The horseshoe prior for edge-preserving Bayesian inversion

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Linear inverse problems

Consider a linear inverse problem in the form:

$$\mathbf{y} = A\bar{\mathbf{x}} + \mathbf{e}.$$

- Noisy observational data: $\mathbf{y} \in \mathbb{R}^m$.
- Forward operator: $A \in \mathbb{R}^{m \times n}$.
- Unknown variables: $\mathbf{x} \in \mathbb{R}^n$ with the ground-truth $\bar{\mathbf{x}}$.
- Measurement noise: $\mathbf{e} \in \mathbb{R}^m$ follows $\mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_m)$.

Goal: Find a good approximation to $\bar{\mathbf{x}}$, which is robust with respect to the measurement noise.

Bayesian inverse problems

- The objective in a Bayesian inverse problem is to find or characterize the *posterior* probability density, defined through Bayes' Theorem as

$$\pi_{\text{pos}}(\mathbf{x}|\mathbf{y}) = \frac{1}{Z}\,\pi_{\text{lk}}(\mathbf{y}|\mathbf{x})\,\pi_{\text{pr}}(\mathbf{x}).$$

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- The *likelihood* density follows from the noise assumption:

$$\pi_{\mathrm{lk}}(\mathbf{y}|\mathbf{x}) = \frac{1}{(2\pi)^{m/2}\sigma^m}\exp\left(-\frac{1}{2\sigma^2}\,\|\mathbf{y} - A\mathbf{x}\|_2^2\right).$$

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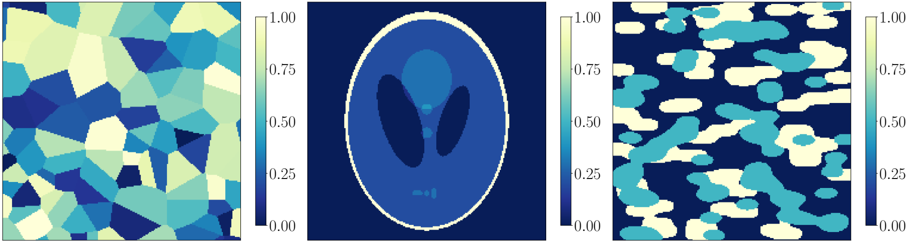
- The *prior* density can be hierarchical:

$$\pi_{\mathrm{pr}}(\mathbf{x},\tau,\boldsymbol{\theta}) = \pi_{\mathrm{pr}}(\mathbf{x}|\tau,\boldsymbol{\theta})\pi_{\mathrm{hpr}}(\tau)\pi_{\mathrm{hpr}}(\boldsymbol{\theta}),$$

Applications that require edge-preserving prior

A few examples (crystallography, medical imaging, geophysics):



Prior models for edge preservation

- Heavy-tailed Markov random fields: increase the probability of large jump events by imposing heavy-tailed distributions on the increments. Some examples: Laplace[1] and Cauchy Markov random fields[2].

[1] J. M. Bardsley. “Laplace-distributed increments, the Laplace prior, and edge-preserving regularization”. In: *Journal of Inverse and Ill-Posed Problems* 20.3 (2012), pp. 271–285.

[2] M. Markkanen et al. “Cauchy difference priors for edge-preserving Bayesian inversion”. In: *Journal of Inverse and Ill-posed Problems* 27.2 (2019), pp. 225–240.

Prior models for edge preservation

- Heavy-tailed Markov random fields: increase the probability of large jump events by imposing heavy-tailed distributions on the increments. Some examples: Laplace and Cauchy Markov random fields .

- Random fields with jump discontinuities: include Besov space priors, Gaussian process, level-set priors[1], etc.

[1] M. M. Dunlop et al. “Hierarchical Bayesian level set inversion”. In: *Statistics and Computing* 27 (2017), pp. 1555–1584.

Prior models for edge preservation

- Heavy-tailed Markov random fields: increase the probability of large jump events by imposing heavy-tailed distributions on the increments. Some examples: Laplace and Cauchy Markov random fields .
- Random fields with jump discontinuities: include Besov space priors, Gaussian process, level-set priors , etc.
- Machine learning-based models: plug-and-play priors and Bayesian neural nets with heavy-tailed weights which promotes edge-preservation[1].

[1] C. Li et al. “Bayesian neural network priors for edge-preserving inversion”. In: *Inverse Problems and Imaging* 0 (2022), pp. 1–26.

