Forward Uncertainty Quantification for the Helmholtz Equation

Workshop: Imaging With Uncertainty Quantification (IUQ)

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CUQI

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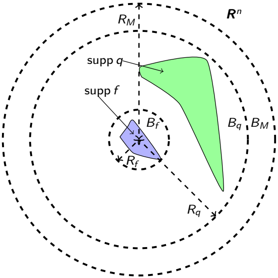
- **spectral properties of radiation for the Helmholtz equation with a random coefficient**
- dimension $n = 2$ or $n = 3$; free-space wavenumber $k_0 = \text{const.} > 0$; $\Delta = \sum_j \partial_{x_j}^2$
- $$(*) \begin{cases} \left(\Delta + (1 + q(x,\omega))\, k_0^2\right) u^{\text{tot}} &= -f \quad \text{in } \boldsymbol{R}^n, \\ \lim_{|x|\to\infty} |x|^{(n-1)/2}(\partial_{|x|} u^{\text{tot}} - ik_0 u^{\text{tot}}) &= 0 \quad \text{uniformly in } x/|x| \in S^{n-1}. \end{cases}$$



- fix $0 < R_f < R_q < R_M$, and write B_f, B_q, B_M for open balls in $\boldsymbol{R}^n$ with radii R_f, R_q and R_M, respectively
- $f \in L^\tau(\boldsymbol{R}^n)$, $\tau > n/2$, is a deterministic source with $\operatorname{supp} f \subseteq \overline{B}_f$
- $q(x,\omega)$ is a stochastic medium a.s. in $L^\infty(\boldsymbol{R}^n)$ with $\operatorname{supp} q(\cdot,\omega) \subseteq \overline{B}_q \setminus B_f$

The medium term $q(x, \omega)$ is a real-valued, second-order, stationary Gaussian random field on $T = \overline{B}_q \setminus B_f$, a.s. bounded on T. This means that there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

- for any $d \in \boldsymbol{N}$ and any $(x_1, \ldots, x_d) \in T^d$, the mapping

$$\Omega \ni \omega \mapsto (q(x_1, \omega), \ldots, q(x_d, \omega))$$

  is a multivariate real-valued Gaussian random variable,
- for each $x \in T$, the expected value $\mathbb{E}_{\omega \in \Omega}[q(x, \omega)]$ is a finite real constant,
- $\|q(x, \cdot)\|_{L^2(\Omega)} < \infty$ for each $x \in T$ (second-order),
- $\mathbb{P}[\{\omega \in \Omega,\ \|q(\cdot, \omega)\|_{L^\infty(T)} = \infty\}] = 0$, and
- the covariance function

$$c(x, y) = \mathbb{E}_{\omega \in \Omega}[(q(x, \omega) - \mathbb{E}_{\omega' \in \Omega}[q(x, \omega')])(q(y, \omega) - \mathbb{E}_{\omega' \in \Omega}[q(y, \omega')])],$$

  defined for $x, y \in T$, depends only on $|x - y|$, that is, $q(x, \cdot)$ is isotropic for each $x \in T$.

- By the Borel-TIS theorem[1], we have $\mathbb{E}_{\omega\in\Omega}[\|q(\cdot,\omega)\|_{L^\infty(T)}] < \infty$ and, for each positive t,

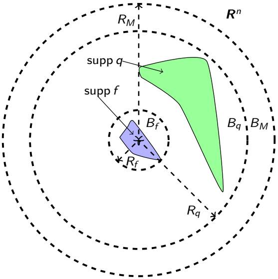
$$\mathbb{P}[\{\omega \in \Omega,\ \|q(\cdot,\omega)\|_{L^\infty(T)} - \mathbb{E}_{\omega'\in\Omega}(\|q(\cdot,\omega')\|_{L^\infty(T)}) > t\}] \\ \leq \exp[-t^2/(2\sup_{x\in T} \mathbb{V}_{\omega\in\Omega}(q(x,\omega)))].$$

- Each $q(x,\cdot)$, $x \in T$, is a Gaussian random variable, so some realizations $q(x,\omega)$ may have (arbitrarily large) negative values. Thus, $k_0^2(1+q(x,\omega))$ may well be negative for some $x \in T$, given a realization $\omega \in \Omega$. In all our numerical examples we have $1 + q(x,\omega) > 0$.
- The field $1 + |q(x,\omega)|$ is not Gaussian and we cannot use the Borell-TIS theorem on it.

[1] Theorem 2.1.1 in Adler and Taylor, *Random fields and geometry*, Springer, 2007.

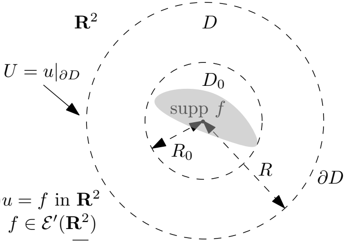
- **we are interested in the properties of the singular value spectrum of the near-field source-to-measurement map (forward map)**

$$F : f \mapsto u|_{\partial B_M}.$$

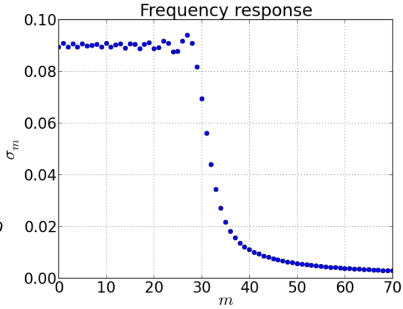


- **motivation: the robustness of solution of inverse source problems in the presence of (random) media**

Robustness of solution of medium-free inverse source problems



$(\Delta + k^2)u = f$ in $\mathbf{R}^2$
$f \in \mathcal{E}'(\mathbf{R}^2)$
supp $f \subseteq \overline{D}_0$
+ radiation conditions



