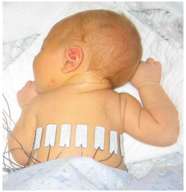
Kim Knudsen - Technical University of Denmark

Sound speed uncertainty in Acousto-Electric Tomography

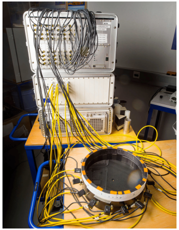
IUQ22, September 29, 2022

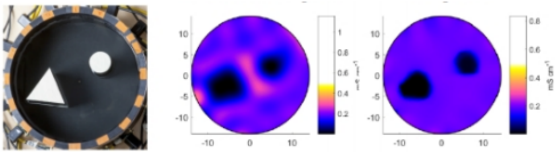
Imaging of electric conductivity



Heinrich *et.al* (2006)

Dräger system

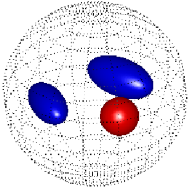


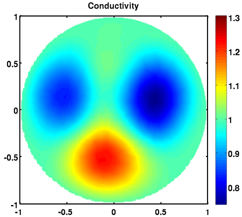


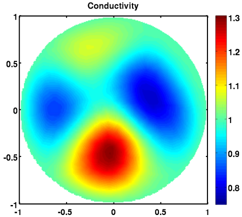
KIT4 system, Kuopio, UEF, Hauptman *et.al.* (2017)

Image quality

Low contrast - low resolution







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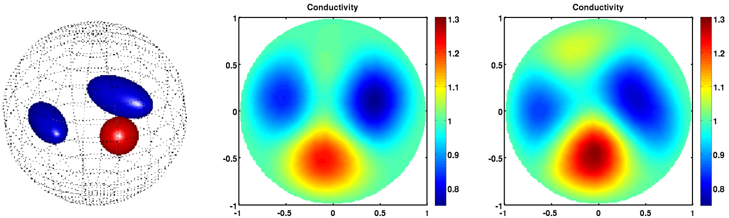


Acousto-electric tomography (AET) excites EIT with ultrasonic waves and claims
high contrast - high resolution

Goal:
Confirm the feasibility of AET through a computational study
Quantify the uncertainty in reconstructions

Outline

1. Mathematical models for Acousto-Electric Tomography
2. Reconstruction method
3. Uncertainty Quantification in AET

Joint work with Bjørn Jensen (Jyväskylä) and Adrian Kirkeby (Oslo)

CUQI
THE VELUX FOUNDATIONS
VILLUM FONDEN VELUX FONDEN

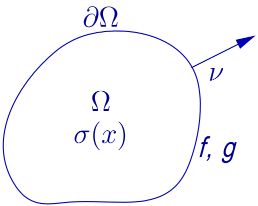
1. Mathematical models for AET

Electrical Impedance Tomography

Smooth bounded domain $\Omega \subset \mathbb{R}^n$ (soon $n = 2$); conductivity coefficient σ.

A voltage potential u in Ω generated by boundary voltage f

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega,$$
$$u|_{\partial\Omega} = f.$$



Measure electric current through the boundary

$$g = \partial_\nu u|_{\partial\Omega}.$$

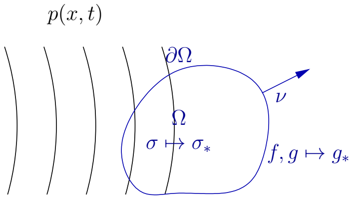
Inverse problem:

Reconstruct σ from measurements of all possible (f, g).

Calderón (1980)

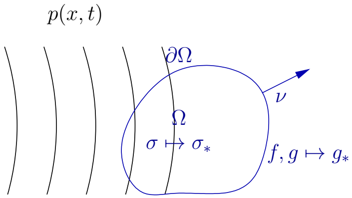
Acousto-Electric Tomography

Acoustic wave excites σ



Acousto-Electric Tomography

Acoustic wave excites σ



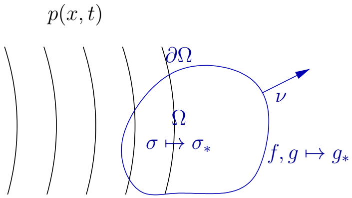
Acousto-Electric effect

$$\sigma_*(x,t) - \sigma(x) = \eta p(x,t)\sigma(x)$$

Excited boundary current $g_* = \partial_\nu u_*|_{\partial\Omega}$

Acousto-Electric Tomography

Acoustic wave excites σ



Acousto-Electric effect

$$\sigma_*(x,t) - \sigma(x) = \eta p(x,t)\sigma(x)$$

Excited boundary current $g_* = \partial_\nu u_*|_{\partial\Omega}$

Inverse problem:

