DTU Compute
Department of Applied Mathematics and Computer Science

Towards UQ for Magnetic Resonance Electrical Impedance Tomography - M.Sc. project

Kenneth Scheel*, **Kim Knudsen and Babak Maboudi Afkham**

Technical University of Denmark, Department of Applied Mathematics and Computer Science, Kgs. Lyngby, Denmark

* Corresponding author, Email: s174488@student.dtu.dk

1 MREIT experiment

The subject is placed inside an MR scanner and pairs of electrodes are attached. Current is then applied through the electrodes.

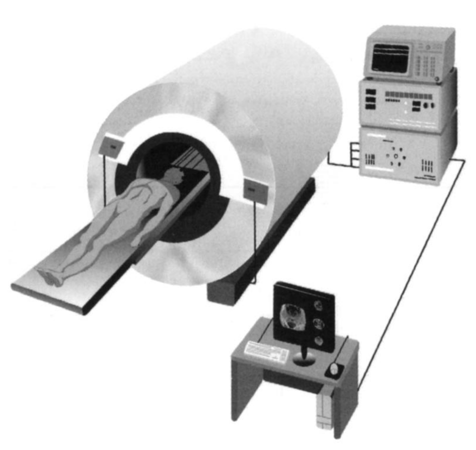
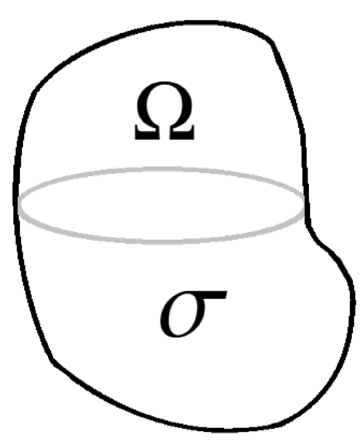


Figure 1: Schematic of MREIT system [1]

The current induces a magnetic field. MR magnitude images are obtained by the MR scanner. Reconstruct cross-sectional conductivity images.

2 Mathematical model

We formulate the MREIT problem in an open, bounded subset Ω of $\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$ and conductivity distribution σ [2].



$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega,$$
$$u = f \quad \text{on } \partial\Omega.$$

Current density $\mathcal{F}(\sigma) = \mathbf{J} = -\sigma\nabla u$. Map from $\mathbf{J}$ to magnetic field $\mathbf{B}$ is the Biot-Savart integral.

$$\mathcal{G} : \mathbf{J}(x) \mapsto \mathbf{B}(x) = \frac{1}{4\pi}\int_\Omega \mathbf{J}(x') \times \frac{x - x'}{|x - x'|^3}dx'.$$

The MR-scanner measures only the z-component of $\mathbf{B}$. The MREIT reconstruction problem is to determine σ from noisy B_3 data.

3 Toy example: ball inside unit cube

Consider a small ball $B(c, r)$ parameterized by its radius $r \in (0, 0.5)$ and center $c \in (0, 1)^3$, with

$$\sigma(x) = 1 + \chi_{B(c,r)}(x).$$

Infeasible to do UQ on discretized σ in 3D with n^3 cells. Reduce dimensionality by parametrization.