K. (2018). *J Phys Commun*

Forward operator: $U(x) = Ff(x) = \int_{y \in D_0} H_0^{(1)}(k|x-y|)f(y)$, $x \in \partial D$
Bao, Lin, & Triki (2010). *J Differ Equ*:

$$F : L^2(D_0) \xrightarrow{\text{cpct.}} L^2(\partial D), \quad F = \sum_{m \in \mathbf{Z}} \sigma_m(\cdot, \psi_m)\phi_m$$

$$\sigma_{-m} = \sigma_m, \quad \psi_m(x) \propto J_m(k|x|)e^{im\angle x}, \quad \phi_m(\angle x) \propto e^{im\angle x}$$

Bounds on the 'bandwidth' $\mathscr{B}$ of F

K. (2018). *J Phys Commun*:

Definition: $\mathscr{B} = \mathrm{argmin}_{m\in\mathbf{N}_0}\{\sigma_{m+n} > \sigma_{m+n+1} \text{ for all } n \in \mathbf{N}_0\}$.

Theorem: $\mathscr{B} \geq \mathrm{argmin}_{m\in\mathbf{N}_0}\{j_{m,1} \geq kR_0\}$ (tight)

Conjecture: $\mathscr{B} \leq \mathrm{argmin}_{m\in\mathbf{N}_0}\{y_{m,1} \geq kR_0\}$ (tight)

Theorem: For the source-to-*far*-field operator, $\sigma_m = \mathcal{O}((kR_0/2)^m/m!)$ when $m \geq \mathrm{argmin}_{m\in\mathbf{N}_0}\{y_{m,1} \geq kR_0\}$ (with explicit bound)

Kirkeby, Henriksen, & K. (2020). *Inverse Probl*:

Theorem: For the Helmholtz equation in $\mathbf{R}^3$, we have $\psi_{m,n}(x) \propto j_m(k|x|)Y_m^n(x/|x|)$ and $\phi_{m,n} \propto Y_m^n(x/|x|)$.

Theorem: $\mathscr{B} \geq \mathrm{argmin}_{m\in\mathbf{N}_0}\{j_{m+1/2,1} \geq kR_0\}$.

Kirkeby, Henriksen, & K. (2020); K., Kirkeby, & Knudsen (2018). *Inverse Probl*:

Stability of reconstruction from a finite number of measurements in the multi-frequency ISP.

Some related work

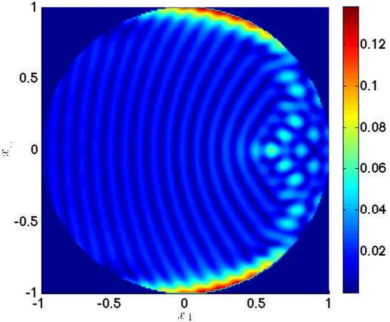
Griesmaier & Sylvester (2017). *SIAM J Appl Math*
Griesmaier & Sylvester (2016). *SIAM J Appl Math*
Griesmaier, Hanke, & Sylvester (2014). *SIAM J Numer Anal*
Griesmaier, Hanke, & Raasch (2012). *SIAM J Sci Comput*

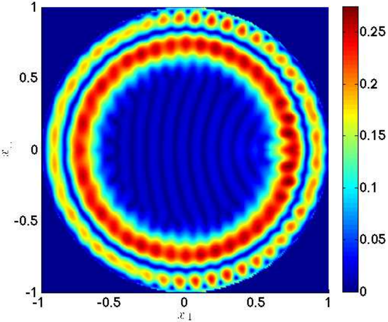
- spectral cutoff of the source-to-far-field operator ("restricted Fourier transform") in $\mathbf{R}^2$ and $\mathbf{R}^3$; the singular values decay rapidly when $|m| \geq kR_0$.
- windowed Fourier transform
- far-field splitting and uncertainty principles for the inverse source problem

Pierri & Moretta (2020,2021). *Electronics*
Xu & Janaswamy (2006). *IEEE Trans Antennas Propag*

- spectral analysis of electromagnetic radiation operators
- applications in antenna design and measurements

Robustness of solution of medium-free inverse source problems





- $f^\dagger = F^\dagger U \approx \sum_{|m|\le C} \sigma_m^{-1} (U, \phi_m)_{L^2(\partial D)} \psi_m$
- $kR = kR_0 = 10\pi$
- $\mathscr{B} \ge 26$ (K. (2018). *J Phys Commun*)
- $m_{\text{noise}} = 26$ vs. $m_{\text{noise}} = 30$, for same amplitude of noise component

- with $a \in L^\infty(B_M)$, the volume potential (the Lippmann-Schwinger operator) is

$$V_a w(x) = \int_{y \in B_M} \Phi^{\{n\}}(x-y) a(y) w(y) dy, \quad x \in \boldsymbol{R}^n$$

- $\Phi^{\{n\}}(x) = \begin{cases} (\mathrm{i}/4) H_0^{(1)}(k_0|x|), & x \in \boldsymbol{R}^2 \setminus \{0\},\ n = 2, \\ \exp(\mathrm{i}k_0|x|)/(4\pi|x|), & x \in \boldsymbol{R}^3 \setminus \{0\},\ n = 3, \end{cases}$
  is the unique outgoing fundamental solution of the Helmholtz operator in $\boldsymbol{R}^n$
- $(\Delta + k_0^2)\Phi^{\{n\}} = -\delta$ in $\boldsymbol{R}^n$
- since $\tau > n/2 \geq 1$ and $\|aw\|_{L^\tau(B_M)} \leq \|a\|_{L^\infty(B_M)} \|w\|_{L^\tau(B_M)}$, the mapping $w \mapsto V_a w$ is continuous from $L^\tau(B_M)$ to $W^{2,\tau}(B_M)$ (Lechleiter, Kazimierski, & K., 2013, Lemma 1).
- the Helmholtz problem $(*)$ is equivalent with the Lippmann-Schwinger equation

$$(**) \quad (I - k_0^2 V_q) u(x) = V_1 f(x), \quad x \in \boldsymbol{R}^n,$$

  which is uniquely solvable in $L^\tau(B_M)$.
- in particular, there is a unique solution $u \in W^{2,\tau}(B_M)$ of $(*)$

Lemma. The Lippmann-Schwinger equation $(**)$ is uniquely solvable in $L^\tau(B_M)$.

