DTU Compute
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How Quantifying Model Error Helps Improve Tomography

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1 Contributions

1. Reviewed state-of-art methods under Bayesian framework, and proposed the recursive updated error model;
2. Applied projected Gaussian to constrain the posterior, and discuss its advantage;
3. Implemented the algorithm onto MREIT problem, and show advantages with comparison.

2 Problem Description

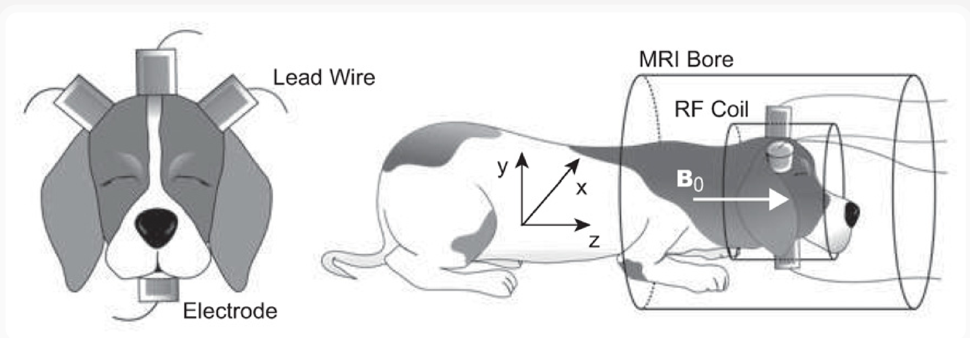

Figure 1: Recessed electrode configurations for MREIT experiments on the canine head[2].

The mathematical modelling of MREIT could be divided into two part:

1. Injected currents generates current density:

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0, & in\ \Omega \\ u = f, & on\ \partial\Omega \\ J = \sigma E = -\sigma \nabla u, & in\ \Omega \end{cases} \quad (1)$$

2. Current density generates magnetic flux density

$$J = \frac{1}{\mu_0} \nabla \times B, \quad in\ \Omega \quad (2)$$

The inverse problem is to approximate the conductivity σ with measurements of perturbed interior current $J_e = J + e$ since the second part is linear.

Given $J_e := F(\sigma) + e$, the linearization is denoted as

$$f\sigma = F(\sigma^*) + DF(\sigma^*)(\sigma - \sigma^*) \quad (3)$$

Then the accurate model goes like

$$J_e = f\sigma + m + e := f\sigma + v \quad (4)$$
$$m = F(\sigma) - f\sigma \quad (5)$$

According to Bayes' Theorem the posterior would be formulated as

$$\pi(\sigma|J_e) = \frac{1}{\pi(J_e)} \pi(\sigma)\pi(J_e|\sigma) \propto \pi(\sigma)\pi(J_e|\sigma) \quad (6)$$

The prior $\pi(\sigma)$ indicates our pre-knowledge of conductivity. As for the likelihood $\pi(J_e|\sigma)$ we have

$$\pi(J_e|\sigma) = \pi_{v|\sigma}(J_e - f\sigma|\sigma) \quad (7)$$

3 Method

Given σ and e gaussian distributed, assuming that m is independent from σ and Gaussian, then v is Gaussian. Following (6) and (7), we have gaussian posterior. To get better reconstruciton, we iteratively updated error model by pushing forward samples from the previous posterior, and updated σ^* with stable posterior estimate. Moreover, since the conductivity is always positive, we project the distribution onto some positive constrained set.

