)

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in collaboration with L. Borcea (Univ. Michigan), A. Mamonov (Univ. Houston), J. Zimmerling (Uppsala Univ.).

Velocity estimation problem

- *Direct problem:* Given the velocity map $c = (c(x))_{x\in\Omega}$ compute the wavefield solution of the wave equation

$$[\partial_t^2 - c^2(x)\Delta]p^{(s)}(t,x) = f(t)\delta(x - x_s), \quad t \in \mathbb{R}, x \in \Omega,$$

starting from $p^{(s)}(t,x) = 0$, $t \ll 0$, + boundary conditions at $\partial\Omega$.
At the locations of the receivers:

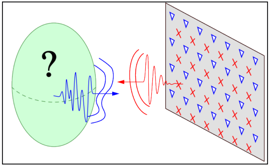
$$d_{r,s}(t) = p^{(s)}(t, x_r), \quad r, s = 1, .., N$$

$\hookrightarrow$ forward map

$$\mathcal{D} : c \mapsto \mathbf{d}$$

where $\mathbf{d} = ((d_{r,s}(t))_{r,s=1}^N)_{t\in[t_{\min},t_{\max}]}$, is the array response matrix.

- *Inverse problem:*
Given the time-resolved measurements $\mathbf{d}$,
determine the velocity map c.



Full Waveform Inversion (FWI)

- FWI fits data with its model prediction in L^2-norm:

$$\hat{c} = \underset{c}{\operatorname{argmin}} \left\{ \mathcal{O}_{FWI}[c] + \operatorname{Reg}[c] \right\},$$

$$\mathcal{O}_{FWI}[c] = \|\mathbf{d}_{meas} - \mathcal{D}[c]\|_2^2 = \sum_{r,s=1}^{N} \int_{t_{\min}}^{t_{\max}} |d_{meas}(t)_{r,s} - \mathcal{D}[c](t)_{r,s}|^2 dt$$

  Cf [Virieux and Operto 2009].
- Bayesian interpretation: Maximum A Posteriori (with Gaussian additive noise).
- The objective function $\mathcal{O}_{FWI}[c]$ is not convex in c.
  $\hookrightarrow$ optimization needs hard to get good initial guess.

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  Cf [Virieux and Operto 2009].
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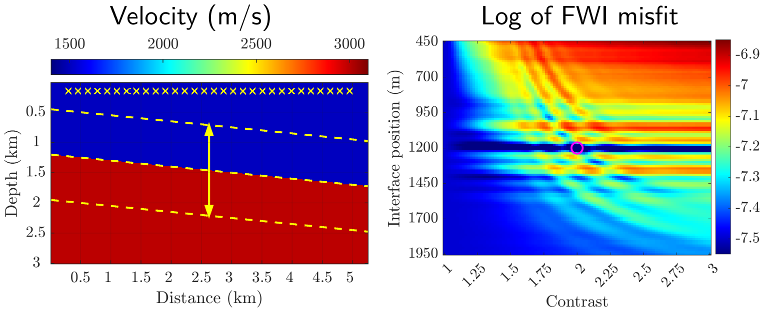
- The objective function $\mathcal{O}_{FWI}[c]$ is not convex in c.
  $\hookrightarrow$ optimization needs hard to get good initial guess.

- Layer stripping: Proceed hierarchically from the shallow part to the deep part [Wang et al. 2009]
- Frequency hopping: Successive inversion of subdata sets of increasing high-frequency content [Bunks et al. 1995]
- Optimal transport: Wasserstein distance instead of L^2-norm [Engquist et al. 2016, Yang et al. 2018, Métivier et al. 2019]

Topography of the FWI objective function



- Probing pulse is a modulated Gaussian pulse with central frequency $6Hz$ and bandwidth $4Hz$ ($\lambda \simeq 300m$ at $10Hz$).
- $N = 30$ sensors and $N_{\mathrm{t}} = 39$ time samples at interval $\tau = 0.0435$s.
- Search velocity has two parameters: the bottom velocity and depth of the interface (the angle and top velocity are known).
- Objective function:

$$\mathcal{O}_{FWI}[c] = \|\mathbf{d}_{meas} - \mathcal{D}[c]\|_2^2$$

$\hookrightarrow$ Cycle skipping issues (many local minima).