Proof. The result follows as a special case of the analysis in Lechleiter, Kazimierski, & K. (2013). Indeed, B_M is relatively compact and $q \in L^\infty(B_M)$, so $q \in L^p(B_M)$ for every $p \geq 1$. Then, by Proposition 2(c) in Lechleiter *et al.* (2013) and the fact that $\tau > n/2 \geq 1$, the mapping $V_q : L^\tau(B_M) \to L^\tau(B_M)$ is compact. Next, if $v \in L^\tau(B_M)$ satisfies $(*)$ with $f = 0$ then $v = k_0^2 V_q v$ in B_M so $v \in W^{2,\tau}(B_M)$, and since v is real analytic in the complement of supp q, it can be extended uniquely to any $B_{\widetilde{R}}$, $R_M \leq \widetilde{R} < \infty$, such that $v \in W^{2,\tau}(B_{\widetilde{R}})$. By Lemma 3 in Lechleiter *et al.* (2013), we therefore have $v \equiv 0$ in $\boldsymbol{R}^n$, and it remains to invoke the classical Riesz theory, for example Corollary 3.5 in Kress, *Linear Integral Equations*, 2014.

Define

$$C_n = \sup_{x\in B_M} \int_{y\in \text{supp}\, q} |\Phi^{\{n\}}(x-y)|,$$

$$\widetilde{C}_n = \sup_{y\in \text{supp}\, q} \int_{x\in B_M} |\Phi^{\{n\}}(x-y)|,$$

and

$$c(k_0, q, R_M, n) = k_0^2 C_n^{1/\tau} \widetilde{C}_n^{(\tau-1)/\tau} \|q\|_{L^\infty(B_q)}.$$

Main result: deterministic medium $q(x)$

Theorem. (K. & Linder-Steinlein, $n = 2$) If $q \in L^\infty(B_q)$ is deterministic and $f \in L^\tau(B_f)$ with $\tau \geq 2$ then

$$Ff = F_0 f + k_0^2 F_0(qV_\mathbf{1} f) + O\left(C_n^{1/\tau} \widetilde{C}_n^{(\tau-1)/\tau} \|f\|_{L^\tau(B_f)} c(k_0, q, R_M, 2)^2 \right)$$

as $c(k_0, q, R_M, 2) \to 0$, where

$$F_0 f = \sum_{m \in \boldsymbol{Z}} \sigma_m^{\{2,f\}} (f, \psi_m^{\{2,f\}}) \phi_m^{\{2,M\}}$$

and

$$k_0^2 F_0(qV_\mathbf{1} f) = \sum_{m \in \boldsymbol{Z}} \sigma_m^{\{2,q\}} \sum_{\nu \in \boldsymbol{Z}} \lambda_{m,\nu}(q)(f, \psi_\nu^{\{2,f\}}) \phi_m^{\{2,M\}}.$$

Note the 'spectral leakage.' It occurs due to the presence of deterministic (as well as random) media, and it makes the inverse source problem $u|_{\partial B_M} \mapsto f$ more ill-posed.

Here

$$\sigma_m^{\{2,X\}} = \sqrt{2R_M}\pi R_X |H_m^{(1)}(k_0 R_M)| A_m(k_0 R_X), \quad m \in \boldsymbol{Z},$$

$$\psi_m^{\{2,X\}}(x) = \frac{J_m(k_0|x|)e^{\mathrm{i}m\angle x}}{\sqrt{\pi}R_X A_m(k_0 R_X)} \quad \text{for } x \in B_X,$$

$$\phi_m^{\{2,M\}}(\theta) = \frac{\mathrm{e}^{\mathrm{i}\angle H_m^{(1)}(k_0 R_M)}\mathrm{e}^{\mathrm{i}m\theta}}{\sqrt{2\pi R_M}} \quad \text{for } \theta \in \boldsymbol{R},\ m \in \boldsymbol{Z},$$

$$A_m(k_0 R_X) = \sqrt{J_m(k_0 R_X)^2 - J_{m-1}(k_0 R_X) J_{m+1}(k_0 R_X)}, \quad m \in \boldsymbol{Z},$$

and

$$\lambda_{m,\nu}(q) = k_0^2 \sqrt{\pi} R_f A_\nu(k_0 R_f) \int_{B_q \setminus B_f} H_\nu^{(1)}(k_0|y|) e^{\mathrm{i}\nu\angle y} q(y) \overline{\psi_m^{\{2,q\}}}(y) dy.$$

Main result: deterministic medium $q(x)$

Theorem. (K. & Linder-Steinlein, $n = 3$) If $q \in L^\infty(B_q)$ is deterministic and $f \in L^\tau(B_f)$ with $\tau \geq 2$ then

$$Ff = F_0 f + k_0^2 F_0(qV_{\mathbf{1}} f) + O\left(C_n^{1/\tau}\, \widetilde{C}_n^{(\tau-1)/\tau} \|f\|_{L^\tau(B_f)} c(k_0, q, R_M, 3)^2\right)$$

as $c(k_0, q, R_M, 3) \to 0$, where

$$F_0 f = \sum_{m\in\mathbf{N}_0} \sum_{\mu=-m}^{m} \sigma_m^{\{3,f\}} (f, \psi_{m,\mu}^{\{3,f\}}) \phi_{m,\mu}^{\{3,M\}}$$

and

$$k_0^2 F_0(qV_{\mathbf{1}} f) = \sum_{m\in\mathbf{N}_0} \sigma_m^{\{3,q\}} \sum_{\mu=-m}^{m} \sum_{\nu\in\mathbf{N}_0} \sum_{\nu'=-\nu}^{\nu} \lambda_{m,\mu,\nu}(q)(f, \psi_{\nu,\nu'}^{\{3,f\}}) \phi_{m,\mu}^{\{3,M\}}.$$

(Note the 'spectral leakage.')