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- Shrinkage priors: are popular in sparse statistics. These models are hierarchical by nature and include for instance, elastic net, spike-slab, Horseshoe, discrete Gaussian mixtures, along with many others[1].

[1] N. G. Polson and V. Sokolov. “Bayesian regularization: from Tikhonov to horseshoe”. In: *WIREs Computational Statistics* 11.4 (2019), e1463.

Horseshoe prior

- The standard horseshoe model[2] imposes a conditionally Gaussian prior to $\mathbf{x}$:

$$\pi_{\mathrm{pr}}(\mathbf{x}|\tau,\boldsymbol{\theta}) = \frac{1}{(2\pi)^{\frac{n}{2}}\det(\Sigma_{\tau,\boldsymbol{\theta}})^{\frac{1}{2}}}\exp\left(-\frac{1}{2}\mathbf{x}^T\Sigma_{\tau,\boldsymbol{\theta}}^{-1}\mathbf{x}\right),$$

where $\Sigma_{\tau,\boldsymbol{\theta}} = \tau^2\mathrm{diag}(\theta_1^2,\ldots,\theta_n^2) \in \mathbb{R}^{n\times n}$ is a prior covariance matrix depending on hyperparameters $\tau \in \mathbb{R}_{>0}$ (global) and $\boldsymbol{\theta} = [\theta_1,\ldots,\theta_n]^T \in \mathbb{R}^n_{>0}$ (local).

[2] C. M. Carvalho et al. “The horseshoe estimator for sparse signals”. In: *Biometrika* 97.2 (2010), pp. 465–480.

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- This hierarchical prior imposes half-Cauchy hyperpriors on the hyperparameters:

$$\pi_{\text{hpr}}(\tau) \propto \frac{1}{\tau_0^2+\tau^2} \quad \text{and} \quad \pi_{\text{hpr}}(\boldsymbol{\theta}) \propto \prod_{i=1}^{n}\frac{1}{1+\theta_i^2} \qquad \text{with } \tau,\theta_i > 0,$$

where τ_0 is a scale parameter.

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Horseshoe prior

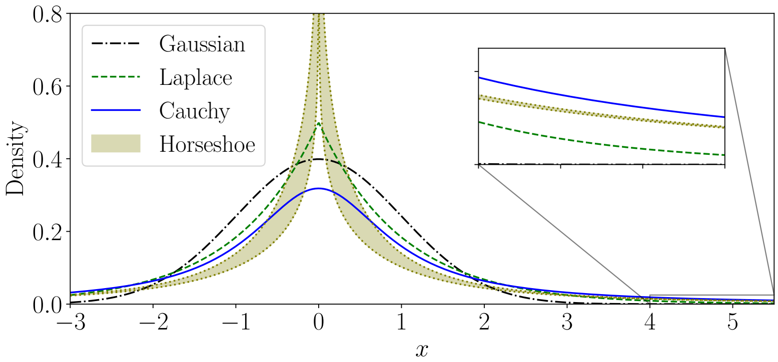

Figure: Comparison of the horseshoe prior bounds with other probability densities. The zoom-in highlights the distributions at the tails.

No closed form for the horseshoe prior, but upper and lower bounds:

$$\frac{1}{2\sqrt{2\pi^3}} \log\left(1 + \frac{4}{x^2}\right) \leq \pi_{\mathrm{pr}}(x) \leq \frac{1}{\sqrt{2\pi^3}} \log\left(1 + \frac{2}{x^2}\right).$$

Shrinkage coefficients

Set $\tau = 1$. The shrinkage coefficients are defined according to the covariance matrix:

$$\phi_i = \frac{1}{1+\theta_i^2} \in [0,1], \quad i = 1,\dots,n.$$

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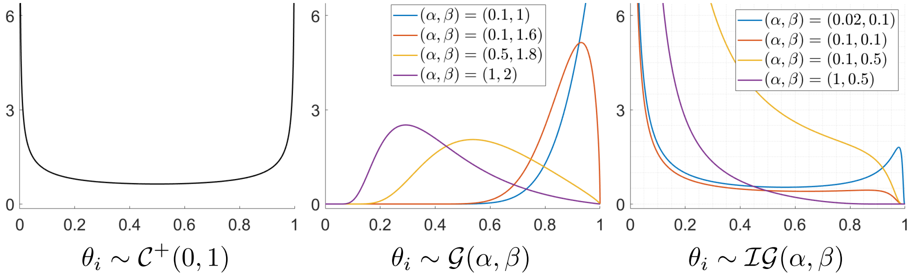
- As $\phi_i \to 1$, the shrinkage occurs.