Reconstruct σ from knowledge of wave $p(x,t)$ and (few) (f, g, g_*)

Chang and Wang (2004); Lavandier, Jossinet and Cathignol (2000); Ammari, Bonnetier, Capdeboscq, Tanter, and Fink (2008).

AET sketch

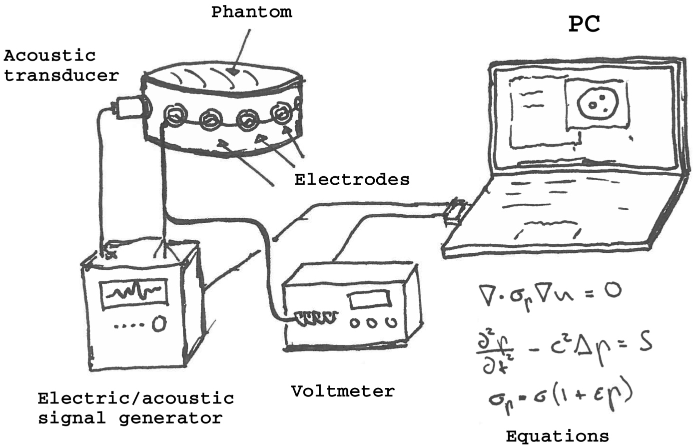
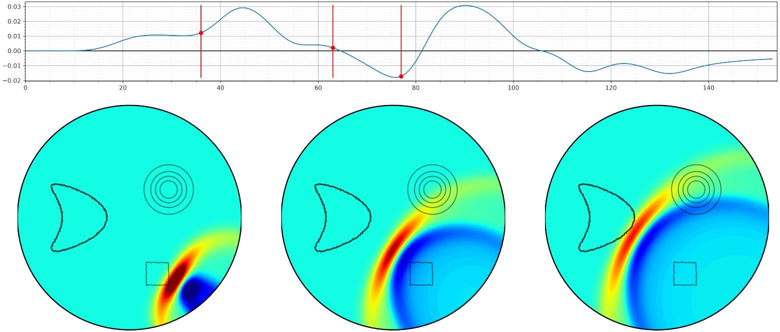


Image courtesy of Adrian Kirkeby

2. Reconstruction method

Total power difference

$$I(t) = \int_{\partial\Omega} f(x)(g_*(x,t) - g(x))\,\mathrm{d}S$$



Power density

Total power difference

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Power density

Total power difference

$$I(t) = \int_{\partial\Omega} f(x)(g_*(x,t) - g(x))\,\mathrm{d}S.$$

Integration by parts

$$I(t) = -\eta \int_{\Omega} p(x,t)\sigma(x)\nabla u(x)\cdot\nabla u_*(x)\,\mathrm{d}x$$

and linearization ($u_* \approx u$ in $H^1(\Omega)$) yields

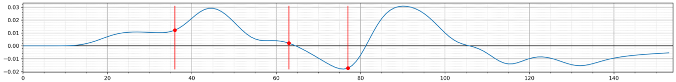
$$I(t) \approx -\eta \int_{\Omega} p(\cdot,t)\sigma|\nabla u|^2\,\mathrm{d}x = -\eta \int_{\Omega} p(\cdot,t)H\,\mathrm{d}x.$$

with **interior power density**

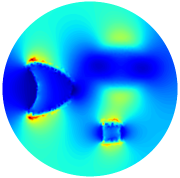
$$H(x) = \sigma(x)|\nabla u(x)|^2 \qquad \text{in } \Omega.$$

To recover H from I we need enough waves $p(x,t)$.

