DTU Compute
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Uncertainty Quantification for Linear Inverse problems with Besov Prior

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1 Introduction

Besov priors are an important prior in imaging, because Besov priors are a consistent model of edges and non-smooth behaviour. The model properties of Besov priors has been applied in imaging, for example in Computed Tomography[1]. The current research on inverse problems with Besov priors does not provide much information about uncertainty quantification and sampling which is the focus of this poster.

2 Linear inverse problem with Besov prior

We consider a discrete linear inverse problem

$$y = Af + \epsilon, \qquad (1)$$

where $A \in \mathbb{R}^{m\times n}$ is the forward operator, $f \in \mathbb{R}^n$ is the unknown, and $y \in \mathbb{R}^m$ is the observed data, that is corrupted by i.i.d. Gaussian noise $\epsilon \sim \mathcal{N}(0, \sigma^2 I_m)$. To solve eq. (1) we take a Bayesian approach, where we assume that the unknown f follows a Besov Prior. With these assumptions the posterior distribution is given by

$$\pi(f|y) = C \exp\left(-\tfrac{1}{2\sigma^2}\|Af - y\|_2^2 - \tfrac{1}{p}\|Bf\|_p^p\right), \qquad (2)$$

where $C > 0$ is a normalization constant, and $B \in \mathbb{R}^{n\times n}$ is the Besov matrix. In the simplest form the action of the Besov matrix is given by

$$(Bf)_h = 2^{j(s+1/2-1/p)} w_{j,h-2^j}, \; s > 0, \; 1 \le p < \infty,$$

where $j = \lfloor \log_2(h) \rfloor$, and $w_{j,k}$ is suitable wavelet coefficients. We want to sample from the posterior in eq. (2).

3 Sampling Method

For sampling we generalize the idea in [2] where a prior transform $g : \mathbb{R}^n \to \mathbb{R}^n$ is used to transform the posterior in eq. (2) into a Gaussian posterior on the form

$$\pi(\tilde{f}|y) \propto \exp\left(-\tfrac{1}{2\sigma^2}\|AB^{-1}g(\tilde{f}) - y\|_2^2 - \tfrac{1}{2}\|\tilde{f}\|_2^2\right). \qquad (3)$$

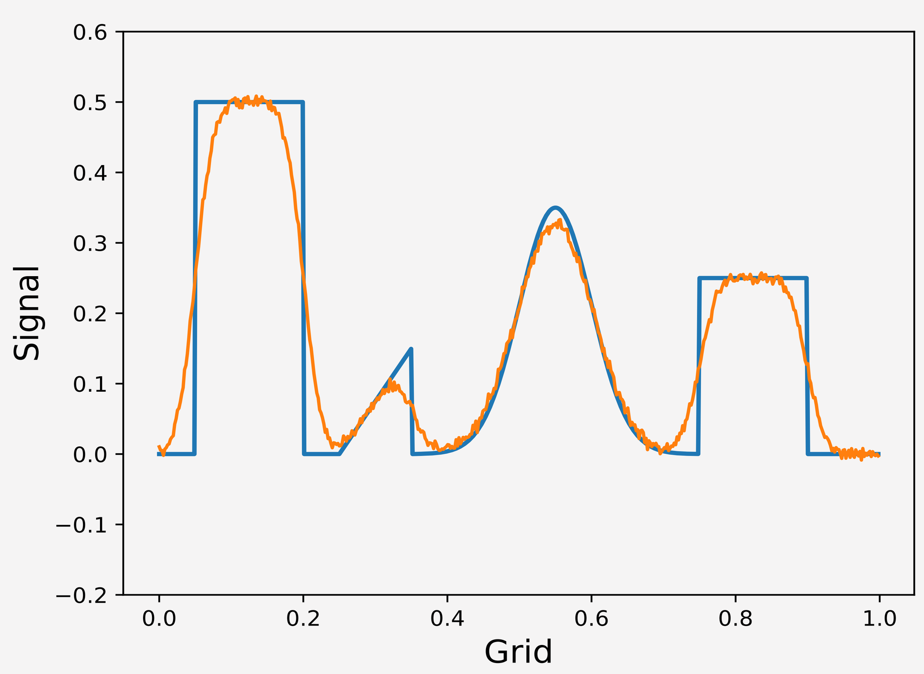
We sample the transformed posterior eq. (3) with a Randomize-Then-Optimize (RTO) approach including a Metropolis-Hastings(MH) acceptance rejection step since the problem is now nonlinear.