Since θ_i follows a half Cauchy distribution with parameter $(0,1)$, we can derive the pdf of ϕ_i:

$$\pi(\phi_i) = \frac{1}{\pi} \frac{1}{\sqrt{\phi_i}} \frac{1}{\sqrt{1-\phi_i}}\,,$$

that is, ϕ_i follows a Beta distribution with a shape parameter equal to $1/2$.

Shrinkage coefficients



Hierarchical posterior density

$$\pi_{\text{pos}}\left(\mathbf{x},\tau,\boldsymbol{\theta}|\mathbf{y}\right)\propto\pi_{\text{lk}}(\mathbf{y}|\mathbf{x})\,\pi_{\text{pr}}(\mathbf{x},\tau,\boldsymbol{\theta}).$$

- The *likelihood* density follows from the noise assumption:

$$\pi_{\text{lk}}(\mathbf{y}|\mathbf{x})\propto\exp\left(-\frac{1}{2\sigma^2}\left\|\mathbf{y}-A\mathbf{x}\right\|_2^2\right).$$

- The *prior* density is hierarchical horseshoe prior on $D\mathbf{x}$:

$$\pi_{\text{pr}}(\mathbf{x},\tau,\boldsymbol{\theta})=\pi_{\text{pr}}(\mathbf{x}|\tau,\boldsymbol{\theta})\pi_{\text{hpr}}(\tau)\pi_{\text{hpr}}(\boldsymbol{\theta}),$$

where we have

- $\pi_{\text{pr}}(\mathbf{x}|\tau,\boldsymbol{\theta})\propto\frac{1}{\det(\Sigma_{\tau,\boldsymbol{\theta}})^{\frac{1}{2}}}\exp\left(-\frac{1}{2}(D\mathbf{x})^T\Sigma_{\tau,\boldsymbol{\theta}}^{-1}(D\mathbf{x})\right)$,
- $\pi_{\text{hpr}}(\tau)\propto\frac{1}{\tau_0^2+\tau^2}$ with $\tau>0$,
- $\pi_{\text{hpr}}(\boldsymbol{\theta})\propto\prod_{i=1}^{n}\frac{1}{1+\theta_i^2}$ with $\boldsymbol{\theta}>0$.

Hierarchical posterior density

$$\pi_{\text{pos}}\left(\mathbf{x},\tau,\boldsymbol{\theta}|\mathbf{y}\right) \propto \frac{1}{\det(\Sigma_{\tau,\boldsymbol{\theta}})^{\frac{1}{2}}} \exp\left(-\frac{1}{2\sigma^2}\left\|\mathbf{y}-A\mathbf{x}\right\|_2^2 - \frac{1}{2}(D\mathbf{x})^T\Sigma_{\tau,\boldsymbol{\theta}}^{-1}\left(D\mathbf{x}\right)\right)$$

$$\times \frac{1}{\tau_0^2+\tau^2}\prod_{i=1}^{n}\frac{1}{1+\theta_i^2} \qquad \text{with} \quad \tau,\boldsymbol{\theta}>0.$$

The main challenges to explore this posterior:

- the dimension of the parameter space is increased;
- the hyperparameters are endowed with heavy-tailed distributions.

MAP estimate

A commonly used point estimate for the posterior density is the maximum a posteriori (MAP) estimate, where one sets the mode of the posterior as the single point representative of the whole density function:

$$\{\mathbf{x}^*, \tau^*, \boldsymbol{\theta}^*\} \in \arg\max_{\mathbf{x},\tau,\boldsymbol{\theta}} \pi_{\text{pos}}(\mathbf{x}, \tau, \boldsymbol{\theta}|\mathbf{y}) = \arg\min_{\mathbf{x},\tau,\boldsymbol{\theta}} -\ln \pi_{\text{pos}}(\mathbf{x}, \tau, \boldsymbol{\theta}|\mathbf{y}) \, .$$