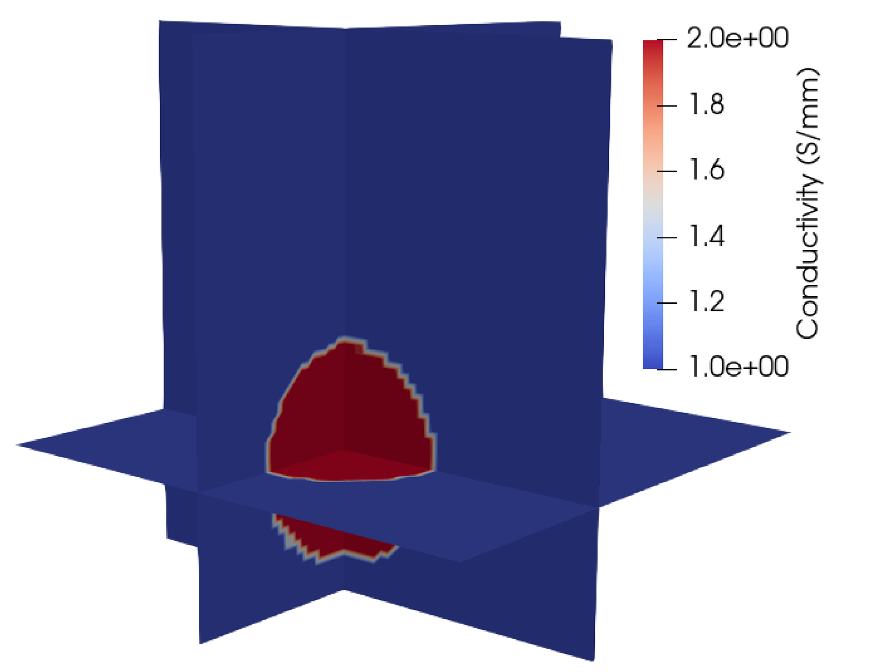


Figure 2: Example conductivity distribution σ in $\Omega = [0, 1] \times [0, 1]$

Forward mapping $\mathcal{F} : \sigma \mapsto \mathbf{J}$ is simulated using FEniCS [3]. Four slices of the x-component of $\mathbf{J}$ from the σ in fig. (2) are shown in fig. (3).

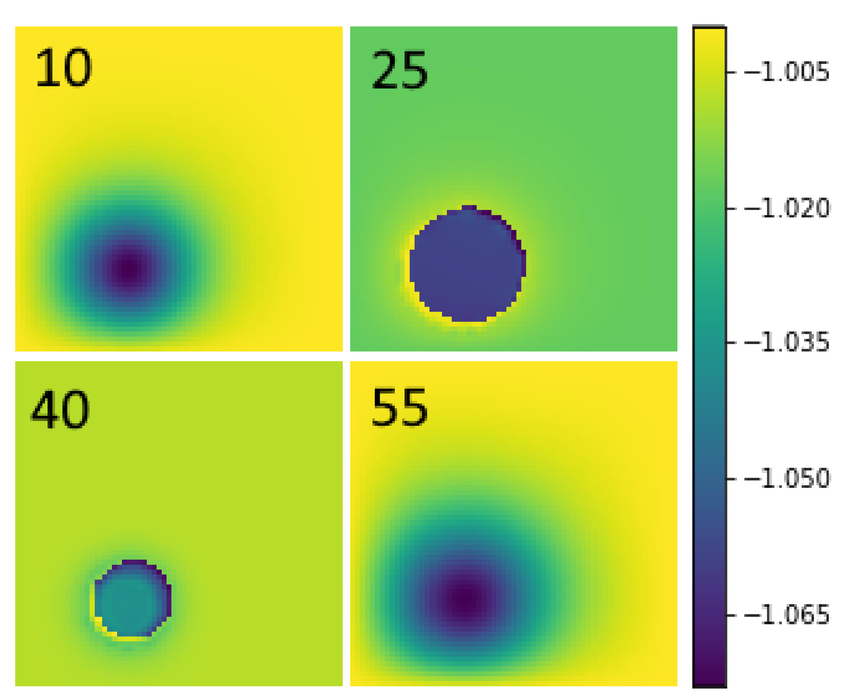


Figure 3: Slice $\{10, 25, 40, 55\}$ of x-component of current density field J_1 with $f = x$ and $n = 64$

The second forward mapping $\mathcal{G} : \mathbf{J} \mapsto \mathbf{B}$, i.e. the Biot-Savart integral, is evaluated using spectral methods [4].

