Inverse moving point source problem for the wave equation

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Collaboration

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Plan

Stationary Sources

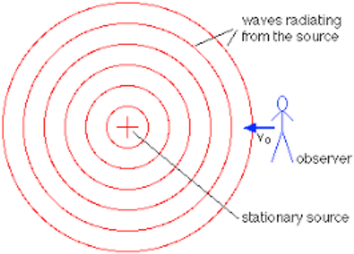
1) Motivation
2) Forward problem
3) Inverse problem
4) Numerical results

Motivation

Motivation for the study of this problem is provided by

- medical imaging: it is desired to use the measurement of the radiated field on the surface of the human brain to infer the abnormalities inside the brain, which produced the measured field
- antenna synthesis: the problem is to find the current distribution along a linear antenna which produces the desired radiated field
- ...

The forward problem



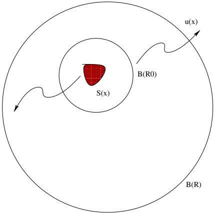
Let u be the wave generated by a stationary source S.

$$\begin{cases} c^{-2}\partial_t^2 u - \Delta u = 0, \\ u(0^+, x) = 0, \quad x \in \boldsymbol{R}^3, \\ \partial_t u(0^+, x) = -S(x) \in C_0(\boldsymbol{R}^3), \quad x \in \boldsymbol{R}^3. \end{cases} \tag{1}$$

The inverse problem

The Fourier transform
of u (still denoted u) satisfies the Helmholtz equation

$$\begin{cases} \Delta u + k^2 u = S(x), \\ \partial_{|x|} u - iku = o(|x|^{-1}), \end{cases} \quad (2)$$



where $k = \frac{\omega}{c}$ is the frequency of the radiated field u.

Let $\Omega = B_R(0)$,
assume that $S(x) \in L^2(\Omega)$, has a compact
support in $B_{R_0}(0) \subset \Omega$, and $\mathrm{dist}\left(B_{R_0}, \partial\Omega\right) > d_0$.

Let $\nu(x)$ be the outward normal derivative at $x \in \partial\Omega$,
and $\Gamma \subset \partial\Omega$ a non-empty connected open subset of $\partial\Omega$.

The inverse problem is to determine the source $S(x)$ that generates the field $(u|_\Gamma, \partial_\nu u|_\Gamma)$ for $k \in [0, k_0]$.

The inverse problem

Define the Green function in the whole space by

$$\Phi(k, r) = \frac{e^{ikr}}{4\pi r}.$$

There is a unique solution u to the system (2)

$$u(x) = \int_{\boldsymbol{R}^3} \Phi(k, |x-y|) S(y) dy.$$

The functions u and $\partial_\nu u$ on Γ determine uniquely u in the whole space $\boldsymbol{R}^3$. For simplicity we further assume that $\Gamma = \partial\Omega$.

The inverse problem

For a fixed wavenumber k, we define the radiation operators
$L_k^{(1)},\ L_k^{(2)}: \quad L^2(B_{R_0}) \to L^2(\Gamma)$ as

$$\begin{aligned} L_k^{(1)}(S)(x) &= \int_{B_{R_0}} \Phi(k, |x-y|)S(y)dy, \quad \text{for } x \in \Gamma \\ L_k^{(2)}(S)(x) &= \int_{B_{R_0}} \partial_{\nu(x)}\Phi(k, |x-y|)S(y)dy, \quad \text{for } x \in \Gamma. \end{aligned}$$

The inverse problem is to find $S(x) \in L^2(B_{R_0})$ such that

$$\begin{cases} L_k^{(1)}(S)(x) = u(x) & \text{for } x \in \Gamma, \\ L_k^{(2)}(S)(x) = \partial_\nu u(x) & \text{for } x \in \Gamma, \end{cases} \tag{3}$$

for all $k \in [0, k_0]$.

The inverse problem

It should be pointed out that the inverse problem with a single frequency data is in general not well-posed:

(i) The solution to the inverse problem at a fixed frequency is in general not unique. This is due to the existence of the non-radiating source, whose radiating field vanishes identically outside the support volume B_{R_0}.

