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Bayesian Approach to Quantitative Photoacoustic Tomography

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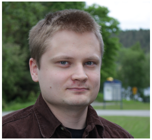    

Contents

Photoacoustic tomography (PAT)

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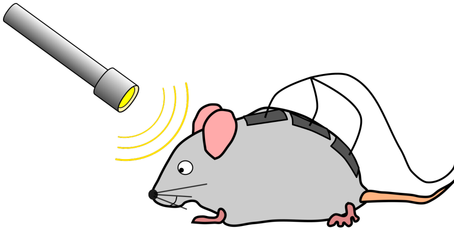
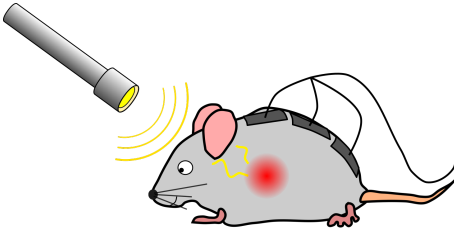
- Tissue is illuminated with a short pulse of light



Photoacoustic tomography (PAT)

- Tissue is illuminated with a short pulse of light
- As light propagates within the tissue, it is absorbed by chromophores
- The absorbed energy causes pressure rise



Photoacoustic tomography (PAT)

- Tissue is illuminated with a short pulse of light
- As light propagates within the tissue, it is absorbed by chromophores
- The absorbed energy causes pressure rise
- This pressure increase propagates through the tissue as an acoustic wave and can be measured on the boundary of the tissue using ultrasound sensors

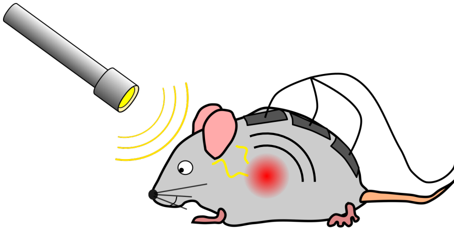
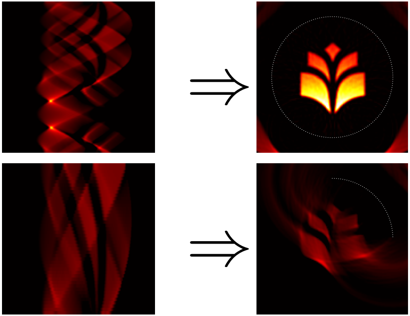
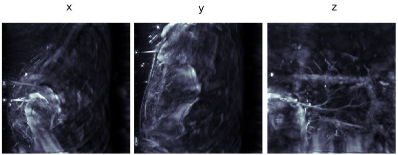


Image reconstruction

- Recover the initial pressure (or absorbed optical energy density) from the photoacoustic signal measured on the boundary of the tissue



- Photoacoustic tomography combines benefits of optical and acoustic methods
- Contrast through optical absorption
  - Tissue chromophores: oxygenated and deoxygenated haemoglobin, water, lipids, melanin
  - Contrast agents
- Resolution by ultrasound
  - Low scattering in soft biological tissue
- Applications in imaging of tissue vasculature, tumours, small animal imaging, etc.



J. Tick et al, Three dimensional photoacoustic tomography in Bayesian framework, *J Acoust Soc Am* 144:2061-2071, 2018

Quantitative photoacoustic tomography (QPAT)

- Aim is to estimate the concentrations of light absorbing molecules
- Two inverse problems:
  - Acoustic inverse problem: estimation of the initial pressure from photoacoustic measurements
  - Optical inverse problem: estimation of the optical parameters from the initial pressure
- Modelling of light propagation, photoacoustic efficiency and ultrasound propagation are needed

$\Downarrow$ **FORWARD PROBLEM**

Illumination, absorption	1. The optical forward problem: Solve the absorbed optical energy density $H(r)$ when the optical properties of the medium $(\mu_a(r), \mu_s(r))$ and the input light sources are given $$-\nabla \cdot \kappa(r)\nabla\Phi(r) + \mu_a(r)\Phi(r) = 0$$ $$H(r) = \mu_a(r)\Phi(r)$$	
Pressure rise	2. Solve the initial acoustic pressure $p_0(r)$ from the absorbed optical energy density $$p_0(r) = G(r)H(r)$$	
Pressure propagation	3. The acoustic forward problem: Solve the time-varying pressure $p_S(t)$ at the sensors (measurable data) when the initial acoustic pressure distribution is given $$\left(\frac{\partial^2}{\partial t^2} - v^2(r)\nabla^2\right) p(r,t) = 0$$	

⇓ FORWARD PROBLEM			INVERSE PROBLEM ⇑
Illumination, absorption	1. The optical forward problem: Solve the absorbed optical energy density $H(r)$ when the optical properties of the medium $(\mu_a(r), \mu_s(r))$ and the input light sources are given $$-\nabla \cdot \kappa(r)\nabla\Phi(r) + \mu_a(r)\Phi(r) = 0$$ $$H(r) = \mu_a(r)\Phi(r)$$	3. The optical inverse problem: Estimate the optical parameters $(\mu_a(r), \mu_s(r))$ when the absorbed optical energy density $H(r)$ and the amount of input light are given $$\arg\min \left\{ \|L_n(H - A_h(x))\|^2 + \|L_x(x - x_*)\|^2 \right\}$$	
Pressure rise	2. Solve the initial acoustic pressure $p_0(r)$ from the absorbed optical energy density $$p_0(r) = G(r)H(r)$$	2. Calculate the absorbed optical energy density $H(r)$ from the estimated initial acoustic pressure $p_0(r)$	
Pressure propagation	3. The acoustic forward problem: Solve the time-varying pressure $p_S(t)$ at the sensors (measurable data) when the initial acoustic pressure distribution is given $$\left(\frac{\partial^2}{\partial t^2} - v^2(r)\nabla^2\right) p(r,t) = 0$$	1 . The acoustic inverse problem: Estimate the initial acoustic pressure distribution $p_0(r)$ when measured acoustic waves $p_S(t)$ on the sensors are given (inverse initial value problem of acoustics)	

Forward problem

Optical forward problem:

- Model light propagation in tissue using either the radiative transfer equation (RTE)

$$\begin{cases} \hat{s}\cdot\nabla\phi(r,\hat{s}) + (\mu_s+\mu_a)\phi(r,\hat{s}) = \mu_s\int_{S^{n-1}}\Theta(\hat{s}\cdot\hat{s}')\phi(r,\hat{s}')\mathrm{d}\hat{s}', & r\in\Omega \\ \phi(r,\hat{s}) = \begin{cases} \phi_0(r,\hat{s}), & r\in\epsilon_j, \quad \hat{s}\cdot\hat{n}<0 \\ 0, & r\in\partial\Omega\backslash\epsilon_j, \quad \hat{s}\cdot\hat{n}<0 \end{cases} \end{cases} \tag{1}$$

or the diffusion approximation (DA)

$$\begin{cases} -\nabla\cdot\kappa(r)\nabla\Phi(r) + \mu_a(r)\Phi(r) = 0 & r\in\Omega \\ \Phi(r) + \frac{1}{2\gamma_n}\kappa(r)A\frac{\partial\Phi(r)}{\partial\hat{n}} = \begin{cases} \frac{I_s}{\gamma_n}, & r\in\epsilon_i \\ 0, & r\in\partial\Omega\setminus\epsilon_i \end{cases} \end{cases} \tag{2}$$

where fluence $\Phi(r) = \int_{S^{n-1}}\phi(r,\hat{s})\mathrm{d}\hat{s}$

- Compute absorbed optical energy density

$$H(r) = \mu_a(r)\Phi(r) \tag{3}$$

- In this work, the solutions of the RTE and DA are numerically approximated using the finite element method and/or Monte Carlo method[1]

Photoacoustic efficiency:

- Calculate the initial acoustic pressure $p_0(r)$ from the absorbed optical energy density

$$p_0(r) = p(r, t = 0) = G(r)H(r) \tag{4}$$

where $G(r)$ is the Grüneisen parameter describing photoacoustic efficiency

[1] Leino AA, Pulkkinen A, Tarvainen T, ValoMC: A Monte Carlo software and MATLAB toolbox for simulating light transport in biological tissue, *OSA Continuum*, 2(3):957-972, 2019

Acoustic forward problem:

- Let us assume a non-attenuating medium with a speed of sound $v(r)$
- Propagation of photoacoustic waves generated by an initial pressure p_0 can be described by the initial value problem

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - v^2(r)\nabla^2\right) p(r,t) = 0 \\ p(r, t=0) = p_0(r) \\ \frac{\partial}{\partial t} p(r, t=0) = 0 \end{cases} \tag{5}$$

  where p is the time-varying pressure
- In this work, the wave equation is solved using a k-space time-domain method implemented with the k-Wave MATLAB toolbox[2] (http://www.k-wave.org/)

[2] Treeby BE and Cox BT, k-Wave: MATLAB toolbox for the simulation and reconstruction of photoacoustic wave fields, *Journal of Biomedical Optics* 15:021314, 2010

Bayesian approach to the inverse problem of PAT (a linear problem)

- Let us consider all parameters as random variables
- Assume measurements (ultrasound pressure wave) $p_t \in \mathbb{R}^m$, unknown quantity of interest (initial pressure) $p_0 \in \mathbb{R}^n$ and measurement noise $e \in \mathbb{R}^m$
- The solution of the inverse problem (posterior probability distribution) given by the Bayes' formula

$$\begin{aligned} \pi(p_0|p_t) &= \frac{\pi(p_t|p_0)\pi(p_0)}{\pi(p_t)} \\ &\propto \pi(p_t|p_0)\pi(p_0) \end{aligned} \tag{6}$$

where $\pi(p_t|p_0)$ is the likelihood and $\pi(p_0)$ is the prior

- A discrete observation model for PAT in the presence of additive noise is a linear model

$$p_t = Kp_0 + e \tag{7}$$

  where $p_t \in \mathbb{R}^m$ is a vector of measured acoustic pressure waves, $p_0 \in \mathbb{R}^n$ is the initial pressure distribution, $K \in \mathbb{R}^{m \times n}$ is the discretised forward model that is assumed to be exact with measurement accuracy, $e \in \mathbb{R}^m$ denotes the noise
- In this work, the matrix K is assembled using the k-Wave MATLAB toolbox in 2D and a matrix free approach is utilised in 3D

- Let us consider the joint probability distribution of all random variables

$$
\begin{aligned}
\pi(p_t, p_0, e) &= \pi(p_t|p_0, e)\pi(p_0, e) \\
&= \pi(p_t|p_0, e)\pi(e|p_0)\pi(p_0) \\
&= \pi(p_t, e|p_0)\pi(p_0)
\end{aligned} \tag{8}
$$

- This gives us

$$
\pi(p_t, e|p_0) = \pi(p_t|p_0, e)\pi(e|p_0) \tag{9}
$$

which in the case of the additive noise model (7) is

$$
\pi(p_t, e|p_0) = \delta(p_t - Kp_0 - e)\pi(e|p_0) \tag{10}
$$

- Now, the likelihood can be written as

$$
\begin{aligned}
\pi(p_t|p_0) &= \int \pi(p_t, e|p_0)\mathrm{d}e \\
&= \int \delta(p_t - Kp_0 - e)\pi(e|p_0)\mathrm{d}e \\
&= \pi_{e|p_0}(p_t - Kp_0|p_0)
\end{aligned} \tag{11}
$$

- This leads to posterior probability distribution

$$
\begin{aligned}
\pi(p_0|p_t) &\propto \pi(p_t|p_0)\pi(p_0) \\
&\propto \pi_{e|p_0}(y - Kp_0|p_0)\pi(p_0)
\end{aligned} \tag{12}
$$