MAP estimate

A commonly used point estimate for the posterior density is the maximum a posteriori (MAP) estimate, where one sets the mode of the posterior as the single point representative of the whole density function:

$$\min_{\mathbf{x},\tau>0,\boldsymbol{\theta}>0} \quad \mathcal{J}(\mathbf{x},\tau,\boldsymbol{\theta}) := \frac{1}{2\sigma^2}\|\mathbf{y}-A\mathbf{x}\|_2^2 + \frac{1}{2}\|\Sigma_{\tau,\boldsymbol{\theta}}^{-\frac{1}{2}} D\mathbf{x}\|_2^2 + \sum_{i=1}^{n}\ln\theta_i + \sum_{i=1}^{n}\ln(1+\theta_i^2) + n\ln\tau + \ln(\tau_0^2+\tau^2).$$

- $\mathcal{J}$ is quadratic with respect to $\mathbf{x}$.
- $\mathcal{J}$ is non-convex with respect to τ and $\boldsymbol{\theta}$. But the global minimizers of the τ- and $\boldsymbol{\theta}$-subproblems have closed-form.
- $\mathcal{J}$ is non-convex with respect to $(\mathbf{x},\tau,\boldsymbol{\theta})$.
- We can prove that the alternating minimization algorithm converges to a stationary point of $\mathcal{J}$.

CT reconstruction

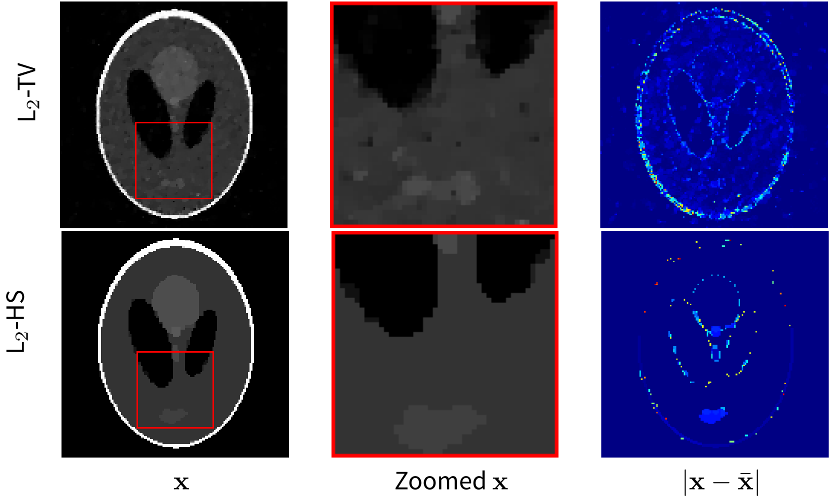


Figure: CT geometry: 45 equidistance angles, 170 detector pixels and 150-by-150 reconstruction resolutions. SNR and SSIM values are: (16.4663, 0.9067) for L_2-TV and (22.8216, 0.9797) for L_2-HS.

Explore the posterior by sampling

$$\pi_{\text{pos}}\left(\mathbf{x},\tau^2,\boldsymbol{\theta}^2|\mathbf{y}\right) \propto \pi_{\text{lk}}(\mathbf{y}|\mathbf{x})\,\pi_{\text{pr}}(\mathbf{x},\tau^2,\boldsymbol{\theta}^2)$$
$$\propto \pi_{\text{lk}}(\mathbf{y}|\mathbf{x})\,\pi_{\text{pr}}(\mathbf{x}|\tau^2,\boldsymbol{\theta}^2)\,\pi_{\text{hpr}}(\tau^2)\,\pi_{\text{hpr}}(\boldsymbol{\theta}^2)$$

We can use Gibbs sampling method to characterize the posterior.

1. Sample $\pi(\mathbf{x}|\mathbf{y},\tau,\boldsymbol{\theta}) \propto \pi_{\text{lk}}(\mathbf{y}|\mathbf{x})\,\pi_{\text{pr}}(\mathbf{x}|\tau,\boldsymbol{\theta})$;
2. Sample $\pi(\tau|\mathbf{x},\boldsymbol{\theta}) \propto \pi_{\text{pr}}(\mathbf{x}|\tau,\boldsymbol{\theta})\pi_{\text{hpr}}(\tau)$;
3. Sample $\pi(\boldsymbol{\theta}|\mathbf{x},\tau) \propto \pi_{\text{pr}}(\mathbf{x}|\tau,\boldsymbol{\theta})\pi_{\text{hpr}}(\boldsymbol{\theta})$.