(ii) The problem is severely ill-posed for low frequencies, that is, an infinitesimal noise in the measurement will give rise to large errors in the reconstruction solution.
In fact, it can be shown that the singular values of forward maps $L_k^{(1)}$ and $L_k^{(2)}$ decay exponentially for low k.

Meanwhile in [Bao-Liu-T21][1] we showed that if $S(x) = \sum_{j=1}^{J} s_j \delta_{x_j}(x)$ and $c = c(x) > c_0$, the inverse problem can be well posed (Lipschitz stability estimate with small Lipschitz constant if the geometry is favorable).

[1] G. Bao, Y. Liu, F. Triki. Recovering point sources for the inhomogeneous Helmholtz equation. Inverse Problems, 37(9), (2021).

The inverse problem

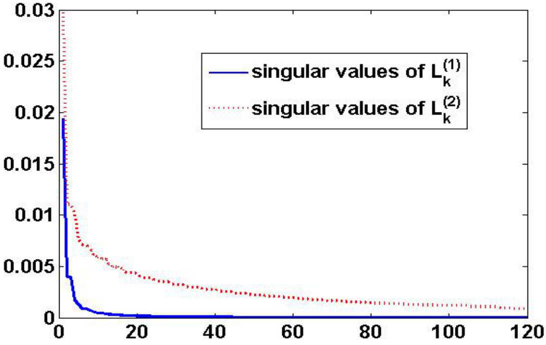


Figure: The first 120 singular values of $L_k^{(1)}$ and $L_k^{(2)}$ when $k = 1$.

Uniqueness

Let $\xi \in \boldsymbol{R}^3$ such that $|\xi| = k$. Multiplying the equation (2) by $e^{-i\xi.x}$ and integrating over Ω, we obtain

$$\widehat{S}(\xi) = \int_\Gamma e^{-i\xi.x}(\partial_\nu u + i\xi.\nu u)d\sigma_x \quad |\xi| = k \in [0, k_0].$$

By collecting the measurements $u,\ \partial_\nu u$ on Γ for $k \in [0, k_0]$, the Fourier transform of S on B_{k_0} can be reconstructed directly.

The inverse problem is now to determine S from the knowledge of $\widehat{S}(\xi)$ for $\xi \in B_{k_0}$.

Theorem (Bao Lin T[2])

Let $(k_j)_j$ be a bounded sequence of positive reals. The measurements $\left(u_{k_j}\right)_j$ determine uniquely the source function S.

[2] J. Differ. Equations, 249 (2010).

Stability estimate

Theorem (Bao-Lin-T[3])
Let S be a function in $C^1(\overline{B}_1)$. Assume that

$$\epsilon = \|\partial_\nu u\|_{L^1(B_{k_0}\times\Gamma)} + \|ku\|_{L^1(B_{k_0}\times\Gamma)} < 1, \qquad \text{and} \quad \|S\|_{C^1(\overline{B}_1)} \leqslant M,$$

where M is a positive constant. The following statements hold:

A) *If $k_0 \geqslant \epsilon^{-4}+1$, then*

$$\|S\|_{C^0(\overline{B}_1)} \quad \leqslant \quad C\Big(1+2M(1+\frac{1}{k_0})+(\frac{1}{16k_0})^2\Big)\epsilon.$$

B) *If $k_0 < \epsilon^{-4}+1$, then*

$$\|S\|_{C^0(\overline{B}_1)} \quad \leqslant \quad C\Big(1+2M(1+\frac{1}{k_0})(\frac{1}{4k_0}+\frac{1}{2\sqrt{k_0}})^{\frac{1}{6}}\Big)\frac{1}{\big(\ln(\epsilon^{-1})\big)^{\frac{1}{6}}},$$

where C is a universal constant.

[3] J. Differ. Equations, 249 (2010).