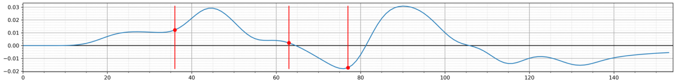
Two step reconstruction



Step 1: From $I(t)$ to $H(x)$
Linear, moderately ill-posed.



Two step reconstruction



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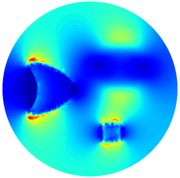
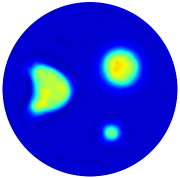
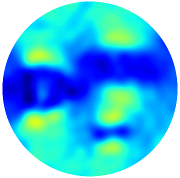


Step 2: From $H(x)$ to $\sigma(x)$.
Non-linear, moderately well-posed.



Step 1: $I \mapsto H$



Integral equation for H

$$I(t) = KH(t) = -\eta \int_{\Omega} p(x,t)H(x)\,\mathrm{d}x$$

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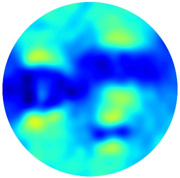

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Reqularized least squares optimization

$$H = \operatorname*{argmin}_H \mathcal{J}_1(H), \qquad \mathcal{J}_1(H) = \|I - KH\|^2_{L^2(0,T)} + \mathcal{R}(H).$$

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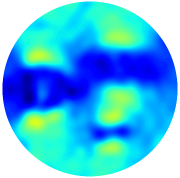

Integral equation for H

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Regularized least squares optimization

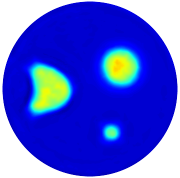
$$H = \operatorname*{argmin}_H \mathcal{J}_1(H), \qquad \mathcal{J}_1(H) = \|I - KH\|^2_{L^2(0,T)} + \mathcal{R}(H).$$

Regularization by $H^1(\Omega)$ norm

$$\mathcal{R}(H) = \alpha\|H_0 - H\|^2_{H^1(\Omega)}.$$

H_0 is the power density for the constant background conductivity σ_0.

Step 2: $H \mapsto \sigma$



Inversion of the nonlinear map

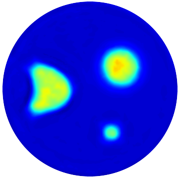
$$\mathcal{H} : L^\infty_+(\Omega) \to L^1(\Omega)$$
$$\sigma \mapsto \sigma|\nabla u|^2 = H$$

Optimization problem

$$\sigma = \underset{\sigma}{\operatorname{argmin}}\, \mathcal{J}_2(\sigma), \qquad \mathcal{J}_2(\sigma) = \|\mathcal{H}(\sigma) - z\|_{L^1} + \beta|\sigma|_{\mathrm{TV}},$$

where z is the reconstructed power density from step 1.

Step 2: $H \mapsto \sigma$



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where z is the reconstructed power density from step 1.

Theorem

$\mathcal{J}_2(\sigma)$ has a minimizer in $BV(\Omega)$.

Capdeboscq et al. (2009); Bal et. al. (2012); Hoffmann and Knudsen (2014); Adesokan et al. (2018); Roy and Borzi (2018)

Step 2: Successive linearization

Linearization around base conductivity σ :
Put $d_\sigma = z - \mathcal{H}(\sigma)$.

$$\mathcal{J}_2(\sigma + \kappa) \approx J_\sigma(\kappa) = \|\mathcal{H}'(\sigma)\kappa - d_\sigma\|_{L^1} + \beta|\sigma + \kappa|_{\mathrm{TV}}.$$

Iteratively minimize J_σ for a step κ and update σ with this step.

Non-differentiability
Replace $|\cdot|_{\mathrm{TV}}$ in J_σ by the standard smoothing $|\cdot|_\varepsilon = \sqrt{|\cdot|^2 + \varepsilon^2}$.