- In case p_0 and e are mutually independent, we get

$$
\pi(p_0|p_t) \propto \pi_e(y - Kp_0)\pi(p_0) \tag{13}
$$

- The unknown p_0 and noise e are modelled as Gaussian distributed

$$p_0 \sim \mathcal{N}(\eta_{p_0}, \Gamma_{p_0}), \quad e \sim \mathcal{N}(\eta_e, \Gamma_e)$$

  where η_{p_0} and η_e are the means and Γ_{p_0} and Γ_e are covariance matrices of the prior and noise
- In this case, the posterior distribution becomes

$$\pi(p_0|p_t) \propto \exp\left\{-\frac{1}{2}\left\|L_e(p_t - Kp_0 - \eta_e)\right\|^2 - \frac{1}{2}\left\|L_{p_0}(p_0 - \eta_{p_0})\right\|^2\right\} \tag{14}$$

  where L_e and L_{p_0} are the square roots of the inverse covariance matrices such as the Cholesky decompositions of the noise $L_e^{\mathrm{T}} L_e = \Gamma_e^{-1}$ and prior $L_{p_0}^{\mathrm{T}} L_{p_0} = \Gamma_{p_0}^{-1}$

Posterior mean and covariance

- Now, in the case of a linear observation model and Gaussian distributed noise and prior, the posterior distribution is also Gaussian

$$p_0|p_t \sim \mathcal{N}(\eta_{p_0|p_t}, \Gamma_{p_0|p_t})$$

where

$$\eta_{p_0|p_t} = (K^{\mathrm{T}}\Gamma_e^{-1}K + \Gamma_{p_0}^{-1})^{-1}(K^{\mathrm{T}}\Gamma_e^{-1}(p_t - \eta_e) + \Gamma_{p_0}^{-1}\eta_{p_0})$$
$$\Gamma_{p_0|p_t} = (K^{\mathrm{T}}\Gamma_e^{-1}K + \Gamma_{p_0}^{-1})^{-1}$$

are the mean and covariance, respectively

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Prior distributions

Some possible choices for the Gaussian prior:

White noise covariance

$$\Gamma_{p_0} = \mathrm{diag}(\sigma^2)$$

Ornstein-Uhlenbeck covariance

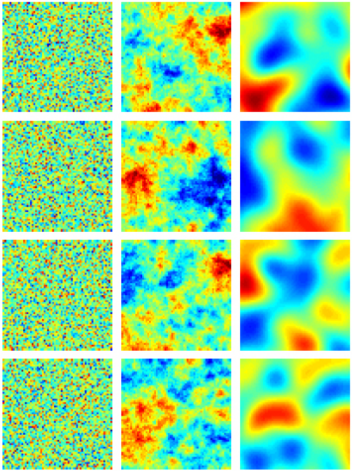
$$\Gamma_{p_0,ij} = \sigma^2 \exp\left\{-\frac{\|r_i - r_j\|}{\zeta}\right\}$$

Squared exponential covariance

$$\Gamma_{p_0,ij} = \sigma^2 \exp\left\{-\frac{\|r_i - r_j\|^2}{2\zeta^2}\right\}$$

where σ is the standard deviation and ζ is the characteristic length scale (controls spatial correlation)

Sample draws



2D simulations

- Two dimensional circular domain with a diameter of $4\,\mathrm{mm}$
- Non-attenuating medium with speed of sound $v = 1500\,\mathrm{m/s}$

Data simulation

- Data was simulated using the acoustic wave equation (5)
- The target domain was discretised using a pixel width $\Delta h = 14.29\,\mu\mathrm{m}$
- The pressure signals were simulated at sensor locations using 701 temporal samples for $7\,\mu\mathrm{s}$ (sampling frequency $100\,\mathrm{MHz}$)
- Uncorrelated Gaussian distributed noise with a zero mean $\eta_e = 0$ and a standard deviation σ_e proportional to $1\,\%$ of the peak amplitude of the simulated pressure signal was added to the signal

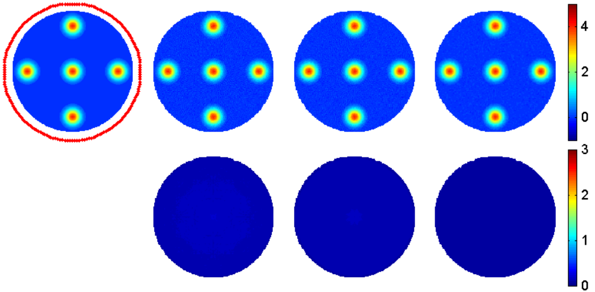
Inverse problem

- The target domain was discretised using a pixel width $\Delta h = 25\,\mu\mathrm{m}$
- We solved the mean and covariance of the posterior distribution $p_0|p_t \sim \mathcal{N}(\eta_{p_0|p_t}, \Gamma_{p_0|p_t})$

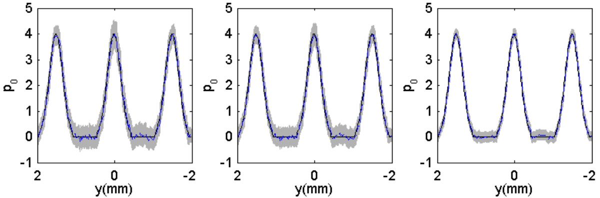
$$\eta_{p_0|p_t} = (K^{\mathrm{T}}\Gamma_e^{-1}K + \Gamma_{p_0}^{-1})^{-1}(K^{\mathrm{T}}\Gamma_e^{-1}(p_t - \eta_e) + \Gamma_{p_0}^{-1}\eta_{p_0})$$
$$\Gamma_{p_0|p_t} = (K^{\mathrm{T}}\Gamma_e^{-1}K + \Gamma_{p_0}^{-1})^{-1}$$

- Three priors were investigated: white noise, Ornstein-Uhlenbeck and squared exponential prior
- The measurement noise was considered to be uncorrelated Gaussian noise with zero mean and standard deviation set to 1 % of the peak amplitude of the noisy simulated data

- Posterior mean (top row) and standard deviation (bottom row). Images from left to right: true target, white noise, Ornstein-Uhlenbeck and squared exponential prior



- Posterior mean with 3 standard deviation credible intervals on a vertical cross-section



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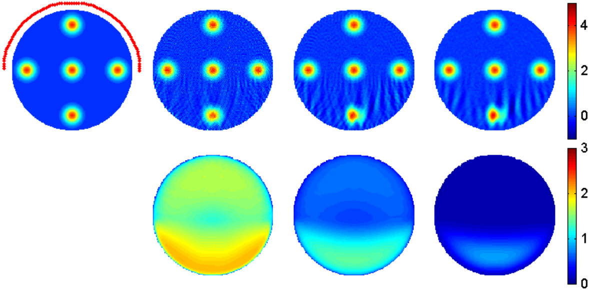