Let $h(x)$ be a radially symmetric normalized cut-off function: $h \in C_0^\infty(B_{k_0})$ and $\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} h(x)dx = 1$.
An analytic approximation of the source function S is given by:

$$S_N(x) := \int_{B_1} \delta_N(x-y)S(y)dy = \frac{1}{(2\pi)^d}\int_{B_{k_0}} h_N(\xi)\hat{S}(\xi)e^{i\xi.x}d\xi,$$

where

$$h_N(\xi) = \int_{\mathbb{R}^3} \delta_N(x)e^{-i\xi.x}dx,$$

and

$$\delta_N(\xi) := \left(\frac{\delta^2 N}{\pi}\right)^{\frac{3}{2}}\left(1-\delta^2|\xi|^2\right)^{N-1}\mathcal{F}^{-1}(h)(\xi), \quad N \geqslant 1.$$

If $\|S\|_{C^1(\overline{B}_1)} \leqslant M$, then $\|S-S_N\|_{C^0(\overline{B}_1)} \leqslant \frac{C_3}{\delta^{\frac{1}{2}}}M(1+\frac{1}{k_0})\frac{1}{N^{\frac{1}{4}}}$, as $N \to \infty$.

$$\|S_N\|_{C^0(\overline{B}_1)} \leqslant C_d\Big(1+(\frac{\delta^2}{k_0^2}N)^2(\frac{\delta(2N)}{ek_0})^{2N}\Big)\|\hat{S}\|_{L^1(B_{k_0})}.$$

Numerical illustration

Assume that at $k = k_m$, the source function has been recovered with $S = S_m$. Then at a higher wavenumber $k = k_{m+1} := k_m + \delta k_m$, where $\delta k_m > 0$ is the increment, the Landweber iteration is applied to solve (3) with $k = k_{m+1}$.

In the example, the support volume of the source function is $[-0.3, 0.3] \times [-0.3, 0.3]$, which lies in the domain Ω such that $dist(B_{R_0}, \Gamma) = 0.05$. The measurements $\{u, \partial_\nu u\}$ are made on Γ for $k \in [k_{min}, k_{max}]$.
We set $k_{min} = 1$ in the following numerical example.
Assume that the true source

$$S(x_1, x_2) = 1.1 e^{-200((x_1-0.01)^2)+(x_2-0.12)^2} - 100(x_2^2 - x_1^2) e^{-90(x_1^2+x_2^2)}.$$

Numerical Illustration

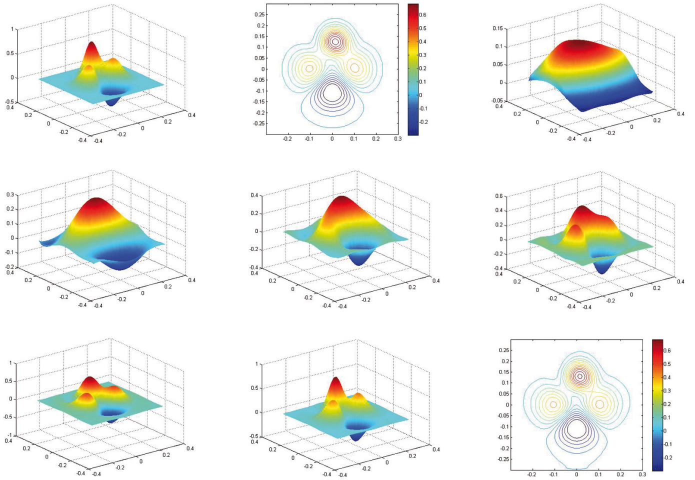


Figure: Reconstruction for different frequency values [Bao Lin T[4]]

[4] Contemp. Math., AMS, 548 (2011)

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Moving Sources

1) Motivation
2) The forward problem
3) The inverse problem
4) Numerical illustrations

Motivation

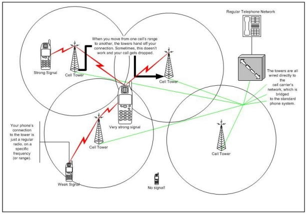


Figure: Tracking a phone position

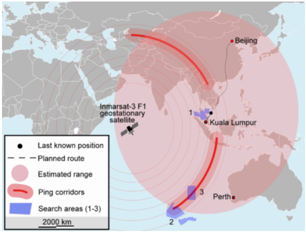


Figure: Searching for MH370 (Wikipedia)