Procedure:

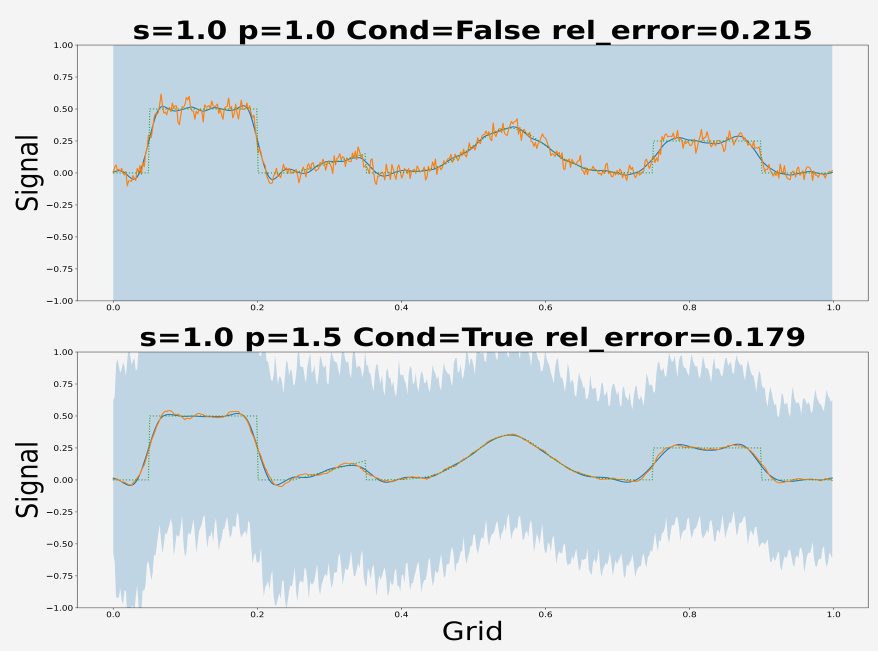
1: Determine the prior transform $g : \mathbb{R}^n \to \mathbb{R}^n$.
2: **for** $i = 1, \ldots, n_{samp}$ **do**
3: Compute the RTO proposal $\tilde{f}^i_{prop}$
4: MH acceptance rejection step on the sample proposal $\tilde{f}^i_{prop}$
5: Transform back $f^i = g(\tilde{f}^i)$.
6: **end for**

4 Numerical Test

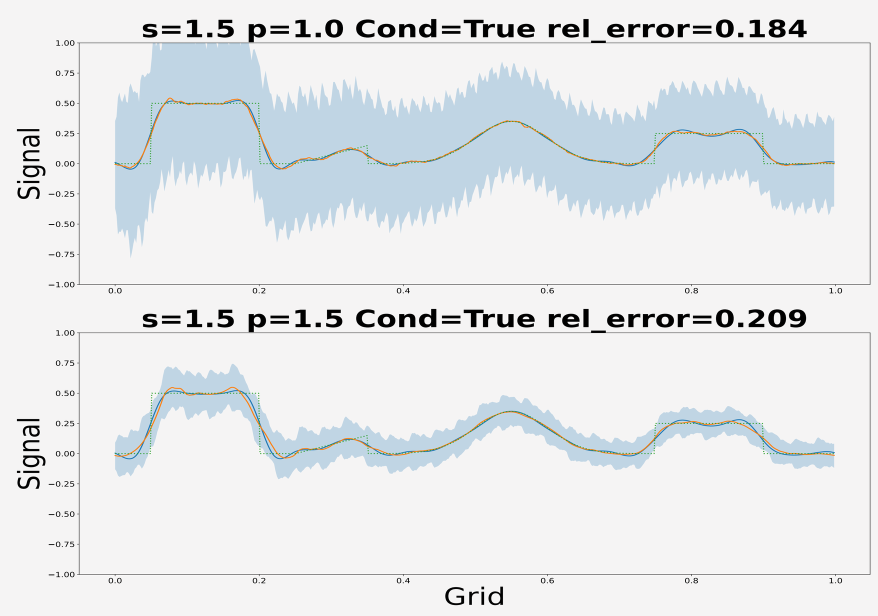
The RTO-MH sampling method with a prior transform is applied on a test signal with Gaussian blur and 2% relative noise.