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3. Sample $\pi(\boldsymbol{\theta}|\mathbf{x}, \tau) \propto \pi_{\text{pr}}(\mathbf{x}|\tau, \boldsymbol{\theta})\pi_{\text{hpr}}(\boldsymbol{\theta})$.

Main challenge is to sample $\pi(\tau|\mathbf{x}, \boldsymbol{\theta})$ and $\pi(\boldsymbol{\theta}|\mathbf{x}, \tau)$.

Extended horseshoe prior

Scale mixture representation of half-Student's t-distribution

If A and B are random variables such that

$$(A^2|B) \sim \mathrm{IG}\left(\frac{\nu}{2}, \frac{\nu}{B}\right) \text{ and } B \sim \mathrm{IG}\left(\frac{1}{2}, \frac{1}{c^2}\right),$$

then $A \sim \mathrm{t}^+(\nu, 0, c)$.

- $\mathrm{IG}(\cdot,\cdot)$ denotes the inverse Gamma distribution depending on shape and scale parameters;
- $\mathrm{t}^+(\cdot,\cdot,\cdot)$ is the half-Student's t-distribution depending on degrees of freedom, location and scale parameters;
- $\mathrm{t}^+(\nu, 0, c)$ with $\nu = 1$ is identical to a half-Cauchy distribution.

M. P. Wand et al. "Mean field variational Bayes for elaborate distributions". In: *Bayesian Analysis* 6.4 (2011), pp. 847–900.

Extended horseshoe prior

- Extended hierarchical horseshoe prior is

$$\pi_{\text{pr}}(\mathbf{x}|\tau^2, \boldsymbol{\theta}^2) = \mathcal{N}\left(\mathbf{0}, (D^T \boldsymbol{\Sigma}_{\tau,\boldsymbol{\theta}}^{-1} D)^{-1}\right),$$

$$\pi_{\text{hpr}}(\tau^2|\gamma) = \text{IG}\left(\frac{1}{2}, \frac{1}{\gamma}\right), \qquad \pi_{\text{hpr}}(\gamma) = \text{IG}\left(\frac{1}{2}, \frac{1}{\tau_0^2}\right),$$

$$\pi_{\text{hpr}}(\theta_i^2|\xi_i) = \text{IG}\left(\frac{1}{2}, \frac{1}{\xi_i}\right), \qquad \pi_{\text{hpr}}(\xi_i) = \text{IG}\left(\frac{1}{2}, 1\right).$$

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- The new posterior density is

$$\pi_{\text{pos}}(\mathbf{x}, \tau^2, \boldsymbol{\theta}^2, \gamma, \boldsymbol{\xi}|\mathbf{y}) \propto$$
$$\pi_{\text{lk}}(\mathbf{y}|\mathbf{x})\, \pi_{\text{pr}}(\mathbf{x}|\tau^2, \boldsymbol{\theta}^2)\, \pi_{\text{hpr}}(\tau^2|\gamma)\, \pi_{\text{hpr}}(\boldsymbol{\theta}^2|\boldsymbol{\xi})\, \pi_{\text{hpr}}(\gamma)\, \pi_{\text{hpr}}(\boldsymbol{\xi}).$$

Gibbs sampler

$$\begin{aligned}
\pi_1(\mathbf{x}|\mathbf{y},\tau^2,\boldsymbol{\theta}^2) &\propto \pi_{\mathrm{lk}}(\mathbf{y}|\mathbf{x})\,\pi_{\mathrm{pr}}(\mathbf{x}|\tau^2,\boldsymbol{\theta}^2)\\
\pi_2(\tau^2|\mathbf{x},\boldsymbol{\theta}^2,\gamma) &\propto \pi_{\mathrm{pr}}(\mathbf{x}|\tau^2,\boldsymbol{\theta}^2)\pi_{\mathrm{hpr}}(\tau^2|\gamma)\\
\pi_3(\boldsymbol{\theta}^2|\mathbf{x},\tau^2,\boldsymbol{\xi}) &\propto \pi_{\mathrm{pr}}(\mathbf{x}|\tau^2,\boldsymbol{\theta}^2)\pi_{\mathrm{hpr}}(\boldsymbol{\theta}^2|\boldsymbol{\xi})\\
\pi_4(\gamma|\tau^2) &\propto \pi_{\mathrm{hpr}}(\tau^2|\gamma)\pi_{\mathrm{hpr}}(\gamma),\\
\pi_5(\boldsymbol{\xi}|\boldsymbol{\theta}^2) &\propto \pi_{\mathrm{hpr}}(\boldsymbol{\theta}^2|\boldsymbol{\xi})\pi_{\mathrm{hpr}}(\boldsymbol{\xi}).
\end{aligned}$$