The forward problem

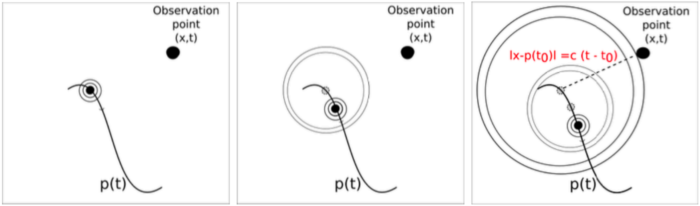


Figure 2.1: *The emission of a wave from a moving source as time increases from left to right.*

$$\left\{\begin{array}{rcl} c^{-2}\partial_t^2 u - \Delta u & = & q(t)\delta_{p(t)} \in C^m(]0,T[, \mathcal{E}'^0(\boldsymbol{R}^3)), \\ u(0+,x) & = & 0, \quad x \in \boldsymbol{R}^3, \\ \partial_t u(0^+,x) & = & 0, \quad x \in \boldsymbol{R}^3. \end{array}\right. \tag{4}$$

• $c \in \boldsymbol{R}_+$ is the constant wave speed, $T \in \boldsymbol{R}_+$ is a fixed final time.
• $p \in C^{m+1}(]0,T[, D)$ is the source trajectory (with $D \subset \boldsymbol{R}^3$ an open, bounded, simply connected domain).
• $q \in C^m(]0,T[)$ is the nonnegative-valued source intensity.
• Only subsonic sources are considered, that is, we assume

$$|p'(t)| < c, \quad t \in]0,T[. \tag{5}$$

The forward problem

$u(t,x)$ can be expressed using the Liénard-Wiechert retarded potential

$$u(t,x) = \begin{cases} (c/4\pi) q(\tau(x,t)) |x - p(\tau(x,t))|^{-1} h(x, \tau(x,t))^{-1}, & |x - p(0)| < ct, \\ 0 & \text{otherwise}, \end{cases} \quad (6)$$

where $\tau(x,t)$ is given implicitly by

$$c(t - \tau(x,t)) = |x - p(\tau(x,t))|, \quad (7)$$

and

$$h(x,\tau) = c - \frac{x - p(\tau)}{|x - p(\tau)|} p'(\tau) \quad \text{for } x \neq p(\tau). \quad (8)$$

Notice that since p satisfies the inequality (5), we have

$$\frac{d}{d\tau} \left(\tau \mapsto t - c^{-1} |x - p(\tau)| \right) \leqslant c^{-1} |p'(\tau)| < 1,$$

so the fixed point problem (7) has a unique solution.

The forward problem

Our first main theoretical result concerns the regularity of solution of the forward problem.

Theorem

Let u be the solution of the system (4)*. Then*

$$u \in C^\infty(]0, T[, H^{m-1/2-\varepsilon}(\boldsymbol{R}^3)),$$

for every strictly positive ε. Moreover,

$$u \in C^\infty(]0, T[, C^\infty(\boldsymbol{R}^3 \setminus \cup_{s=0}^{T_0} \overline{B_{p(s)}(c(T_0 - s))})),$$

and $u(t, \cdot)$ is C^∞ in

$$\Big\{|x - p(s)| < c(t - s),\ 0 \leqslant s \leqslant \min\{t, T_0\}\Big\} \setminus p([0, T_0]).$$

The forward problem

Applying the Fourier transform to equation (4), and using the fact that

$$\int_{\omega\in S^2} \exp\left(-ic(t-s)|\xi|\frac{\xi}{|\xi|}\omega\right) d\omega = \int_0^{\pi}\int_0^{2\pi} \exp\left(-i|\xi|c(t-s)\cos\theta\right)\sin\theta\, d\varphi\, d\theta$$
$$= \frac{4\pi}{c(t-s)}\frac{\sin(|\xi|c(t-s))}{|\xi|},$$

we get

$$\widehat{u}(t,\xi) = \frac{c^2}{4\pi}\int_0^t (t-s)q(s)\int_{\omega\in S^2}(\delta_{c(t-s)\omega+p(s)})_x(e^{-ix\xi})\,d\omega\, ds$$
$$= \frac{c}{|\xi|}\int_0^t q(s)\sin(c|\xi|(t-s))e^{-i\xi p(s)}ds, \qquad (9)$$

Then the regularity of u can be directly deduced from $\widehat{u}(t,\cdot)$ in the Fourier space.