Main objective

- Our main objective: find a convex formulation of FWI.

Proposed approach: Find a nonlinear mapping $\mathcal{R}$: data $\mathbf{d} \mapsto$ reduced order model (ROM) of wave operator $\mathbf{A}^{rom}$ (matrix) such that:

- ROM is computed with efficient numerical linear algebra tools in non-iterative fashion.
- Minimization of ROM misfit is better for velocity estimation.

We can think of the data to ROM mapping $\mathcal{R}$ as a preconditioner of the forward mapping $\mathcal{D}$

$$c \overset{\mathcal{D}}{\mapsto} \mathbf{d} \overset{\mathcal{R}}{\mapsto} \mathbf{A}^{rom}$$

because the composition $\mathcal{R} \circ \mathcal{D}$, which gives $\mathbf{A}^{rom} = \mathcal{R}(\mathcal{D}[c])$, is easier to "invert".

Towards the ROM objective function

- *Ideal objective function 1:*

$$\mathcal{O}[c] = \|c - c^{meas}\|^2$$

but c^{meas} is not observed (i.e., cannot be extracted from $\mathbf{d}_{meas}$) !

Towards the ROM objective function

- Let us consider the wave operator

$$\mathcal{A}[c] = -c(x)\Delta\big[c(x)\,\cdot\,\big]$$

- Galerkin method to approximate the operator $\mathcal{A}$ by a matrix:

- consider a space of (piecewise polynomial) functions with basis $(\Psi_l(x))_{l=1}^L$,
- consider the row vector field $\boldsymbol{\Psi}(x) = (\Psi_1(x), \dots, \Psi_L(x))$ and define:

$$\mathbf{A}^\Psi = \int_\Omega dx\, \boldsymbol{\Psi}(x)^T \mathcal{A} \boldsymbol{\Psi}(x) \in \mathbb{R}^{L\times L}$$

- *Ideal objective function 2:*

$$\mathcal{O}[c] = \|\mathbf{A}^\Psi[c] - \mathbf{A}^{\Psi,meas}\|_2^2$$

but $\mathbf{A}^{\Psi,meas}$ is not observed !

The ROM matrix

- Our Galerkin approximation space:

- consider a time discretization $\{t_j = j\tau\}_{0\leq j\leq N_{\rm t}}$ with uniform stepping τ,
- gather the waves $p^{(s)}(t,x)$ evaluated at $t = t_j$ for all the N sources:

$$\boldsymbol{p}_j(x) = \Big(p^{(1)}(t_j,x),\dots,p^{(N)}(t_j,x)\Big)\,, \quad x \in \Omega.$$

(note: apply first a linear preprocessing).

- organize the first $N_{\rm t}$ snapshots in the $NN_{\rm t}$ dimensional row vector field:

$$\boldsymbol{U}(x) = \left(\boldsymbol{p}_0(x),\dots,\boldsymbol{p}_{N_{\rm t}-1}(x)\right)\,, \quad x \in \Omega.$$

- apply Gram-Schmidt orthogonalization onto $\boldsymbol{U}(x) = \boldsymbol{V}(x)\mathbf{R}$.

- Define ROM matrix:

$$\mathbf{A}^{rom} = \int_\Omega dx\, \boldsymbol{V}(x)^T \mathcal{A} \boldsymbol{V}(x) \in \mathbb{R}^{NN_{\rm t}\times NN_{\rm t}}.$$

The ROM matrix

- Our Galerkin approximation space:

- consider a time discretization $\{t_j = j\tau\}_{0 \le j \le N_t}$ with uniform stepping τ,
- gather the waves $p^{(s)}(t,x)$ evaluated at $t = t_j$ for all the N sources:

$$\boldsymbol{p}_j(x) = \Big(p^{(1)}(t_j,x), \dots, p^{(N)}(t_j,x) \Big) , \quad x \in \Omega.$$

(note: apply first a linear preprocessing).