Step 2: Minimizing J_σ via lagged diffusivity

Derivative

$$J'_\sigma(\kappa)\xi = \int \frac{H'(\sigma)\kappa - d_\sigma}{|H'(\sigma)\kappa - d_\sigma|_\varepsilon} H'(\sigma)\xi \, \mathrm{d}x + \beta \int \frac{\nabla(\sigma+\kappa)}{|\nabla(\sigma+\kappa)|_\varepsilon} \cdot \nabla\xi \, \mathrm{d}x.$$

Freeze the denominator at κ'

$$J'_{\sigma,\kappa'}(\kappa)\xi = \int \frac{H'(\sigma)\kappa - d_\sigma}{|H'(\sigma)\kappa' - d_\sigma|_\varepsilon} H'(\sigma)\xi \, \mathrm{d}x + \beta \int \frac{\nabla(\sigma+\kappa)}{|\nabla(\sigma+\kappa')|_\varepsilon} \cdot \nabla\xi \, \mathrm{d}x$$

Equivalent to weighted quadratic functional

$$J_{\sigma,\kappa'}(\kappa) = \|H'(\sigma)\kappa - d_\sigma\|^2_{L^2_w} + \beta\|\nabla(\sigma+\kappa)\|^2_{L^2_{w_0}}.$$

We minimize J_σ by iteratively minimizing $J_{\sigma,\kappa'}$ while updating κ', w, w_0.

Vogel and Oman (1996)

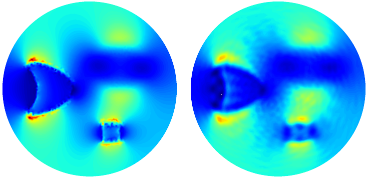
Feasibility of AET

Medical imaging scenario

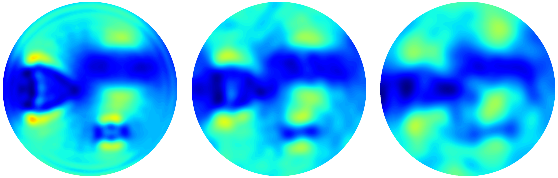
Parameter	Symbol	Values(s)	Unit
Conductivity	σ	$0.1 - 1.5$	S/m
Boundary current amplitude	f_{max}	1×10^{-3}	A
Acousto-electric coupling constant	η	$\{10^{-7}, 10^{-9}, 10^{-11}\}$	Pa^{-1}
Max. acoustic pressure	p_{max}	1.5×10^{6}	Pa
Acoustic wave speed	c_0	1500	m/s
Domain radius	R	0.1	m
Boundary conditions	f_j	$x,\ y,\ x+y$	
Noise level	δ	$\{0, 10^{-2}, 10^{-1}\}$ %	

Software: k-Wave and FEniCS

Reconstructions: Power densities (step 1)

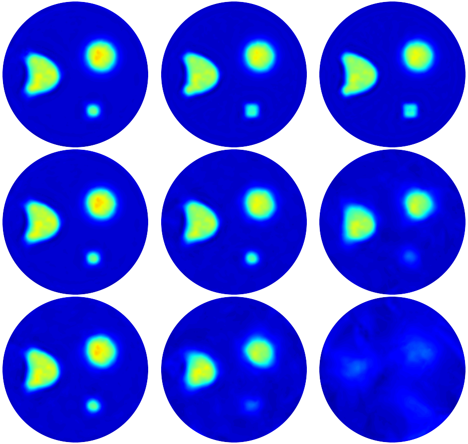


True power density and projection onto basis functions



Reconstructed power density with increasing noise level.

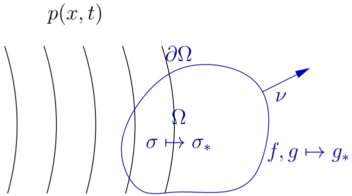
Reconstructions: conductivity (step 2)



Coupling $\eta = 10^{-7}, 10^{-8}, 10^{-9}$ (left to right); noise $\delta = 0, 0.1, 0.01\%$ (top to bottom)

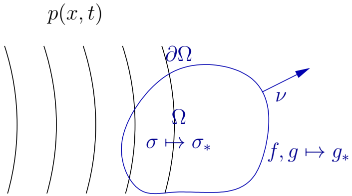
3. Uncertainty Quantification for AET

Uncertainty is all over



- EIT model - Electrode position
- Linearization error
- Coupling coefficients
- Measurement noise