Here

$$\sigma_m^{\{3,X\}} = \frac{R_X R_M \sqrt{k_0 \pi}}{2} |h_m^{(1)}(k_0 R_M)| a_m(k_0 R_X), \quad m \in \boldsymbol{Z},$$

$$\psi_{m,\mu}^{\{3,X\}}(x) = \frac{2\sqrt{k_0/\pi}}{R_X a_m(k_0 R_X)} j_m(k_0 r) Y_m^\mu(x/|x|) \text{ for } x \in B_X, \quad m \in \boldsymbol{Z},\ \mu = -m, \dots, m,$$

$$\phi_{m,\mu}^{\{3,M\}}(\omega) = \frac{\mathrm{i}}{R_M} \mathrm{e}^{\mathrm{i} \arg h_m^{(1)}(k_0 R_M)} Y_m^\mu(\omega) \text{ for } \omega \in S^2, \quad m \in \boldsymbol{Z},\ \mu = -m, \dots, m,$$

$$a_m(k_0 R_X) = A_{m+1/2}(k_0 R_X),$$

and

$$\lambda_{m,\mu,\nu}(q) = \frac{\mathrm{i}\sqrt{\pi} k_0^{5/2} R_f}{2} a_\nu(k_0 R_f) \int_{B_q \setminus B_f} h_\nu^{(1)}(k_0|y|) Y_\nu^{\nu'}(y/|y|) q(y) \overline{\psi_{m,\mu}^{\{3,q\}}(y)} dy.$$

Lemma. (K. & Linder-Steinlein)
$\|V_q\|_{L^\tau(B_M)\to L^\tau(B_M)} \leq C_n^{1/\tau}\, \widetilde{C}_n^{(\tau-1)/\tau} \|q\|_{L^\infty(B_q)} = c(k_0, q, R_M, n)/k_0^2.$

Proof. We have $\Phi^{\{n\}} \in L^1_{\rm loc}(\boldsymbol{R}^n)$ for $n = 2$ and $n = 3$: the Hankel function $H_0^{(1)}$ is real analytic in $\boldsymbol{R}^2 \setminus \{0\}$, has a log-type singularity at the origin, and

$$\int_{\substack{x\in\boldsymbol{R}^2\\|x|<1}} |\ln|x|| = -2\pi \int_{r=0}^1 r \ln r = \pi/2,$$

while

$$\int_{\substack{x\in\boldsymbol{R}^3\\|x|<1}} |\Phi^{\{3\}}(x)| = 4\pi \int_{r=0}^1 \frac{r^2}{4\pi r} = 1/2.$$

Therefore, with $R' = \max\{|x|,\ x \in \operatorname{supp} q\}(< R_M)$, we have $C_n \leq \|\Phi^{\{n\}}\|_{L^1(B_{R_M+R'})} < \infty$ and $\widetilde{C}_n \leq \|\Phi^{\{n\}}\|_{L^1(B_{R_M+R'})} < \infty$, and the function $\boldsymbol{R}^n \times \boldsymbol{R}^n \ni (x, y) \mapsto \Phi^{\{n\}}(x - y)q(y)$ is measurable on $B_M \times B_M$; indeed,

$$\int_{y\in B_q} |\Phi^{\{n\}}(x-y)q(y)| \leq C_n \|q\|_{L^\infty(B_q)}, \quad x \in B_M,$$

and

$$\int_{x\in B_M} |\Phi^{\{n\}}(x-y)q(y)| \leq \widetilde{C}_n \|q\|_{L^\infty(B_q)}, \quad y \in B_q.$$

The result now follows from Proposition 5.1 on p. 573 in Taylor, *Partial Differential Equations I: Basic Theory*, 2011.

The solution of $(**)$ is given by $u = (I - k_0^2 V_q)^{-1} V_{\mathbf{1}} f$. Thus, if

$$c = c(k_0, q, R_M, n) = k_0^2 \|q\|_{L^\infty(B_q)} C_n^{1/\tau} \widetilde{C}_n^{(\tau-1)/\tau} < 1$$

then the inverse of $I - k_0^2 V_q$ is expressible in terms of a convergent Neumann series, and

$$u = \sum_{j=0}^{\infty} k_0^{2j} V_q^j V_{\mathbf{1}} f = V_{\mathbf{1}} f + k_0^2 V_q V_{\mathbf{1}} f + O\left(C_n^{1/\tau} \widetilde{C}_n^{(\tau-1)/\tau} \|f\|_{L^\tau(B_f)} c^2 \right) \qquad (1)$$

as $c \to 0$. In the following let $\tau \geq 2$ and define formally the trace operator γ_0^+ by

$$\gamma_0^+ u(x_0) = \lim_{x \nearrow x_0} u(x), \quad x_0 \in \partial B_M,$$

with the limit taken from B_M. Application of γ_0^+ to (1) yields

$$\gamma_0^+ u = F_0 f + F_1 f + O(c^2) = F_0(f + k_0^2 q V_{\mathbf{1}} f) + O(c^2) \text{ as } c \to 0,$$

with the 'medium-free' source-to-measurement map $F_0 = \gamma_0^+ V_{\mathbf{1}}$ and the 'first-order medium' source-to-measurement map $F_1 = k_0^2 \gamma_0^+ V_q V_{\mathbf{1}} = k_0^2 \gamma_0^+ V_{\mathbf{1}} q V_{\mathbf{1}} = k_0^2 F_0 q V_{\mathbf{1}}$.