- organize the first N_t snapshots in the NN_t dimensional row vector field:

$$\boldsymbol{U}(x) = \left(\boldsymbol{p}_0(x), \dots, \boldsymbol{p}_{N_t-1}(x) \right) , \quad x \in \Omega.$$

- apply Gram-Schmidt orthogonalization onto $\boldsymbol{U}(x) = \boldsymbol{V}(x)\mathbf{R}$.

- Define ROM matrix:

$$\mathbf{A}^{rom} = \int_\Omega dx \, \boldsymbol{V}(x)^T \mathcal{A} \boldsymbol{V}(x) \in \mathbb{R}^{NN_t \times NN_t}.$$

- *Ideal objective function 3:*

$$\mathcal{O}[c] = \| \mathbf{A}^{rom}[c] - \mathbf{A}^{rom,meas} \|_2^2$$

but $\mathbf{A}^{rom,meas}$ is not observed (neither $\mathcal{A}$ nor $\boldsymbol{V}(x)$ is observed) !

The ROM matrix

- Our Galerkin approximation space:

- consider a time discretization $\{t_j = j\tau\}_{0\leq j\leq N_{\rm t}}$ with uniform stepping τ,
- gather the waves $p^{(s)}(t,x)$ evaluated at $t = t_j$ for all the N sources:

$$\boldsymbol{p}_j(x) = \Big(p^{(1)}(t_j,x),\dots,p^{(N)}(t_j,x)\Big)\,,\quad x\in\Omega.$$

(note: apply first a linear preprocessing).
- organize the first $N_{\rm t}$ snapshots in the $NN_{\rm t}$ dimensional row vector field:

$$\boldsymbol{U}(x) = \left(\boldsymbol{p}_0(x),\dots,\boldsymbol{p}_{N_{\rm t}-1}(x)\right)\,,\quad x\in\Omega.$$

- apply Gram-Schmidt orthogonalization onto $\boldsymbol{U}(x) = \boldsymbol{V}(x)\mathbf{R}$.

- Define ROM matrix:

$$\mathbf{A}^{rom} = \int_\Omega dx\, \boldsymbol{V}(x)^T\mathcal{A}\boldsymbol{V}(x) \in \mathbb{R}^{NN_{\rm t}\times NN_{\rm t}}.$$

- **Main result**: *The ROM matrix $\mathbf{A}^{rom}$ can be extracted from the measurements* $\mathbf{d}$, *without knowing $\mathcal{A}$ nor $\boldsymbol{V}(x)$.*
$\hookrightarrow \mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}^{rom,meas}\|_2^2$ *is a legitimate objective function.*

First step: *Linear preprocessing*.

- Define the new data matrix $\mathbf{D}(t)$:

$$\mathbf{D}(t) = \mathbf{d}^f(t) + \mathbf{d}^f(-t), \quad \text{with } \mathbf{d}^f(t) = -f'(-t) *_t \mathbf{d}(t).$$

Second step: *Expression of the new data entries as wave correlations*.

- Introduce the solution $u^{(s)}(t,x)$ of the homogeneous wave equation

$$(\partial_t^2 + \mathcal{A})u^{(s)}(t,x) = 0, \qquad t > 0, \quad x \in \Omega,$$

with boundary conditions on $\partial\Omega$, with initial state

$$u^{(s)}(0,x) = u_0^{(s)}(x) = \left|\hat{f}(\sqrt{\mathcal{A}})\right| \delta(x - x_s), \qquad \partial_t u^{(s)}(0,x) = 0.$$

It has the form

$$u^{(s)}(t,x) = \cos\left(t\sqrt{\mathcal{A}}\right) u_0^{(s)}(x).$$

$\rightarrow$ The entries of $\mathbf{D}(t)$ can be expressed as wave correlations:

$$D_{r,s}(t) = \int_\Omega dx\, u_0^{(r)}(x) u^{(s)}(t,x).$$

Third step: *Definition of the ROM.*
Let $\tau > 0$ be fixed.