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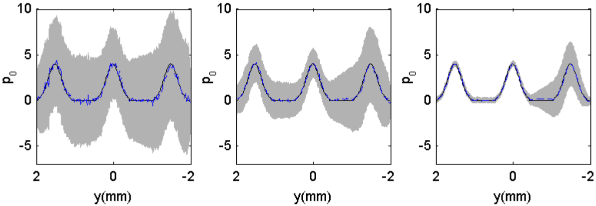

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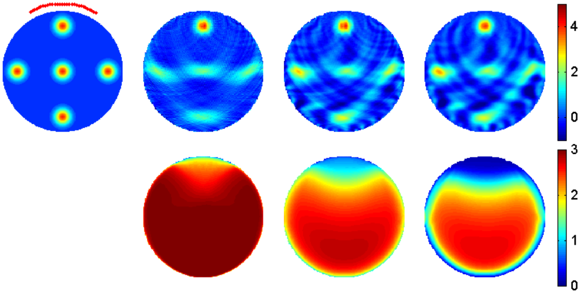

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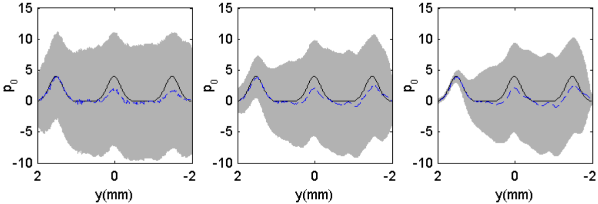
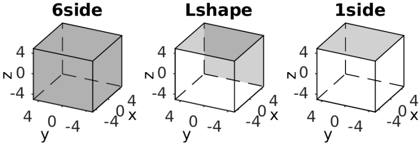


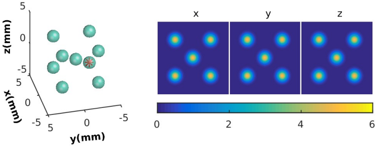
3D simulations

- Three dimensional cube with a side length 10 mm
- Non-attenuating medium with speed of sound $v = 1500\,\mathrm{m/s}$
- Three types of detector geometries were considered: full view (6-side), L-shaped, one-side



Data simulation

- Data was simulated using the acoustic wave equation (5)
- The target domain was discretised into 306 $\times$ 306 $\times$ 306 voxels (voxel side length $\Delta h = 32.7\,\mu\mathrm{m}$)
- The pressure signals were simulated at sensor locations using 849 temporal samples from 0 μs to 14.1 μs (sampling frequency 60 MHz)
- Uncorrelated Gaussian distributed noise with a zero mean $\eta_e = 0$ and a standard deviation σ_e proportional to 1 % of the peak amplitude of the simulated pressure signal was added to the signal



Inverse problem

- The target domain was discretised into $204 \times 204 \times 204$ pixels (pixel width $\Delta h = 49\,\mu\mathrm{m}$)
- The MAP estimates of the initial pressure distribution were computed iteratively using a biconjugate gradient stabilized method built-in Matlab with an adjoint operator[3]
- Ornstein-Uhlenbeck prior, 1% of noise

[3] Arridge SR, Betcke MM, Cox BT, Lucka F, Treeby BE, On the Adjoint Operator in Photoacoustic Tomography, *Inverse Problems* 32:115012, 2016

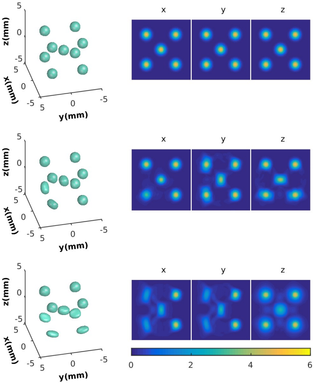
Bayesian approach

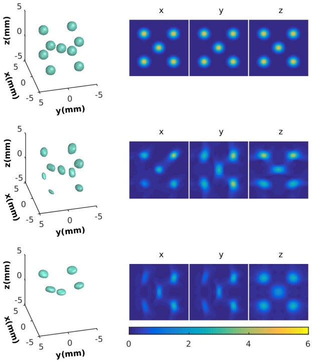
Time reversal

Full-view:

L-shape:

1-side:

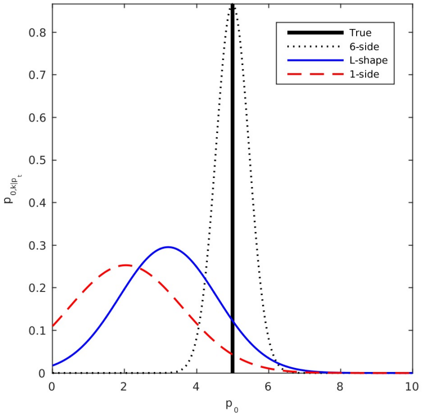




Relative errors (%)

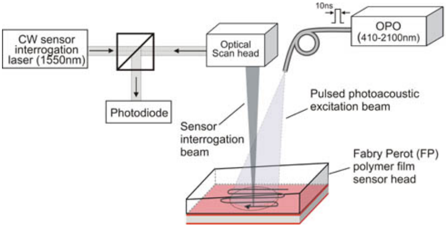
	full-view	L-shape	1-side
Bayesian approach	3.8	24.9	48.4
Time-reversal	4.0	61.5	79.9

Marginal density of the posterior distribution



Experiments

- PAT system of University College London[4]
- Measurements were done by Robert Ellwood[5]
- Planar Fabry-Pérot sensor

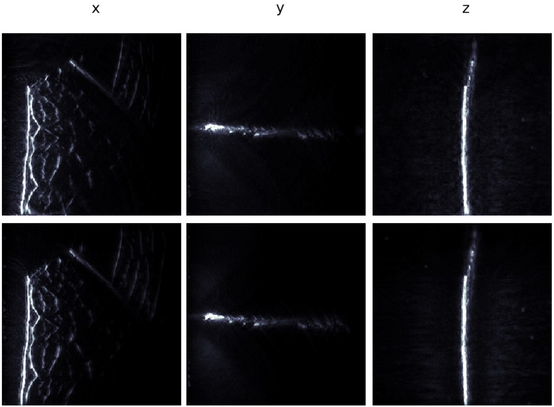


[4] Zhang E, Laufer J, Beard P, Backward-mode multiwavelength photoacoustic scanner using a planar Fabry-Perot polymer film ultrasound sensor for high-resolution three-dimensional imaging of biological tissues, *Applied Optics*, 47:561-577, 2008