We have already characterized spectrally F_0 for $n = 2$ (K., 2018) and for $n = 3$ (Kirkeby, Henriksen, & K., 2020). Our problem geometry is such that $x \in (\operatorname{supp} q)^\circ$, $y \in (\operatorname{supp} f)^\circ$ implies $|x| > |y|$, so, in case $n = 2$, the Graf addition theorem (Eq. 9.1.79 on p. 363 in Abramowitz and Stegun, 1972) gives

$$
\begin{aligned}
V_{\mathbf{1}} f(x) &= \sum_{\nu \in \boldsymbol{Z}} H_\nu^{(1)}(k_0|x|) \mathrm{e}^{\mathrm{i}\nu \angle x} \int_{B_f} J_\nu(k_0|y|) \mathrm{e}^{-\mathrm{i}\nu \angle y} f(y) dy \\
&= \sqrt{\pi} R_f \sum_{\nu \in \boldsymbol{Z}} A_\nu(k_0 R_f) H_\nu^{(1)}(k_0|x|) \mathrm{e}^{\mathrm{i}\nu \angle x} (f, \psi_\nu^{\{2,f\}}) \quad \text{for } x \in (\operatorname{supp} q)^\circ.
\end{aligned}
$$

This, in turn, implies

$$
\begin{aligned}
F_0^{\{2\}}(q V_{\mathbf{1}} f) = \sqrt{\pi} R_f \sum_{m \in \boldsymbol{Z}} \sum_{\nu \in \boldsymbol{Z}} & \sigma_m^{\{2,q\}} A_\nu(k_0 R_f) (f, \psi_\nu^{\{2,f\}}) \\
& \times \left(\int_{B_q \setminus B_f} H_\nu^{(1)}(k_0|y|) e^{\mathrm{i}\nu \angle y} q(y) \overline{\psi_m^{\{2,q\}}(y)} dy \right) \phi_m^{\{2,M\}}.
\end{aligned}
$$

The analysis of the case $n=3$ is similar. Since $x \in (\operatorname{supp} q)^\circ$, $y \in (\operatorname{supp} f)^\circ$ implies $|x| > |y|$, we have by Theorem 2.11 on p. 31 of Colton and Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 2013, that

$$\begin{aligned} V_{\mathbf{1}} f(x) &= \mathrm{i}k_0 \sum_{\nu \in \boldsymbol{N}_0} \sum_{\nu'=-\nu}^{\nu} h_\nu^{(1)}(k_0|x|) Y_\nu^{\nu'}(x/|x|) \int_{B_f} j_\nu(k_0|y|) \overline{Y_\nu^{\nu'}(y/|y|)} f(y) dy \\ &= \frac{\mathrm{i}\sqrt{\pi k_0} R_f}{2} \sum_{\nu \in \boldsymbol{N}_0} \sum_{\nu'=-\nu}^{\nu} a_\nu(\kappa_f) h_\nu^{(1)}(k_0|x|) Y_\nu^{\nu'}(x/|x|)(f, \psi_{\nu,\nu'}^{\{f\}}) \quad \text{for } x \in (\operatorname{supp} q)^\circ. \end{aligned}$$

This, in turn, implies

$$\begin{aligned} F_0^{\{3\}}(qV_{\mathbf{1}} f) = \frac{\mathrm{i}\sqrt{\pi k_0} R_f}{2} & \sum_{m \in \boldsymbol{N}_0} \sum_{\mu=-m}^{m} \sum_{\nu \in \boldsymbol{N}_0} \sum_{\nu'=-\nu}^{\nu} \sigma_m^{\{3,q\}} a_\nu(\kappa_f)(f, \psi_{\nu,\nu'}^{\{3,f\}}) \\ & \times \left(\int_{B_q \setminus B_f} h_\nu^{(1)}(k_0|y|) Y_\nu^{\nu'}(y/|y|) q(y) \overline{\psi_{m,\mu}^{\{3,q\}}(y)} dy \right) \phi_{m,\mu}^{\{3,M\}}. \end{aligned}$$

The case with random medium $q(x,\omega)$

If q is a centered Gaussian random field then the Borell-TIS theorem[2] implies

$$\begin{aligned}
&\mathbb{P}\left(\|q(\cdot,\omega)\|_{L^\infty(B_R)}^{1/2} < C_n^{-1/(2\tau)}\,\widetilde{C}_n^{(1-\tau)/(2\tau)}/k_0\right)\\
&\quad = 1-\mathbb{P}\left(\|q(\cdot,\omega)\|_{L^\infty(B_R)}^{1/2} \geq C_n^{-1/\tau}\,\widetilde{C}_n^{(1-\tau)/\tau}/k_0^2\right)\\
&\quad \geq 1-\exp\left(-\left(C_n^{-1/\tau}\,\widetilde{C}_n^{(1-\tau)/\tau}/k_0^2-\mathbb{E}\left[\|q(\cdot,\omega)\|_{L^\infty(B_R)}^{1/2}\right]\right)^2 \Big/ \left(2\sigma_{B_M}^2\right)\right),
\end{aligned}$$

with $\sigma_{B_M}^2 := \sup_{x\in B_M}\mathbb{E}\left[q^2\right]$. Also, realizations of q are L^∞ a.s. on compact subsets of $\boldsymbol{R}^n$.