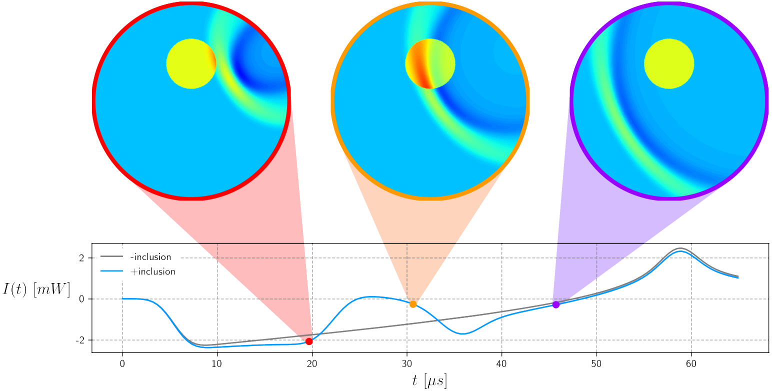
Uncertainty is all over



- EIT model - Electrode position
- Linearization error
- Coupling coefficients
- Measurement noise
- **Wave model - sound speed and sound source**

Uncertain wave speed

$$I(t) = KH(t) = -\eta \int_{\Omega} p(x,t) H(x) \mathrm{d}x$$



Scalar wave equation

Ground truth

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)p(x,t) = S(x,t) \\ p(x,0) = 0 \\ \partial_t p(x,t)|_{t=0} = 0 \end{cases}$$

Scalar wave equation

Ground truth

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)p(x,t) = S(x,t) \\ p(x,0) = 0 \\ \partial_t p(x,t)|_{t=0} = 0 \end{cases}$$

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Model

$$\begin{cases} (\partial_t^2 - \widetilde{c}^2(x)\Delta)p(x,t) = S(x,t) \\ p(x,0) = 0 \\ \partial_t p(x,t)|_{t=0} = 0 \end{cases}$$

Errors $\widetilde{c} - c$, $\widetilde{p} - p$ and $\widetilde{K} - K$.

Errors as perturbations

Wave difference

$$(\partial_t^2 - \widetilde{c}^2\Delta)(\widetilde{p} - p) = (\widetilde{c} - c)(\widetilde{c} + c)\Delta p$$

Errors as perturbations

Wave difference

$$(\partial_t^2 - \widetilde{c}^2\Delta)(\widetilde{p} - p) = (\widetilde{c} - c)(\widetilde{c} + c)\Delta p$$

Propositions ($m > (n+1)/2$)

Wave

$$\|\widetilde{p} - p\|_{L^\infty(0,T;H^{m+1}(B))} \leq C\|\widetilde{c} - c\|_{H^m(B)}$$

Operator

$$\|\widetilde{K} - K\|_{\mathcal{L}(L^2(\Omega),L^2(0,T))} \leq C'\|\widetilde{c} - c\|_{H^m(B)}$$

Reconstruction

Optimization problem

$$\operatorname*{argmin}_h \frac{1}{2}\|\widetilde{K}h - I\|^2_{L^2(0,T)} + \frac{\beta}{2}\|h\|^2_{L^2(\Omega)}$$

Operator

$$R_\beta = (\widetilde{K}^*\widetilde{K} + \beta\mathcal{I})^{-1}\widetilde{K}^*$$

Reconstruction

Optimization problem

$$\underset{h}{\operatorname{argmin}} \frac{1}{2}\|\widetilde{K}h - I\|^2_{L^2(0,T)} + \frac{\beta}{2}\|h\|^2_{L^2(\Omega)}$$

Operator

$$R_\beta = (\widetilde{K}^*\widetilde{K} + \beta\mathcal{I})^{-1}\widetilde{K}^*$$

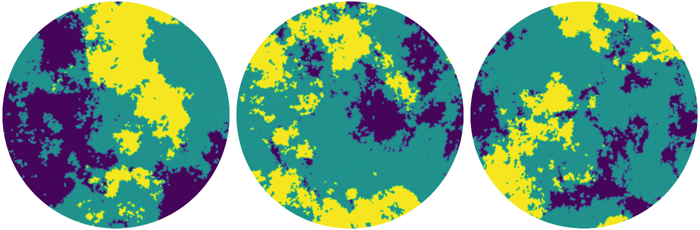
Regularization strategy

For $\beta \propto \|\widetilde{c} - c\|_{H^m(B)}$

$$\|H - R_\beta I\|_{L^2(\Omega)} \leq C\|\widetilde{c} - c\|^{\frac{1}{2}}_{H^m(B)}.$$