The inverse problem

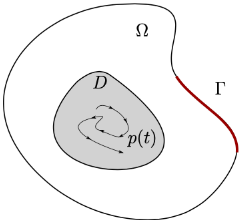


Figure 3.1: *The regions D and Ω with the partial boundary Γ.*

- Let Ω be a bounded and connected domain with a C^2 boundary $\partial\Omega$, and such that $D \subset \Omega$.
- Let $\Gamma \subseteq \partial\Omega$ be an open set that is not included in any plane intersecting Ω.

For $T_0 < T$, the inverse moving source problem we consider is to reconstruct $p|_{[0,T_0]}$ and $q|_{[0,T_0]}$, given a measurement $u|_{[0,T]\times\Gamma}$.

The inverse problem

- How much information about the source do the boundary measurements hold? How large T has to be in terms of the geometry of the trajectory, the wave speed, the source speed, and T_0?
- We must *turn off* the source at a time $T_0 < T$ (i.e., $q(t) = 0$ for $t \in [T_0, T]$) to have any chance of uniqueness of solution of the inverse problem, due to the finite speed of wave propagation.
- We consider T large enough to allow the information about p and q for $t \in [0, T_0]$ to propagate towards the measurement set Γ, that is,

$$T > T_0 + \sup\{|x - y|/c,\ x \in \Gamma,\ y \in \Omega_0\} := T^*. \tag{10}$$

- We further assume that $\{t \in [0, T_0],\ q(t) = 0\}$ has zero measure. This prevents the situation where two trajectories are indistinguishable because the source does not radiate where the trajectories are not aligned.
- if Γ is included in a plane P intersecting Ω then one is unable to distinguish the field measurements from pairs of sources constructed using the same intensity, and by translating a trajectory confined in $P \cap \Omega$ to two different planes parallel to P and at the same orthogonal distance from P.

Uniqueness

Notice that

$$u(t,\cdot)|_{\Gamma} = 0 \quad \text{for } t \in]T^*, +\infty[,$$

with $T^* = T_0 + \sup\{|x-y|/c,\ x \in \Gamma,\ y \in \Omega_0\}$.

Theorem (Karamehmedovic-Wang-T22)

Assume that $T > T^*$.
- $(u|_{[0,T]\times\Gamma}, \partial_\nu u|_{[0,T]\times\Gamma})$ *determines the source trajectory* $p|_{[0,T_0]}$ *and intensity* $q|_{[0,T_0]}$ *uniquely.*
- *Assume that $\partial\Omega$ is real analytic. Then the field measurement $u|_{[0,T]\times\Gamma}$ determines the source trajectory p and intensity q uniquely.*

Theorem (El Badia-Jebawy-T22)

Assume that $T > T^*$, and $q(t)$ is a known constant.
Let $\{\mathbf{x}_i\}_{1\leqslant i\leqslant 6}$ be fixed points on Γ chosen such that the matrix $X = (\mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_3, \mathbf{x}_6 - \mathbf{x}_5)^T$ is invertible. Then the knowledge of $u(\mathbf{x}_i, t),\ i = 1, \cdots, 6$ for $t \in [0,T]$, determines uniquely the trajectory of the point source, that is $p|_{[0,T_0]}$.