The convergence of the Neumann series for $(I-k_0^2V_q)^{-1}$ constrains the variability of the random fields from which $q(x,\omega)$ is allowed to originate.

[2] Theorem 2.1.1 in Adler and Taylor, *Random fields and geometry*. Springer, 2007.

The covariance function C of the medium $q(x,\omega)$ is a positive definite function which depends on the underlying physics of the problem at hand. The associated covariance operator is defined by

$$(\mathcal{C}g)(x) := \int C(x,y)g(y)dy$$

Now $q(x,\omega)$ is a second-order random field, and we use the eigensystem $\{\alpha_j, \varphi_j\}_{j=1}^{\infty}$ of the covariance operator for a Karhunen-Loève expansion of $q(x,\omega)$:

$$q(x,\omega) = \eta(x) + \sum_{j=0}^{\infty} \sqrt{\alpha_j}\varphi_j(x)\xi_j(\omega), \quad x \in \overline{B}_q \setminus B_f,\ \omega \in \Omega.$$

Here $\eta(x) = \mathbb{E}_{\omega\in\Omega}\left[q(x,\omega)\right]$, and $\xi_j(\omega)$ are pairwise uncorrelated $\mathcal{N}(0,1)$ random variables given by

$$\xi_j(\omega) := \frac{1}{\sqrt{\alpha_j}} \left(q(x,\omega) - \eta(x), \varphi_j(x)\right)_{L^2(\overline{B}_q \setminus B_f)}.$$

Remember that, in the deterministic case,

$$k_0^2 F_0(qV_{\mathbf{1}}f) = \sum_{m\in\mathbf{Z}} \sigma_m^{\{2,q\}} \sum_{\nu\in\mathbf{Z}} \lambda_{m,\nu}(q)(f, \psi_\nu^{\{2,f\}})\phi_m^{\{2,M\}},$$

with

$$\lambda_{m,\nu}(q) = k_0^2\sqrt{\pi}R_f A_\nu(k_0 R_f)\int_{B_q\setminus B_f} H_\nu^{(1)}(k_0|y|)e^{\mathrm{i}\nu\angle y}q(y)\overline{\psi_m^{\{2,q\}}}(y)dy.$$

When q is stochastic, we compute the resulting stochastic integral above by inserting the Karhunen-Loève expansion for q in the integrand. Our KL expansion converges in the L^2 sense on the compact set $\overline{B}_q \setminus B_f$; recall that $q(\cdot,\omega)$ is a.s. in $L^\infty(\overline{B}_q \setminus B_f)$.

Main result: stochastic medium $q(x,\omega)$

Write $s_{m,\nu}^{\{2\}}(y) = H_\nu^{(2)}(k_0|y|)e^{-\mathrm{i}\nu\angle\mathrm{y}}\psi_m^{\{2,q\}}(y)$ and
$s_{m,\mu,\nu}^{\{3\}}(y) = \overline{h_\nu^{(1)}(k_0|y|)Y_\nu^{\nu'}(y/|y|)}\psi_{m,\nu}^{\{3,q\}}(y)$.

Theorem. (K. & Linder-Steinlein, $n=2$) For second order stationary Gaussian random fields q, the forward operator satisfies $Ff = F_0 f + k_0^2 F_0(qV_\mathbf{1} f) + \varepsilon$ with $F_0 f$ as in the deterministic case, and where

$$
\begin{aligned}
k_0^2 F_0(qV_\mathbf{1} f) =& k_0^2\sqrt{\pi}R_f \sum_m \sigma_m^{\{2,q\}} \sum_\nu A_\nu(k_0 R_f)(f,\psi_\nu^{\{2,f\}}) p_{m,\nu}^{\{2\}} \cdot n\phi_m^{\{2,M\}} + \\
& k_0^2\sqrt{\pi}R_f \sum_m \sigma_m^{\{2,q\}} \sum_\nu A_\nu(k_0 R_f)(f,\psi_\nu^{\{2,f\}}) \tilde{A}\left(\theta_{m,\nu}^{\{2\}}\right)\begin{pmatrix}\bar{\xi}_{m,\nu}^{\{2\}} \\ 0\end{pmatrix} \cdot n\phi_m^{\{2,M\}}
\end{aligned}
$$

(deterministic mean value + stochastic component; currently tractable only numerically)

Randomness in the medium q causes randomness in the spectral leakage. The eigenfunctions and eigenvalues of the covariance operator $\mathcal{C}$ determine the statistical properties of the leakage.

Here,

$$\theta_{m,\nu}^{\{2\}} = \arctan\left(\frac{a_{m,\nu}^{\{2\}}}{b_{m,\nu}^{\{2\}}}\right), \qquad \bar{\xi}_{m,\nu}^{\{2\}} \sim \mathcal{N}\left(0, a_{m,\nu}^{\{2\},2} + b_{m,\nu}^{\{2\},2}\right),$$

$$a_{m,\nu}^{\{2\}} = \sum_{j=1}^{\infty} \sqrt{\alpha_j}\Re\left[\left(\varphi_j, s_{m,\nu}^{\{2\}}\right)_{L^2(D)}\right] \qquad b_{m,\nu}^{\{2\}} = \sum_{j=1}^{\infty} \sqrt{\alpha_j}\Im\left[\left(\varphi_j, s_{m,\nu}^{\{2\}}\right)_{L^2(D)}\right]$$

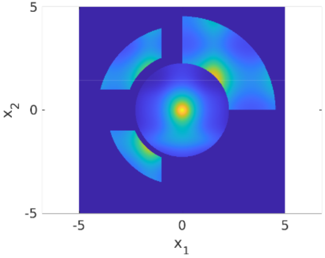
$$p_{m,\nu}^{\{2\}} = \begin{pmatrix} \Re\left[\left(\eta, s_{m,\nu}^{\{2\}}\right)_{L^2(D)}\right] \\ \Im\left[\left(\eta, s_{m,\nu}^{\{2\}}\right)_{L^2(D)}\right] \end{pmatrix}, \qquad n = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Idea of the proof. Under the assumptions on q, the random field, it is possible to substitute the *KLE* for the general Gaussian setting and the modified spectral values become