Recursive Updated Error Model

Take $\pi(\sigma) = \mathcal{N}(\sigma_0, \Gamma_\sigma)$, $\pi(e) = \mathcal{N}(0, \Gamma_e)$. Input sample number N, constrain set C, outside loops L, inside loops L_r. Set $\sigma^* = \sigma_0$, $l = 0$, $l_r = 0$, $\sigma^0 = \sigma_0$, $\Gamma^0 = \Gamma_\sigma$;
Sample $\left\{\sigma^0_{(i)}\right\}_{i=1,\dots,N} \sim \Pi_C(\pi(\sigma))$;
if $l_r < L_r$ **then**
  if $l < L$ **then**
    Calculate linearized values of samples as $f_h\sigma^l_{(i)}$ and get model error samples $m^l_{(i)}$;
    Evaluate model error sample mean m^l_0 and sample covariance Γ^l_v;
    Set $l = l + 1$, solve the posterior $\pi^l_{l_r}(\sigma^l_h|J_e)$;
    Sample $\left\{\sigma^l_{(i)}\right\}_{i=1,\dots,N} \sim \Pi_C\left(\pi^l_{l_r}\right)$;
  end
  Calculate the median of $\left\{\sigma^L_{(i)}\right\}_{i=1,\dots,N}$ and update σ^*;
  Set $l = 0$, $l_r = l_r + 1$;
end
Return $\pi^L_{L_r}$ as the result.

4 Result

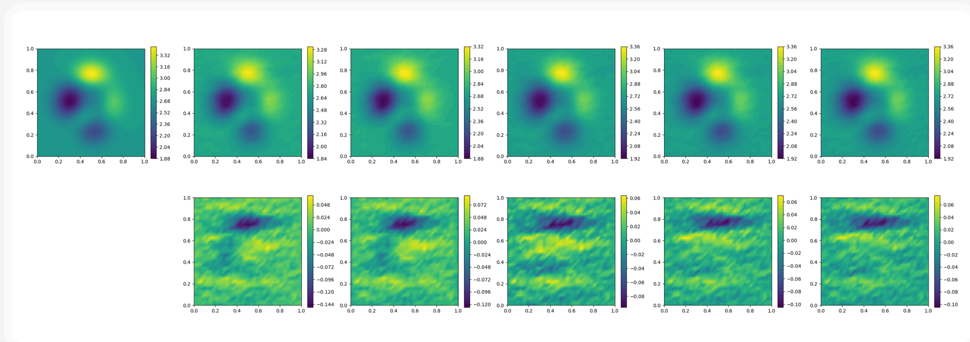

Figure 2: Figures in the first row from left to right are (1) the exact conductivity, (2) reconstruction from linear approximation without model error, (3) reconstruction from linear approximation with model error, (4) reconstruction from linear approximation with 2-time updated model error, (5) reconstruction from linear approximation with 2-time updated model error and updated reference point, (6) reconstruction from linear approximation with 2-time updated model error and 2-time updated reference point respectively. Figure in the second row are corresponding difference between the upper reconstruction and the exact conductivity.

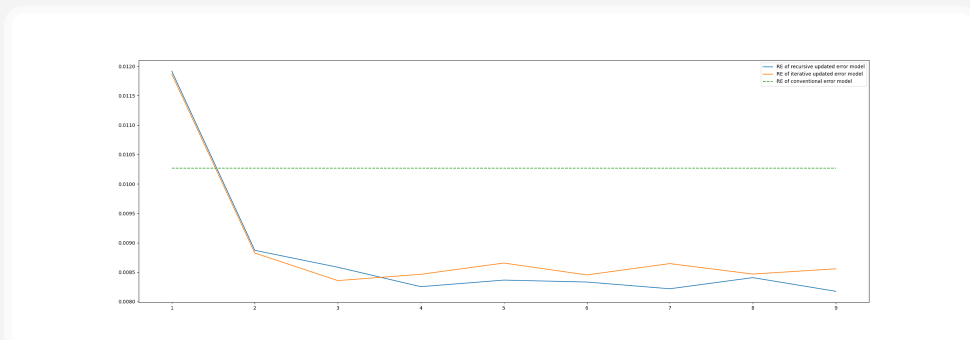

Figure 3: Relative error of the conventional error model, the iterative updated error model[1] and the recursive updated error model with respect to the exact conductivity.

References

[1] D. Calvetti, M. Dunlop, E. Somersalo, and A. Stuart. "Iterative updating of model error for Bayesian inversion". In: *Inverse Problems* 34.2 (2018), p. 025008.

[2] E. J. Woo and J. K. Seo. "Magnetic resonance electrical impedance tomography (MREIT) for high-resolution conductivity imaging". In: *Physiological Measurement* 29.10 (2008), R1–R26.

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