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\end{aligned}
$$

- $\pi_1(\mathbf{x}|\mathbf{y},\tau^2,\boldsymbol{\theta}^2)$ follows Gaussian distribution with the mean $\widetilde{\boldsymbol{\mu}}$ and the covariance $\widetilde{\Lambda}$:

$$
\widetilde{\Lambda}_{\tau,\boldsymbol{\theta}} = \frac{1}{\sigma^2}A^TA + D^T\Sigma_{\tau,\boldsymbol{\theta}}^{-1}D, \qquad \widetilde{\boldsymbol{\mu}}_{\tau,\boldsymbol{\theta}} = \widetilde{\Lambda}_{\tau,\boldsymbol{\theta}}^{-1}\left(\frac{1}{\sigma^2}A^T\mathbf{y}\right).
$$

- The conditional densities on all hyperparameters, i.e., π_2 to π_5, follow inverse Gamma distribution with closed forms.

1D deconvolution

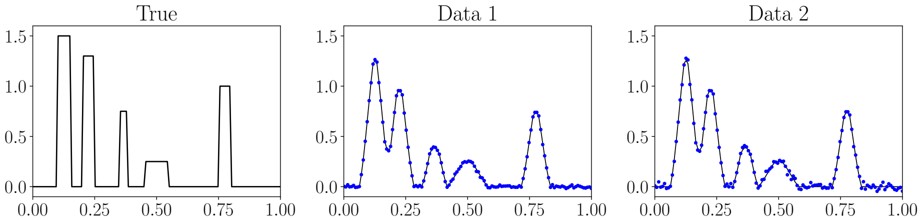


Figure: The original signal and noisy observed data with 2% and 5% noise, respectively.

- A is from Gaussian blurring.
- $\mathbf{x}$ is sparse under the first order derivative, so we use the first order derivative operator as D.
- Samples: $n_s = 2 \times 10^4$, $n_b = 2 \times 10^3$ and $n_t = 40$.

Hyperparameters τ and θ

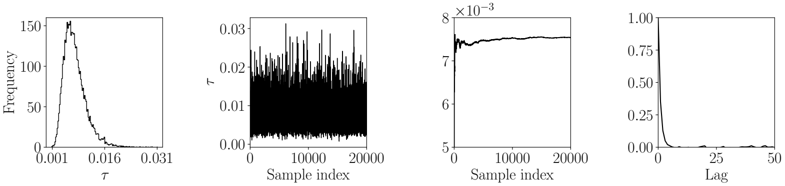


Figure: Posterior of τ with low noise level.

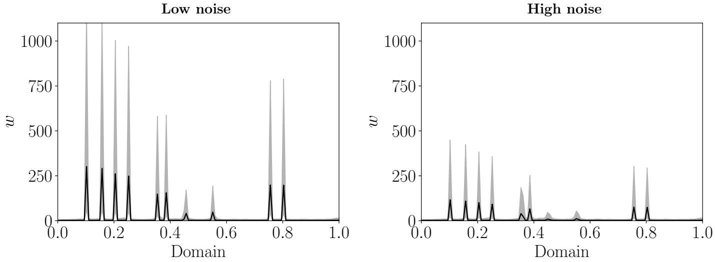


Figure: Posterior mean and 95% CI for $\boldsymbol{\theta}$.