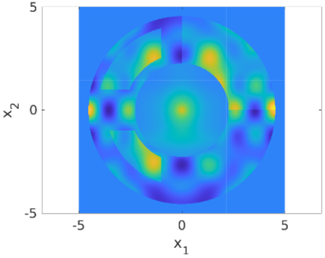
$$\lambda_{m,\nu} = \sqrt{\pi} R_f A_\nu(\kappa_f)\left[\left(\eta, s_{m,\nu}^{\{2\}}\right)_{L^2(D)} + \lim_{k\to\infty}\sum_{j=1}^{k}\sqrt{\lambda_j}\xi_j\left(\varphi_j, s_{m,\nu}^{\{2\}}\right)_{L^2(D)}\right]$$

$$\lambda_{m,\mu,\nu} = \frac{i\sqrt{\pi}k_0^{5/2}R_f}{2} a_\nu(\kappa_f)\left[\left(\eta, s_{m,\mu,\nu}^{\{3\}}\right)_{L^2(D)} + \lim_{k\to\infty}\sum_{j=1}^{k}\sqrt{\lambda_j}\xi_j\left(\varphi_j, s_{m,\mu,\nu}^{\{3\}}\right)_{L^2(D)}\right]$$

Here $\xi_j \sim \mathcal{N}(0,1)$.



(a) Deterministic setup.



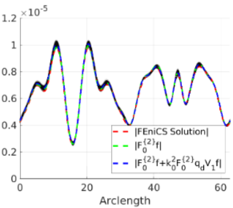
(b) Stochastic setup, truncated white noise.

Fig. 2: Configuration of source and medium for numerical results in the deterministic case 2a and stochastic case 2b. Here $k_0 = 2\pi$, $R_m = 5$, $R_q = R_m/1.1$, $R_f = R_q/2$, $\|q_d\|_\infty \approx 0.0030$, $\|q_s\|_\infty \approx 0.0045$, $c_d \approx 0.442$, and $c_s \approx 0.666$.

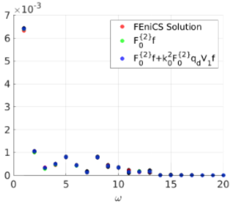
$$q(x) = \eta(x) + s(x)\frac{dW(x)}{dx}, \quad \operatorname{supp} s \subset B_q \setminus \overline{B_f}, \quad x \in B_M.$$

Here $dW(x)/dx$ is the formal derivative of the white noise, in the sense that it is the derivative of the Karhunen-Loéve expansion of the Brownian sheet.

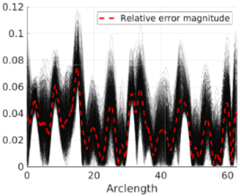
Figs. 2–7 from K. & Linder-Steinlein, Spectral properties of radiation for the Helmholtz equation with a random coefficient (2022), *submitted*



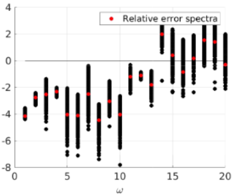
(a) Absolute value $|u|_{\partial B_M}|$ of the measurement.



(b) Spectrum $|(u|_{\partial B_M}, \exp(im\theta))_{L^2(S^1)}|$ of measurements.



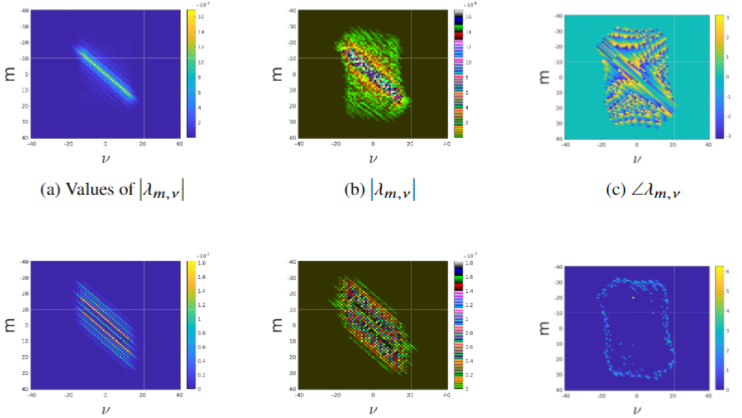
(c) Absolute relative error w.r.t. FEniCS in measurements.



(d) Log-plot of relative error in the spectra.

Fig. 3: Comparison of numerical solutions to the Helmholtz equation with deterministic medium present, obtained for $k_0 = 2\pi$, $R_m = 5$, $R_q = R_m/1.1$, $R_f = R_q/2$, $\|q\|_\infty \approx 0.0030$ and $c \approx 0.442$. Black curve and spots indicates values from sampling. Stochasticity is Brownian based.