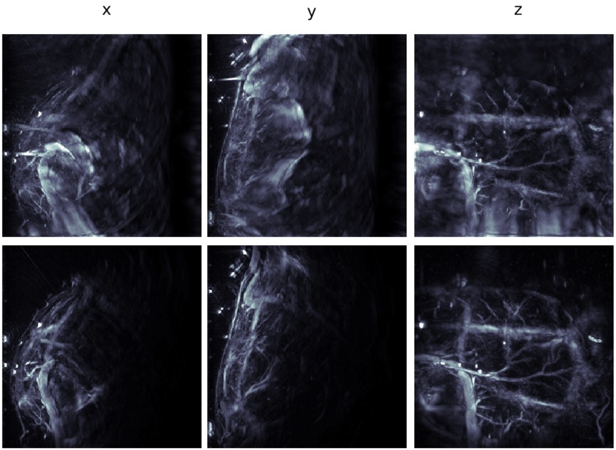
[5] Ellwood R, Ogunlade O, Zhang E, Beard P, Cox B, Photoacoustic tomography using orthogonal Fabry-Pérot sensors, *Journal of Biomedical Optics*, 22:041009, 2017

- Wave equation (5) as the model
- Non-attenuating medium with a speed of sound $v = 1488\,\mathrm{m/s}$
- Noise statistics (mean and standard deviation) were estimated from the data
- Ornstein-Uhlenbeck prior
- The MAP estimates of the initial pressure distribution were computed iteratively using a biconjugate gradient stabilized method built-in Matlab with an adjoint operator

Leaf phantom



Mouse head



Inverse problem in the presence of modelling errors

- Let us now consider a situation, where the forward operator K is parameterized by some parameter φ
- The observation model can be written in the form

$$p_t = K(\varphi)p_0 + e \tag{15}$$

- In practice, however, the random variable φ is often fixed to some constant $\varphi \to \varphi_0$ and an approximate forward operator $K(\varphi_0)$ is used instead

- The observation model (15) can be written utilising Bayesian approximation error modelling[6] as

$$
\begin{aligned}
p_t &= K(\varphi)p_0 + K(\varphi_0)p_0 - K(\varphi_0)p_0 + e \\
&= K(\varphi_0)p_0 + (K(\varphi)p_0 - K(\varphi_0)p_0) + e \\
&= K(\varphi_0)p_0 + \varepsilon + e \\
&= K(\varphi_0)p_0 + n
\end{aligned}
\tag{16}
$$

  where $\varepsilon = K(\varphi)p_0 - K(\varphi_0)p_0$ is the modelling error that describes the discrepancy between the exact and reduced models
- Thus, if $K(\varphi) = K(\varphi_0)$, forward model is exact (within measurement precision) and $\varepsilon = 0$

[6] Kaipio J and Somersalo E, *Statistical and Computational Inverse Problems*, Springer, 2005

- In this work, we study three situations:
  1. Inverse problem using an 'exact' forward model $K(\varphi) = K(\varphi_0)$ and $\varepsilon = 0$
  2. Inverse problem using a reduced forward model $K(\varphi_0)$ with a Gaussian approximation for the modelling errors $\varepsilon \sim \mathcal{N}(\eta_\varepsilon, \Gamma_\varepsilon)$
  3. Inverse problem using a reduced forward model $K(\varphi_0)$ and ignoring the modelling errors ($\varepsilon = 0$)

- Let us assume that p_0 and e are mutually independent and Gaussian distributed

$$p_0 \sim \mathcal{N}(\eta_{p_0}, \Gamma_{p_0}), \quad e \sim \mathcal{N}(\eta_e, \Gamma_e)$$

- Approximate the modelling error ε and the total error $n = \varepsilon + e$ as Gaussian

$$\varepsilon \sim \mathcal{N}(\eta_\varepsilon, \Gamma_\varepsilon), \quad n \sim \mathcal{N}(\eta_n, \Gamma_n)$$

- Let us ignore the mutual dependence of p_0 and modelling error ε (so called enhanced error model)

- Following a similar derivation as in the case of the exact forward model, the posterior distribution can be derived

$$\pi(p_0|p_t) \propto \exp\left\{-\frac{1}{2}\|L_n(p_t - K(\varphi_0)p_0 - \eta_n)\|^2 - \frac{1}{2}\|L_{p_0}(p_0 - \eta_{p_0})\|^2\right\} \tag{17}$$

where η_n and η_{p_0} are the means and L_n and L_{p_0} are the Cholesky decompositions of the inverse covariance matrices of the noise and prior $L_n^{\mathrm{T}}L_n = \Gamma_n^{-1}$ and $L_{p_0}^{\mathrm{T}}L_{p_0} = \Gamma_{p_0}^{-1}$, and $\eta_n = \eta_\varepsilon + \eta_e$ and $\Gamma_n = \Gamma_\varepsilon + \Gamma_e$

- Approximation error ε can be determined by, for example, using simulations and a sample based approximation as follows
- Let $S = \{s(1), s(2), \ldots, s(N)\}$ be a set of samples drawn from prior distribution of p_0 and φ^ℓ a sampled value of the uncertain parameter φ
- Mean and covariance of the approximation error ε can then be computed using the accurate and reduced forward models as

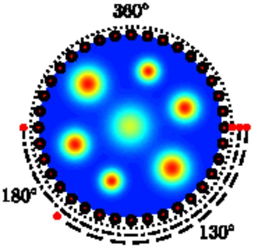
$$\varepsilon^\ell = K(\varphi^\ell)p_0^\ell - K(\varphi_0)p_0^\ell$$

$$\eta_\varepsilon = \frac{1}{L}\sum_{\ell=1}^{L}\varepsilon^\ell$$

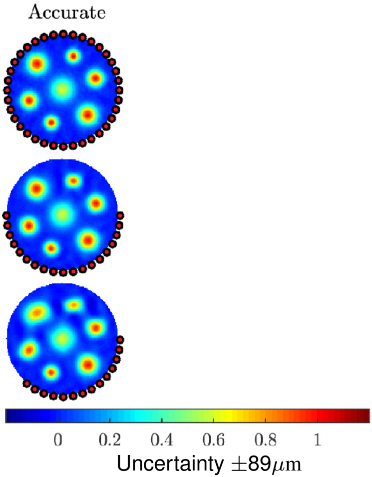
$$\Gamma_\varepsilon = \frac{1}{L-1}\sum_{\ell=1}^{L}(\varepsilon^\ell - \eta_\varepsilon)(\varepsilon^\ell - \eta_\varepsilon)^T$$