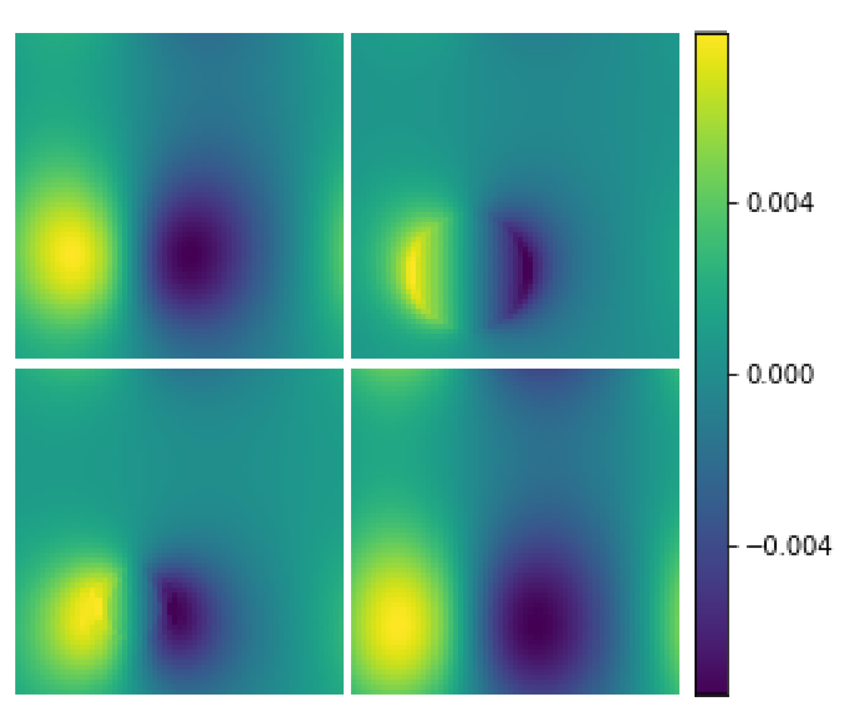


Figure 4: Slice $\{10, 25, 40, 55\}$ of z-component of induced magnetic field B_3

4 Bayesian inverse problem

We seek to identify σ from noisy measurements.

$$Y = \mathcal{G}(\mathcal{F}(\sigma)) + E, \quad E \sim N(0, \epsilon^2 I).$$

$\mathcal{F}$ is the map from conductivity σ to current density $\mathbf{J}$. $\mathcal{G}$ is the map from $\mathbf{J}$ to magnetic field $\mathbf{B}$.

Setting
- $\mathcal{F}$ nonlinear, continuous: $\sigma \mapsto \mathbf{J}$
- $\mathcal{G}$ linear, continuous: $\mathbf{J} \mapsto \mathbf{B}$
- $\sigma \in L^\infty_+(\Omega)$, $\mathbf{J} \in [L^2(\Omega)]^3$, $\mathbf{B} \in [L^2(\Omega)]^3$

We consider a piecewise constant prior for sigma with spherical inclusions. Bayes' rule gives the posterior.

$$\pi_{\text{post}}(r, c|y) \propto \pi_{\text{like}}(y|r, c)\pi_{\text{pr}}(r, c)$$

We use a Gaussian likelihood and place a uniform, non-informative prior on r and c.

5 Implementation

The CUQI team have developed CUQIpy, a package for Bayesian modelling of inverse problems. We intend to use CUQIpy as a key modelling tool.

UQ plan: step-by-step
- Sampling the posterior using MCMC
- Computing the posterior mean as an estimate for σ
- Computing credibility intervals to quantify uncertainties

References

[1] Seo, J. K., Woo, E. J. (2011). Magnetic Resonance Electrical Impedance Tomography (MREIT). SIAM Review, 53(1), 40–68.

[2] Yazdanian, H., Knudsen, K. (2021). Numerical conductivity reconstruction from partial interior current density information in three dimensions. Inverse Problems, 37(10), [105010].

[3] M. S. Alnaes, J. Blechta, J. Hake, A. Johansson, B. Kehlet, A. Logg, C. Richardson, J. Ring, M. E. Rognes and G. N. Wells. The FEniCS Project Version 1.5, Archive of Numerical Software 3 (2015).

[4] Yazdanian H., Saturnino G. B., Thielscher A. and Knudsen K. (2020), Fast evaluation of the Biot–Savart integral using FFT for electrical conductivity imaging, J. Comput. Phys. 411.

CUQI

THE VELUX FOUNDATIONS
VILLUM FONDEN VELUX FONDEN