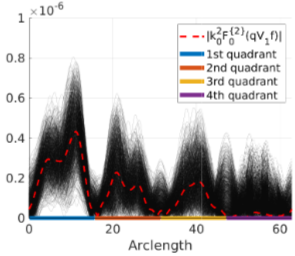
$$\frac{\|F_0 f - U_{\text{FEniCS}}\|_{L^2}}{\|U_{\text{FEniCS}}\|_{L^2}} = 0.185; \qquad \frac{\|F_0(Id + k_0^2 q V_1)f - U_{\text{FEniCS}}\|_{L^2}}{\|U_{\text{FEniCS}}\|_{L^2}} = 0.175$$



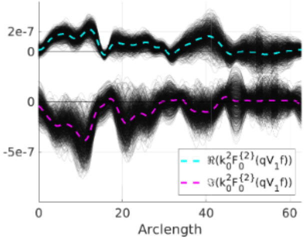
(a) Values of $|\lambda_{m,\nu}|$ (b) $|\lambda_{m,\nu}|$ (c) $\angle\lambda_{m,\nu}$

(d) $\left|\lambda_{m,\nu} - \mathbb{E}\left[\lambda^s_{m,\nu}\right]\right|$ (e) $\left|\lambda_{m,\nu} - \mathbb{E}\left[\lambda^s_{m,\nu}\right]\right|$ (f) $\left|\angle\lambda_{m,\nu} - \angle\mathbb{E}\left[\lambda^s_{m,\nu}\right]\right|$

Fig. 4: Illustration of the $\lambda_{m,\nu}$ for a truncation of $|M| = 40$, $|N| = 40$, $k_0 = 2\pi$, $R_m = 5$, $R_q = R_m/1.1$, $R_f = R_q/2$, $\|q_d\|_\infty \approx 0.0030$, $\|q_s\|_\infty \approx 0.0045$, $c_d \approx 0.442$ and $c_s = 0.666$. Calculating the angle of the spectrum has been done by setting every element 2000 times smaller than the absolute maximum to zero. The notation $\lambda^s_{m,\nu}$ is used for the case with stochasticity. Figure 4b and 4e show the same information as figure 4a and 4d but with enhanced details.



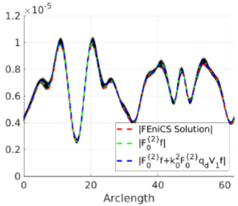
(a) Magnitude of perturbation.



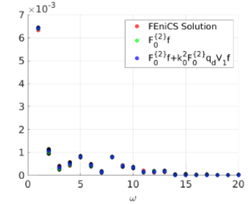
(b) Real and imaginary part of perturbation.

Fig. 5: Contribution of the perturbation term arising when a medium q is present. The quadrants are those of the circle. Here $k_0 = 2\pi$, $R_m = 5$, $R_q = R_m/1.1$, $R_f = R_q/2$, $\|q\|_\infty \approx 0.0030$ and $c \approx 0.442$. See e.g. Figure 2a for source and medium configuration.

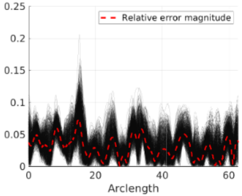
Exponential covariance: $C(x,y) = e^{-\sum_{j=1}^{d} |x_j - y_j|/\ell_j}$



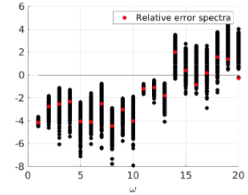
(a) Absolute value $|u|_{\partial B_M}|$.



(b) Spectrum $|(u|_{\partial B_M}, \exp(\mathrm{i}\omega\theta))_{L^2(S^1)}|$ of solutions.

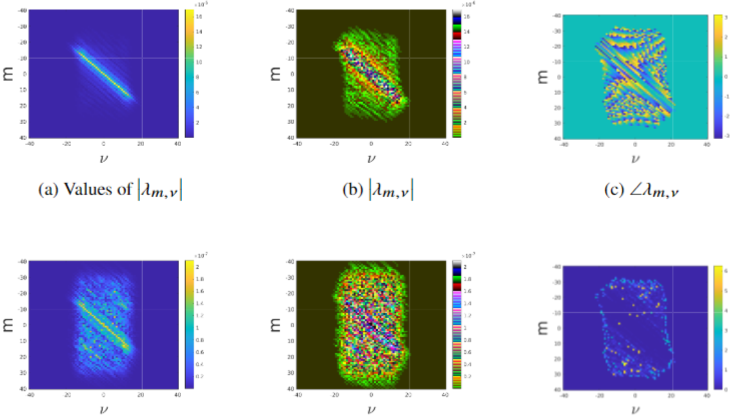


(c) Absolute relative error w.r.t. FEniCS.



(d) Log-plot of relative error of the spectra.

Fig. 6: Comparison of numerical solutions to the Helmholtz equation with deterministic medium present, obtained for $k_0 = 2\pi$, $R_m = 5$, $R_q = R_m/1.1$, $R_f = R_q/2$, $\|q\|_\infty \approx 0.0030$ and $c \approx 0.4412$. Black curve and spots indicates values from sampling. Stochasticity with exponential covariance.



(a) Values of $|\lambda_{m,\nu}|$ (b) $|\lambda_{m,\nu}|$ (c) $\angle\lambda_{m,\nu}$

(d) $|\lambda_{m,\nu} - \mathbb{E}[\lambda^s_{m,\nu}]|$ (e) $|\lambda_{m,\nu} - \mathbb{E}[\lambda^s_{m,\nu}]|$ (f) $|\angle\lambda_{m,\nu} - \angle\mathbb{E}[\lambda^s_{m,\nu}]|$

Fig. 7: Illustration of the $\lambda_{m,\nu}$ for a truncation of $|M| = 40$, $|N| = 40$, $k_0 = 2\pi$, $R_m = 5$, $R_q = R_m/1.1$, $R_f = R_q/2$, $\|q_d\|_\infty \approx 0.0030$, $\|q_s\|_\infty \approx 0.004$, $c_d \approx 0.4412$ and $c_s \approx 0.67$. Calculating the angle of the spectrum has been done by setting every element 2000 times smaller than the absolute maximum to zero. The notation $\lambda^s_{m,\nu}$ is used for the case with stochasticity. Figure 7b and 7e show the same information as figure 7a and 7d but with enhanced details.

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