Simulations

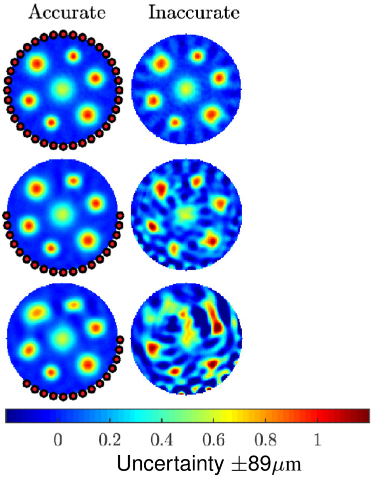
- We studied modelling of errors related to uncertainties in sensor positions in PAT
- The approach was evaluated using three measurement geometries
  - 130°, 180° and 360°
  - 10° increments
- Sensor locations were modelled to be on a circle of 5 mm radius
- Data was simulated using radially altered sensor locations
  - Uncertainties: $\pm 89 \mu\mathrm{m}$ and $\pm 177 \mu\mathrm{m}$



- Posterior mean solved using accurate sensor positions, inaccurate sensor positions without modelling of uncertainties, and inaccurate sensor positions with modelling of uncertainties (AE)



- Posterior mean solved using accurate sensor positions, inaccurate sensor positions without modelling of uncertainties, and inaccurate sensor positions with modelling of uncertainties (AE)



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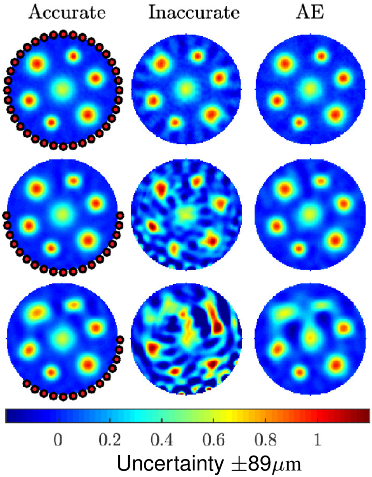
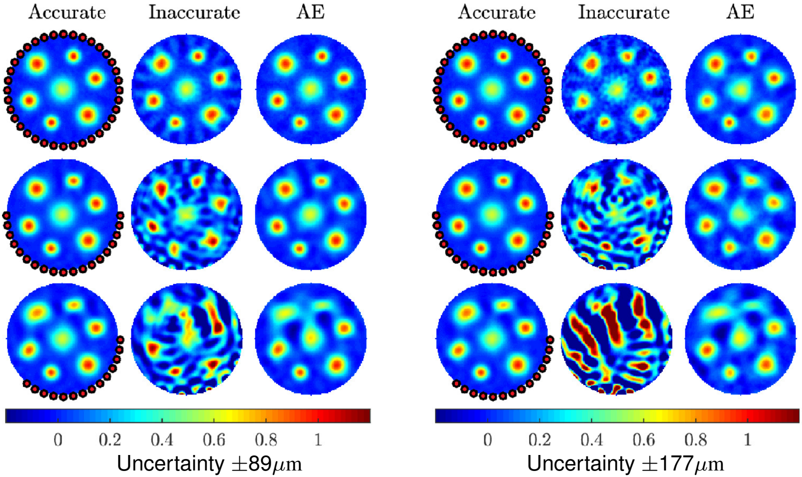
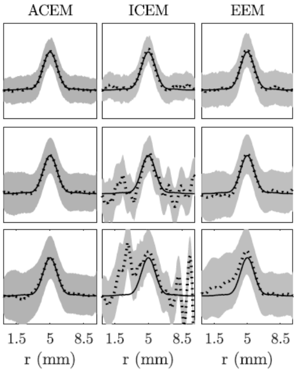


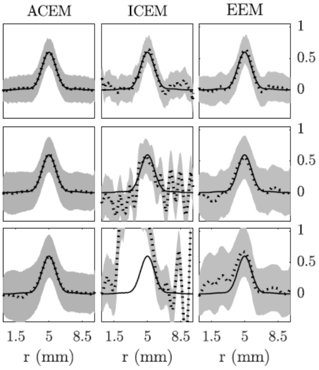
- Posterior mean solved using accurate sensor positions, inaccurate sensor positions without modelling of uncertainties, and inaccurate sensor positions with modelling of uncertainties (AE)



- Posterior mean with 3 standard deviations on a cross-section computed using accurate sensor positions (ACEM), inaccurate sensor positions without modelling of uncertainties (ICEM), and inaccurate sensor positions with modelling of uncertainties (EEM)



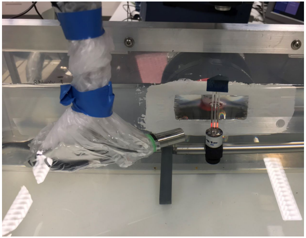
Uncertainty $\pm 89 \mu\mathrm{m}$



Uncertainty $\pm 177 \mu\mathrm{m}$

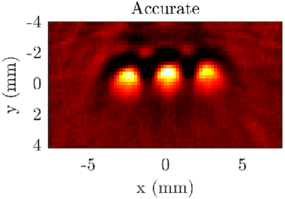
Experiments

- PAT system of UEF with a LED illumination[7]
- Measurement setup with a piezoelectric ultrasound sensor
- 180° rotation, 5° increments, 1024 illuminations per angle
- Target: 3 plastic microcapillary tubes with inner diameter 0.85 mm



[7] Leskinen J, Pulkkinen A, Tick J, Tarvainen T, Photoacoustic tomography setup using LED illumination, *In Proc. SPIE 11077*, 110770Q, 2019