- Gather the snapshots for all the N sources in the row vector fields

$$\boldsymbol{u}_j(x) = \Big(u^{(1)}(j\tau, x), \dots, u^{(N)}(j\tau, x) \Big) , \qquad 0 \leq j \leq N_{\rm t} .$$

- Organize the first $N_{\rm t}$ snapshots in the $NN_{\rm t}$ dimensional row vector field:

$$\boldsymbol{U}(x) = \left(\boldsymbol{u}_0(x), \dots, \boldsymbol{u}_{N_{\rm t}-1}(x) \right) , \quad x \in \Omega.$$

- Apply Gram-Schmidt orthogonalization onto $\boldsymbol{U}(x) = \boldsymbol{V}(x)\mathbf{R}$.

(note: we have $\int_\Omega dx\, \boldsymbol{V}(x)^T \boldsymbol{V}(x) = \mathbf{I}_{NN_{\rm t}}$).

- Define

$$\mathbf{A}^{rom} = \int_\Omega dx\, \boldsymbol{V}(x)^T \mathcal{A} \boldsymbol{V}(x)$$

Fourth step: *Expression of the ROM in terms of mass and stiffness*.

- Define the $NN_{\rm t} \times NN_{\rm t}$ "mass" and "stiffness" matrices:

$$\mathbf{M} = \int_\Omega dx\, \boldsymbol{U}^T(x)\boldsymbol{U}(x), \qquad \mathbf{S} = \int_\Omega dx\, \boldsymbol{U}^T(x)\mathcal{A}\boldsymbol{U}(x)$$

- Since $\boldsymbol{U}(x) = \boldsymbol{V}(x)\mathbf{R}$, we get

$$\begin{aligned}\mathbf{M} &= \mathbf{R}^T \int_\Omega dx\, \boldsymbol{V}^T(x)\boldsymbol{V}(x)\mathbf{R} \\ &= \mathbf{R}^T\mathbf{R}\end{aligned}$$

and

$$\begin{aligned}\mathbf{A}^{rom} &= \int_\Omega dx\, \boldsymbol{V}(x)^T\mathcal{A}\boldsymbol{V}(x) = \mathbf{R}^{-T}\int_\Omega dx\, \boldsymbol{U}(x)^T\mathcal{A}\boldsymbol{U}(x)\mathbf{R} \\ &= \mathbf{R}^{-T}\mathbf{S}\mathbf{R}\end{aligned}$$

$\hookrightarrow$ $\mathbf{A}^{rom}$ can be expressed in terms of $\mathbf{M}$ and $\mathbf{S}$.

Fifth step: *Expression of the ROM in terms of data.*
The $N \times N$ blocks of the mass matrix $\mathbf{M}$ are

$$
\begin{aligned}
\mathbf{M}_{i,j} &= \langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle_{L^2(\Omega)} = \langle \cos\big(i\tau\sqrt{\mathcal{A}}\big)\boldsymbol{u}_0, \cos\big(j\tau\sqrt{\mathcal{A}}\big)\boldsymbol{u}_0 \rangle_{L^2(\Omega)} \\
&= \langle \boldsymbol{u}_0, \cos\big(i\tau\sqrt{\mathcal{A}}\big)\cos\big(j\tau\sqrt{\mathcal{A}}\big)\boldsymbol{u}_0 \rangle_{L^2(\Omega)} \\
&= \frac{1}{2}\langle \boldsymbol{u}_0, \big[\cos\big((i+j)\tau\sqrt{\mathcal{A}}\big) + \cos\big(|i-j|\tau\sqrt{\mathcal{A}}\big)\big]\boldsymbol{u}_0 \rangle_{L^2(\Omega)} \\
&= \frac{1}{2}\langle \boldsymbol{u}_0, \boldsymbol{u}_{i+j} + \boldsymbol{u}_{|i-j|} \rangle_{L^2(\Omega)} \\
&= \frac{1}{2}\Big(\mathbf{D}_{i+j} + \mathbf{D}_{|i-j|}\Big), \quad 0 \le i, j \le N_{\rm t} - 1.
\end{aligned}
$$

Idem for the stiffness matrix **S**.
$\hookrightarrow$ **M** and **S** can be expressed in terms of the data **D**.

Algorithm for data-driven ROM matrix

Input: The matrices $\mathbf{d}(t) = (d_{r,s}(t))_{r,s=1}^{N}$ of measurements.