Uniqueness

Writing $u = u_1 - u_2$ and $f(t,\cdot) = q_1(t)\delta_{p_1(t)} - q_2(t)\delta_{p_2(t)}$, we have

$$\left\{\begin{array}{rcll} c^{-2}\partial_t^2 u - \Delta u &=& f(t,\cdot) & \text{in } C_0(\boldsymbol{R}, \mathcal{E}'^0(\boldsymbol{R}^3)), \\ u(x,0) &=& 0, & x \in \boldsymbol{R}^3, \\ \partial_t u(x,0) &=& 0, & x \in \boldsymbol{R}^3, \\ u|_{\Gamma\times\boldsymbol{R}} &\equiv& 0. & \end{array}\right. \tag{11}$$

Since $T > T^*$, taking the Fourier transform with respect to the time, we get $\widehat{u} = \partial_\nu \widehat{u} = 0$ at $\partial\Omega$.
On the other hand $\widehat{u}$ solves the interior Helmholtz problem

$$\left\{\begin{array}{rcll} (\Delta + (\omega/c)^2)\widehat{u}(\omega,x) &=& -\widehat{f}(\omega,x), & \text{in } \Omega, \\ \widehat{u}(\omega,w) &=& 0, & x \in \partial\Omega, \\ \partial_\nu \widehat{u}(\omega,x) &=& 0, & x \in \partial\Omega. \end{array}\right. \tag{12}$$

Since $\widehat{f}(\omega,\cdot)$ has a compact support $p_1([0,T_0]) \cup p_2([0,T_0])$, which has a zero two-dimensional Hausdorff measure, we deduce from the uniqueness of Cauchy problem, $\widehat{u} = 0$ a. e. in Ω (and then in $\boldsymbol{R}^3$). Therefore $\widehat{f}(\omega,\cdot) = f(t,\cdot) = 0$.

Stability estimate

Theorem (Karamehmedovic-Wang-T22)

Assume that $\Gamma = \partial\Omega$. Let u_j be the solution of the system (4) *with $q = q_j$, and $p = p_j$ for $j = 1, 2$. Assume that $\|q_j\|_{C^1([0,T_0])} \leqslant M_q$ and $\|p_j\|_{C^2([0,T_0])} \leqslant M_p$ for $j = 1, 2$.*
Let $\rho \in]0, T_0[$, q_0 and c_0 be strictly positive constants. Assume

$$q_j(t) \geqslant q_0, \quad \forall t \in [0, T_0 - \rho], \tag{13}$$

and

$$1 - c^{-1} p_j'(t) \geqslant c_0, \quad \forall t \in [0, T], \; j = 1, 2. \tag{14}$$

Let

$$\varepsilon = \sup_{\hat{k}\in S^2} \int_{\boldsymbol{R}\times\partial\Omega} \Big(c^{-1} |\partial_t u(t - c^{-1}\hat{k}\cdot x, x)| + |\partial_\nu u(t - c^{-1}\hat{k}\cdot x, x)| \Big) H(t - c^{-1}\hat{k}\cdot x)\, d\sigma(x)\, dt,$$

where $u = u_1 - u_2$. Then for $\varepsilon \in]0, q_0(T_0 - \rho)[$, we have

$$\|q_1 - q_2\|_{C^0([0,T_0-\rho])} + \|p_1 - p_2\|_{C^0([0,T_0-\rho])} \leqslant C\varepsilon, \tag{15}$$

where H is the Heaviside function, and $C > 0$ is a constant that only depends on c_0, q_0, T_0, c, M_q, and M_p.

Numerical results

We consider a circular arc trajectory in the *xy*-plane parametrized by

$$p(t) = (\cos(0.3t) - 1, \sin(0.3t), 0), t \in [0, T_0] \tag{16}$$

and with non-constant polynomial intensity

$$q(t) = \begin{cases} -28.44(t/T_0)^4 + 56.89(t/T_0)^3 - 39.11(t/T_0)^2 + 10.67(t/T_0), & t \in [0, T_0] \\ 0, & \text{otherwise} \end{cases}. \tag{17}$$