- Uncertainty in modelling of ultrasound sensor position $\pm 188\mu$m in the radial direction
- Mean of the posterior distribution computed in three cases
  - Accurately modelled sensor locations
  - Inaccurately modelled sensor locations
  - Inaccurately modelled sensor locations with Bayesian approximation error modelling



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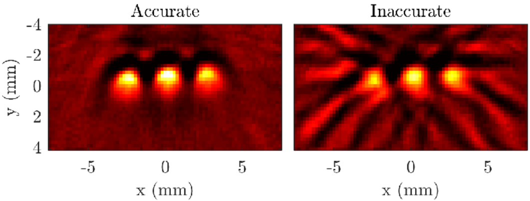

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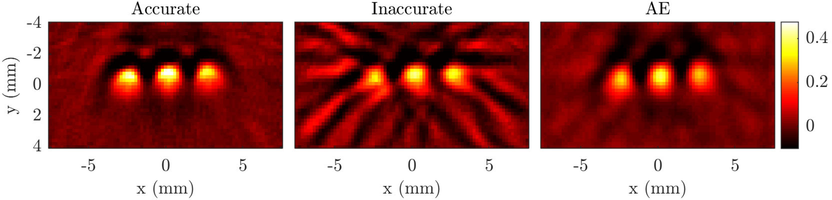


Non-linear case: the optical inverse problem of QPAT

- Let us now study the optical inverse problem of QPAT
- Assume data (absorbed optical energy density) $y \in \mathbb{R}^M$, unknown quantities of interests $x = [\mu_a, \mu_s] \in \mathbb{R}^{2N}$ and noise $e \in \mathbb{R}^M$
- Observation model

$$y = f(x) + e \tag{18}$$

- Following the Bayesian framework, similarly as in the linear case, we model all parameters as random variables

- Model unknown x and noise e as Gaussian distributed

$$x \sim \mathcal{N}(\eta_x, \Gamma_x), \quad e \sim \mathcal{N}(\eta_e, \Gamma_e)$$

  and ignore their mutual dependence
- The posterior distribution can be derived to have the form

$$\begin{aligned} \pi(x|y) &\propto \pi(y|x)\pi(x) \\ &\propto \exp\left\{-\frac{1}{2}\left\|L_e(y - f(x) - \eta_e)\right\|^2 - \frac{1}{2}\left\|L_x(x - \eta_x)\right\|^2\right\} \end{aligned} \tag{19}$$

- In large dimensional tomographic inverse problems, we compute point estimates to approximate the posterior distribution
- We calculate the *maximum a posteriori* (MAP) estimate

$$\hat{x} = \arg\min_{x} \left\{ \frac{1}{2} \left\| L_e(y - f(x) - \eta_e) \right\|^2 + \frac{1}{2} \left\| L_x(x - \eta_x) \right\|^2 \right\} \quad (20)$$

- The minimisation problem can be solved using methods of non-linear computational optimisation

- To evaluate the credibility of the estimates, we approximate the forward solution using the first order Taylor series

$$f(x) \approx f(\hat{x}) + J(\hat{x})(x - \hat{x}) \tag{21}$$

where $J(\hat{x})$ is the Jacobian matrix of $f(x)$ evaluated at $\hat{x}$

- By substituting the Taylor approximation into the observation model, a Gaussian approximation for the posterior distribution can be achieved

$$\pi(x|y) \sim \mathcal{N}(\hat{\eta}, \hat{\Gamma})$$

where $\hat{\eta} = \hat{x}$ is the MAP estimate and

$$\hat{\Gamma} = (J(\hat{x})^{\mathrm{T}} \Gamma_e^{-1} J(\hat{x}) + \Gamma_x^{-1})^{-1} \tag{22}$$

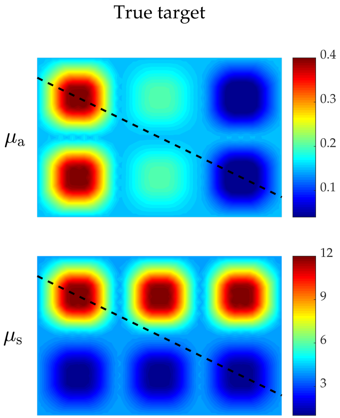
is the covariance matrix

- Credible intervals $[\hat{x} - p\sigma_{\hat{x}}\, ,\ \hat{x} + p\sigma_{\hat{x}}]$ where $\sigma_{\hat{x}_j} = \sqrt{\hat{\Gamma}(j,j)}$

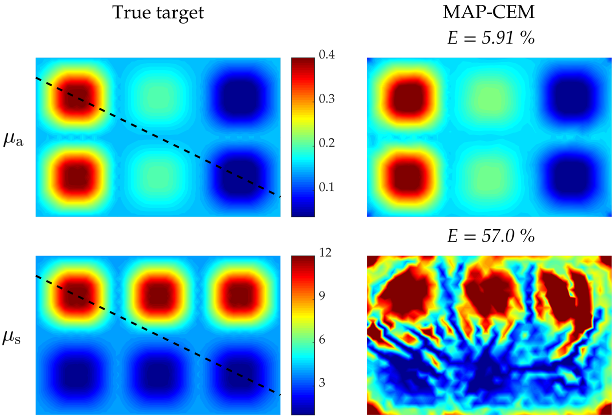
Simulations

- We studied the optical inverse problem of QPAT
- The data was simulated using Monte Carlo method for light transport
- In the inverse problem, the diffusion approximation (DA) was used
- Modelling of errors due to using the DA as the forward model was studied using Bayesian approximation error modelling