Target parameter x

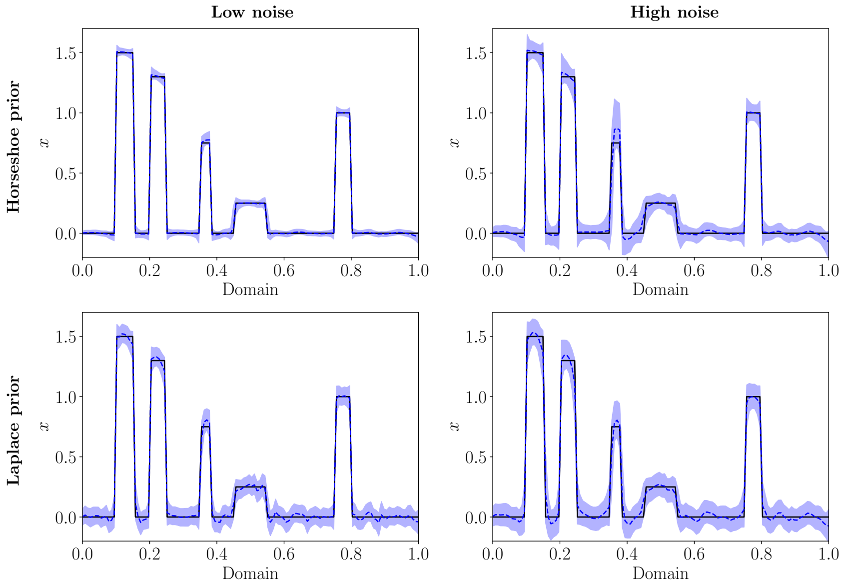


Figure: The relative errors are: (0.0154, 0.0663) for HS and (0.0536, 0.0927) for Laplace.

CT reconstruction

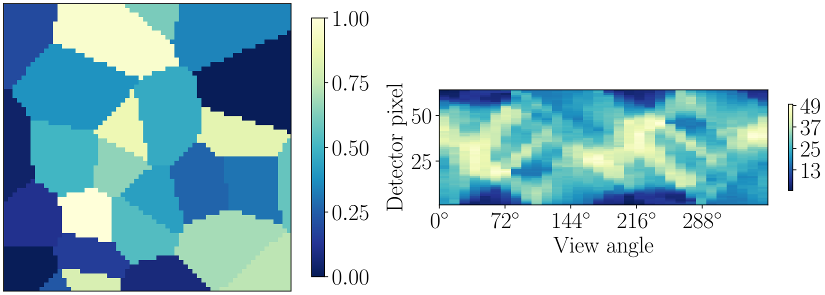
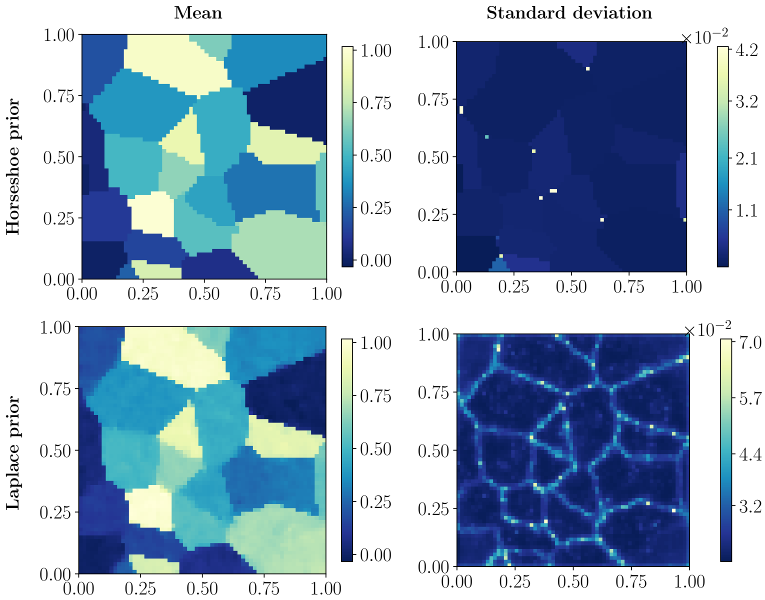


Figure: The ground truth and the sinogram.

- A is from Radon transform with 32 equidistance angles and 64 detector pixels.
- $\mathbf{x}$ is sparse under the first order derivatives, so we use the gradient operator as D.
- The resolution of the reconstruction is 64-by-64. The noise level is 1%.

CT reconstruction: Target parameter $\mathbf{x}$



Thank you!

For your interest:

- **Reference**:
  - Uribe, F., Dong, Y. and Hansen, P. C.: Horseshoe priors for edge-preserving linear Bayesian inversion, *arXiv:2207.09147*, (2022).
  - Dong, Y. and Pragliola, M.: Inducing sparsity via horseshoe prior in imaging problems, *submitted*, (2022).
- **CUQI project**: https://www.compute.dtu.dk/english/cuqi
- **Open postdoc positions**: TAPP-project, "Tomography of Alpha Particles in Fusion Plasma".