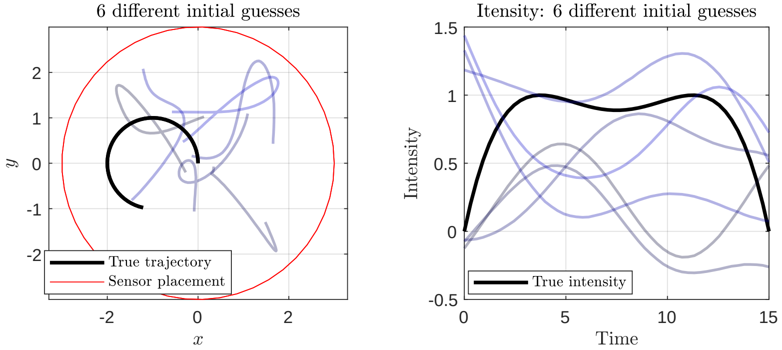
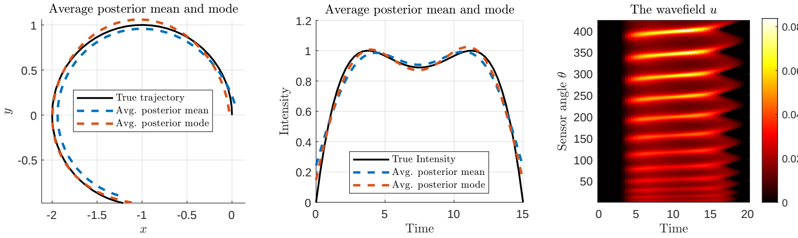
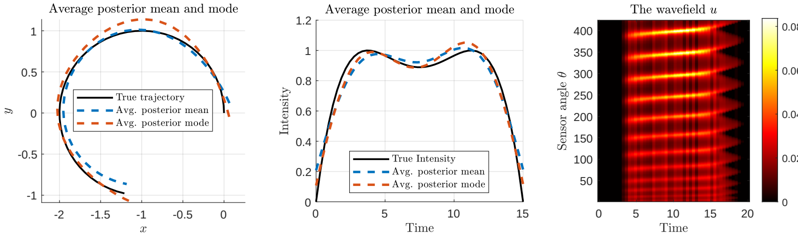


Figure: Initial samples from the GP prior for the trajectory (left) and intensity (right).

Numerical illustration

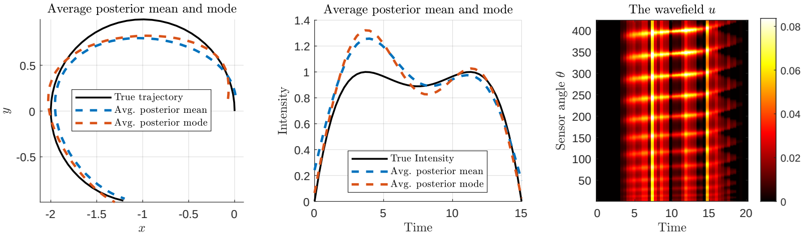


No noise

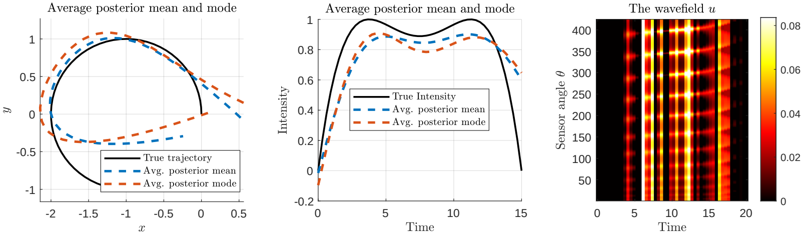


5% noise

Numerical illustration



25% noise



50% noise

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- Wang, Sara, Mirza Karamehmedovic, and Faouzi Triki. Localization of moving sources: uniqueness, stability, and Bayesian inference. arXiv preprint arXiv:2204.04465 (2022).

Conclusion

- The generalization of the uniqueness result to a non-constant wave speed $c(x) > c_0$ is forward.
- The stability estimate for the supersonic case ($|\{|p'(t)| \geqslant c,\ t \in]0, T[\}| \neq 0$) is more challenging.
- Future works will deal with more than one source $\sum_{j=1}^{J} q_j(t)\delta_{p_j(t)}$ with $J \geqslant 2$.

Thanks!