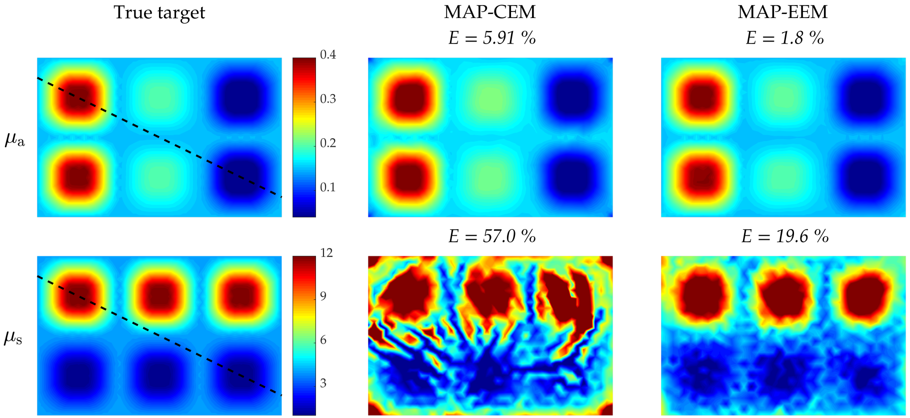
MAP-estimates



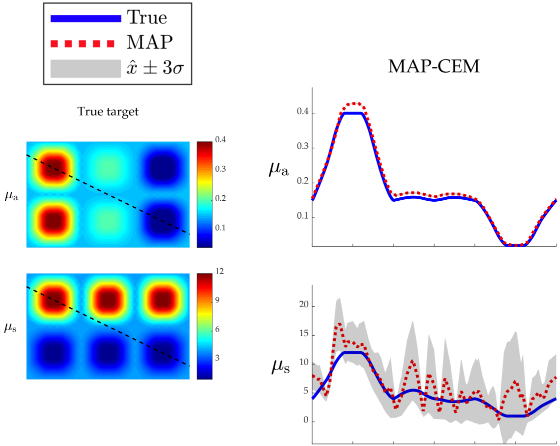
MAP-estimates



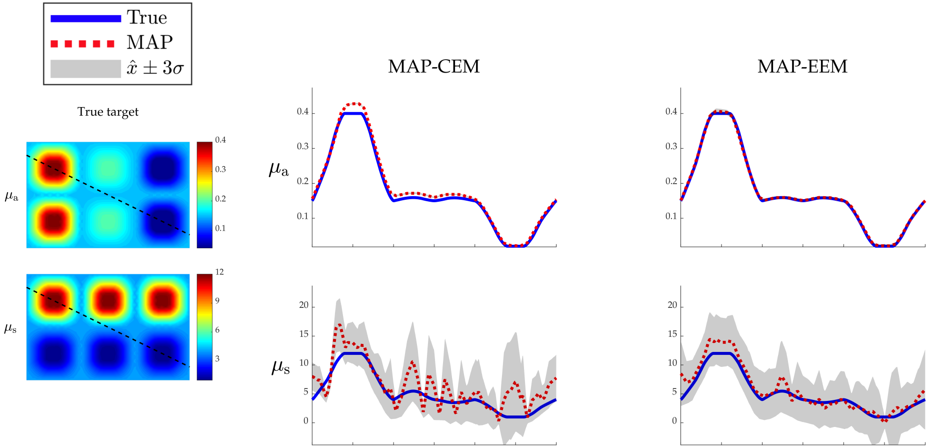
MAP-estimates



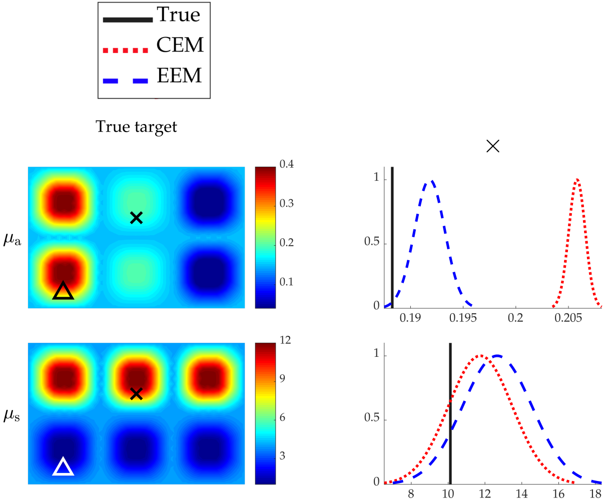
Cross-section



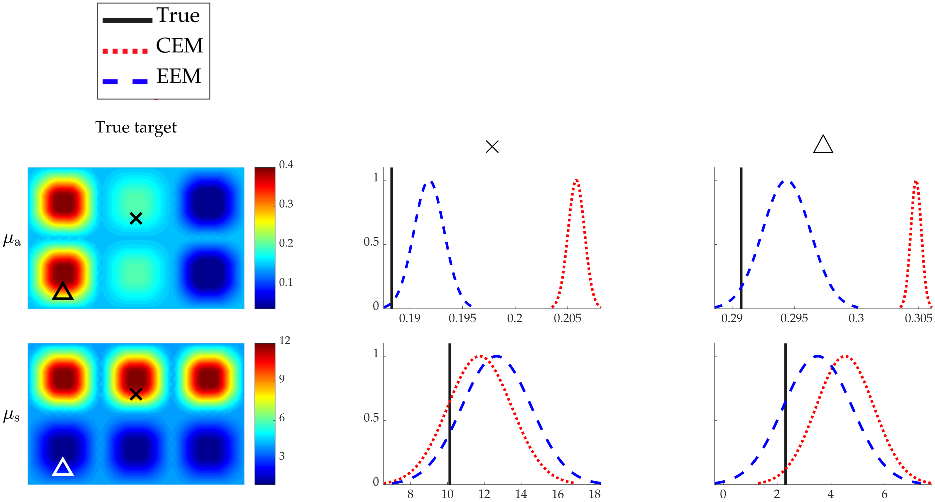
Cross-section



Marginal densities



Marginal densities



Summary

- Inverse problem of PAT was approached in the Bayesian framework
- The posterior distribution as well as point estimates and their reliability were investigated
- Prior information and modelling of errors are important in quantitative tomography: if inaccurately modelled, the photoacoustic image may look qualitatively good but they may not be quantitatively reliable

Summary

- Inverse problem of PAT was approached in the Bayesian framework
- The posterior distribution as well as point estimates and their reliability were investigated
- Prior information and modelling of errors are important in quantitative tomography: if inaccurately modelled, the photoacoustic image may look qualitatively good but they may not be quantitatively reliable
- Future work and challenges:
  - Computationally efficient methods for 3D
  - Examining the safety of the uncertainty estimates
  - Realistic imaging situations
  - More accurate forward models

Thank you for your attention!

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