Sampling sound speed structure

Treat uncertain sound speed and solution as random variables.
Samples s

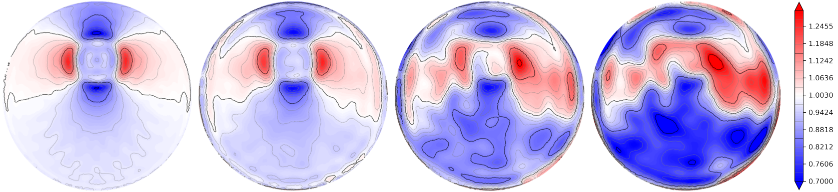


Take constant sound speed $\tilde{c} = c_0$ and

$$c(x) = c_0(1 + \mu s(x))$$

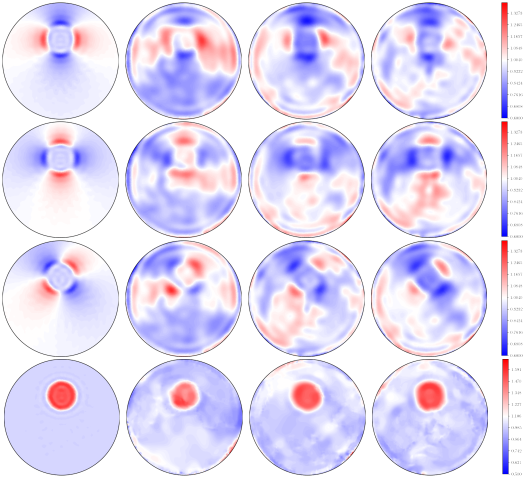
Reiser, Edwards, and Nishikawa (2013)

Error due to uncertain sound speed



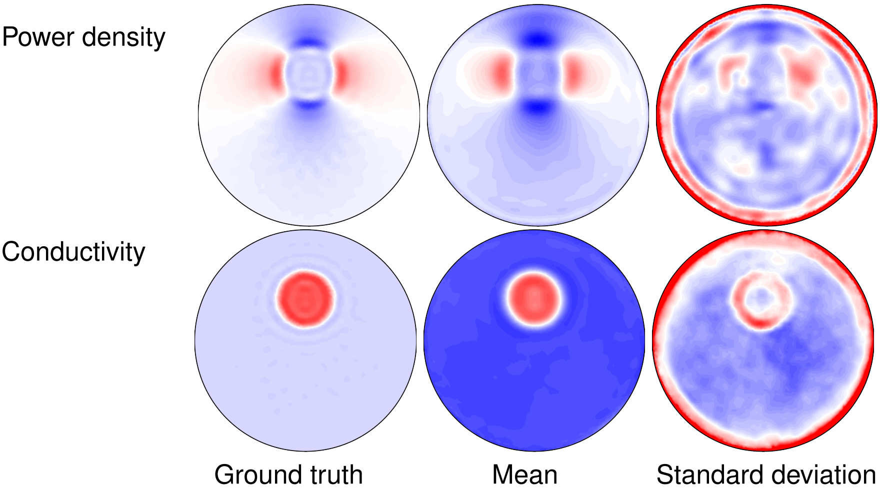
Reconstructions of H for (left-to-right) $\mu = 0.00$, 0.01, 0.05 and 0.10.

Results



H and σ for 3 different BC and $\mu = 0.00, 0.01, 0.05$ and 0.10.

Computational Uncertainty Quantification



Take home message

- Acousto-electric tomography: model problem
- Uncertainty handled as operator perturbations
- Quantification via reconstrcution method

References:
Adesokan, Jensen, Jin, and Knudsen, *Acousto-electric tomography with total variation regularization*, Inverse Problems (2019)
Jensen, Kirkeby, and Knudsen, *Feasibility of Acousto-Electric Tomography* , arXiv:1908.04215, (2019)
Jensen and Knudsen, *Sound speed uncertainty in Acousto-Electric Tomography*, arXiv:2003.14114, (2020).

Thank you for the attention