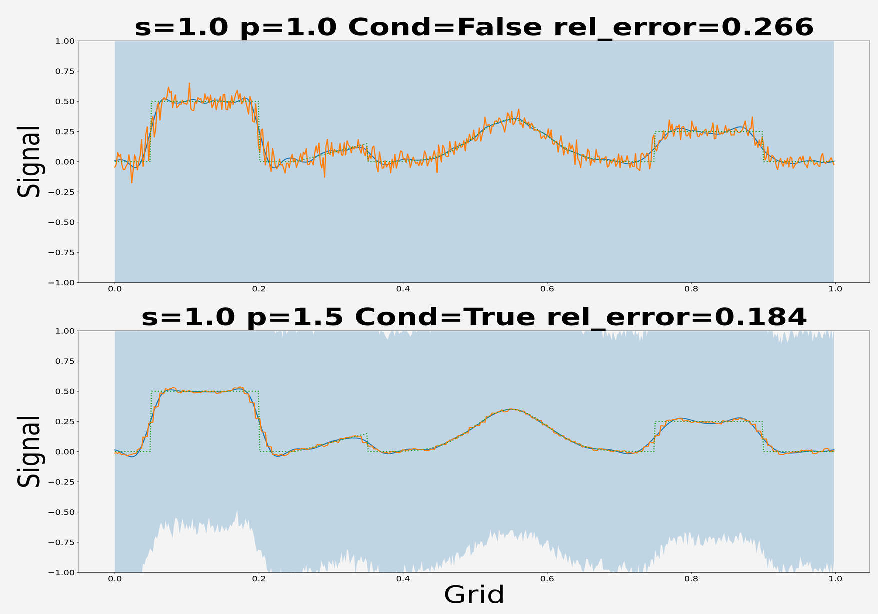
— True signal, — Data. The test signal and the generated noisy data

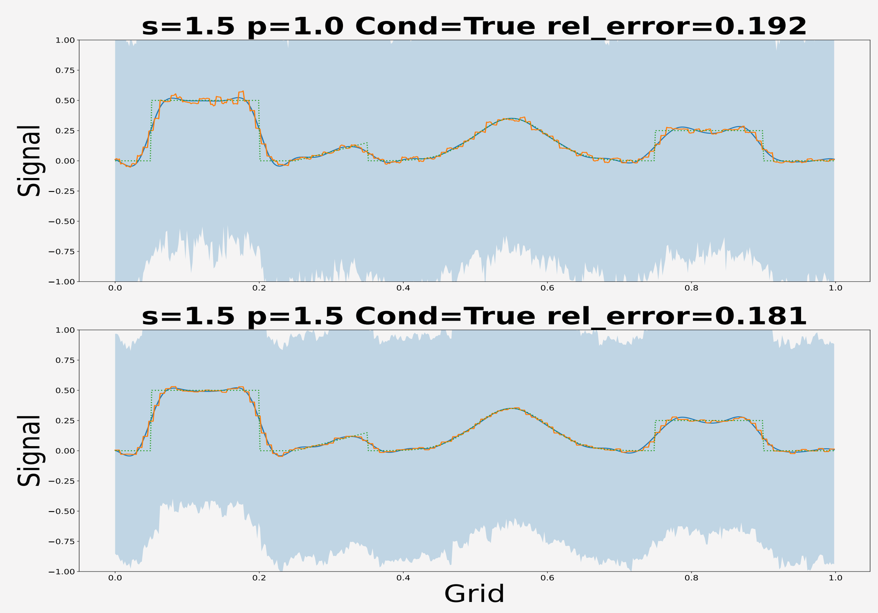


··· True signal, — Posterior mean. — MAP Sampling using the Daubechies4-wavelet where the Besov parameters, Besov condition $s - 1/p > 0$, and the relative error are in the titles



··· True signal, — Posterior mean. — MAP Sampling using the Daubechies4-wavelet





··· True signal, — Posterior mean. — MAP Sampling using the Daubechies1-wavelet.

References

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[2] Z. Wang, J. M. Bardsley, A. Solonen, T. Cui, and Y. M. Marzouk. "Bayesian Inverse Problems with l_1 Priors: A Randomize-Then-Optimize Approach". In: *SIAM Journal on Scientific Computing* 39.5 (2017), S140–S166.

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