1. Compute $d^f_{r,s}(t) = -f'(-t) *_t d_{r,s}(t)$ and
$$\mathbf{D}_j = \mathbf{d}^f(j\tau) + \mathbf{d}^f(-j\tau), \quad 0 \le j \le 2N_\mathrm{t} - 2.$$
2. Compute $\ddot{\mathbf{D}}_j = \ddot{\mathbf{f}}(j\tau) + \ddot{\mathbf{f}}(-j\tau)$, for $j = 0, \dots, 2N_\mathrm{t} - 2$ with $\ddot{f}_{r,s}(t) = f''_{r,s}(t)$ using, e.g., the Fourier transform.
3. Calculate $\mathbf{M}, \mathbf{S} \in \mathbb{R}^{NN_\mathrm{t} \times NN_\mathrm{t}}$ with the block entries
$$\mathbf{M}_{i,j} = \frac{1}{2}(\mathbf{D}_{i+j} + \mathbf{D}_{|i-j|}) \in \mathbb{R}^{N\times N},$$
$$\mathbf{S}_{i,j} = -\frac{1}{2}(\ddot{\mathbf{D}}_{i+j} + \ddot{\mathbf{D}}_{|i-j|}) \in \mathbb{R}^{N\times N},$$

for $0 \le i, j \le N_\mathrm{t} - 1$.

4. Perform block Cholesky factorization $\mathbf{M} = \mathbf{R}^T\mathbf{R}$.

Output: $\mathbf{A}^{rom} = \mathbf{R}^{-T}\mathbf{S}\mathbf{R}^{-1}$.

ROM objective function

- ROM misfit function:

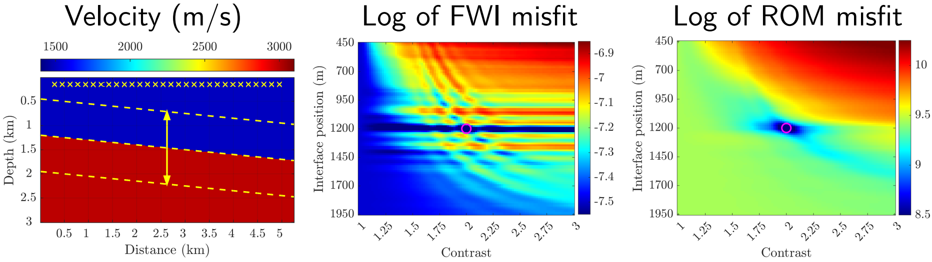
$$\mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}^{rom,meas}\|_2^2$$

  where $\mathbf{A}^{rom}[c]$ is computed from $\mathcal{D}[c]$ and $\mathbf{A}^{rom,meas}$ is computed from $\mathbf{d}_{meas}$.

- For a rich enough space of snapshots, the ROM matrix $\mathbf{A}^{rom}$ contains roughly the same information as $\mathcal{A} = -c(x)\Delta\big[c(x)\cdot\big]$.
  $\hookrightarrow$ The ROM misfit function should have nice convexity properties.

- Conjecture: "rich enough" means for sensors all around the domain of interest, separated by roughly half a wavelength, for time sampling satisfying the Nyquist criterium.
  $\hookrightarrow$ Conjecture proved only in special situations.

Topographies of the FWI and ROM objective functions



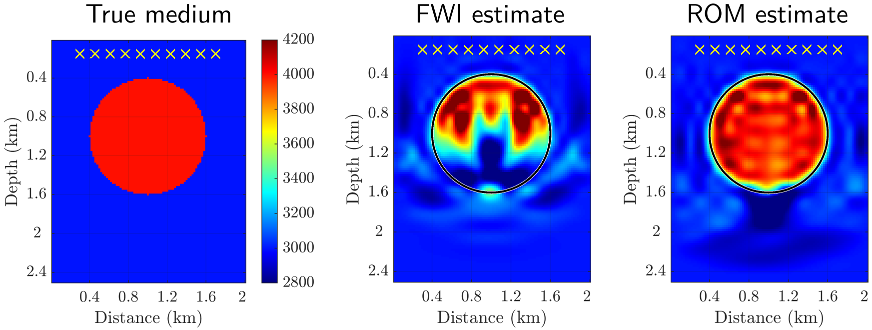
- Search velocity has two parameters: the contrast and the depth of the interface (the angle and top velocity are known).
- FWI objective function:

$$\mathcal{O}_{FWI}[c] = \|\mathcal{D}[c] - \mathbf{d}^{meas}\|_2^2$$

- ROM objective function:

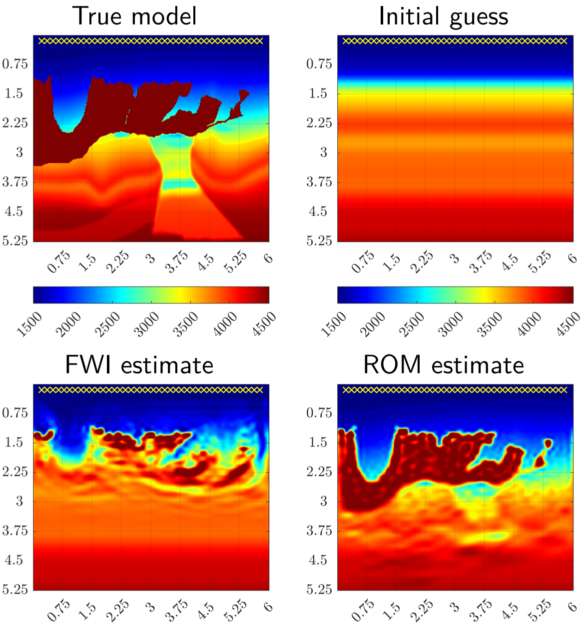
$$\mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}^{rom,meas}\|_2^2$$

Camembert model



- Probing pulse is a modulated Gaussian pulse with central frequency $6Hz$ and bandwidth $4Hz$ ($\lambda = 300m$ at $10Hz$).
- Search velocity: $c(x, \boldsymbol{\eta}) = c_o + \sum_l \eta_l \phi_l(x)$, $\boldsymbol{\eta} = (\eta_l)_{l=1}^L$.
- $\phi_l(x)$ are Gaussian peaks with centers on a regular grid, $L = 400$, with width 60m (0.2λ).
- FWI minimizes $\mathcal{O}_{FWI}(\boldsymbol{\eta}) = \|\mathcal{D}[c(\boldsymbol{\eta})] - \mathbf{d}^{meas}\|_2^2 + \mu\|\boldsymbol{\eta}\|_2^2$
- ROM minimizes $\mathcal{O}_{ROM}(\boldsymbol{\eta}) = \|\mathbf{A}^{rom}[c(\boldsymbol{\eta})] - \mathbf{A}^{rom,meas}\|_2^2 + \mu\|\boldsymbol{\eta}\|_2^2$

Salt body (BP - model)



A limitation and an extension to passive imaging

- One limitation of the method:
  We need co-located sources and receivers.
- Extension to passive imaging:
  Consider a receiver array recording signals transmitted by noise sources.
  Compute the cross correlation matrix of the recorded signals.
  $\rightarrow$ The cross correlation matrix is related to the Green's function (virtual active array) [Shapiro et al. 2005; Garnier et al. 2016].
  $\rightarrow$ The ROM procedure is natural in the passive framework, since the cross correlation matrix gives directly the data matrix $\mathbf{D}(t)$.
  - the virtual sources and receivers are naturally co-located,
  - the signals are even (because cross correlations are even).

Conclusions

- The ROM is an approximation of the wave operator on a space defined by the snapshots of the wavefield at uniformly spaced time instants.
- This space is not known and neither is the wave operator.
- Yet, we can compute the ROM from the data !
- We can then formulate a velocity estimation algorithm that minimizes the ROM misfit and that avoids cycle skipping and other problems.
- Other formulations: See also

L Borcea, J Garnier, AV Mamonov, J Zimmerling, When data driven reduced order modeling meets waveform inversion, arXiv:2302.05988

L Borcea, J Garnier, AV Mamonov, J Zimmerling, Waveform inversion with a data driven estimate of the internal wave, SIAM Journal on Imaging Sciences 16 (1), 2023, 280-312.

L Borcea, J Garnier, AV Mamonov, J Zimmerling, Waveform inversion via reduced order modeling, Geophysics 88 (2), 2023, R175-R191.