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Stability Estimates Of a Stochastic Inverse Source Problem For a Class of Elliptic Multipliers

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1 Introduction

The challenge of recovering statistical information about an unknown, uncertain source from indirect and noisy observations is addressed within a mathematical modeling framework. The study focuses on an inverse source problem for differential operators involving high powers of the Laplacian. The source is modeled as a stochastic process, and explicit formulas are derived to reconstruct its mean and variance. Stability to noise is established, and the method is validated numerically.

2 Problem Setup

Source Term
$\mu(x) + \sigma(x)W(x)$
$\mu \in C_c^\infty(\Omega)$: Mean, $\sigma^2 \in C_c^\infty(\Omega)$: Variance
$W(x)$: White Noise

Forward Operator
$(\lambda I + \sum c_k(-\Delta)^{s_k})\, u(x,\lambda)$
$s_k \in \mathbb{N}$, $c_k > 0$
$\lambda \in \mathbb{R} \backslash \{0\}$

Solution
$u(x,\lambda)$

Boundary Measurements
$U_\ell = (-\Delta)^\ell u|_{\partial\Omega}$, $\ell = 0, \ldots, \max_k s_k - 1$

Goal
Reconstruct $\mu(x)$ and $\sigma(x)$

3 Remark

Under the chosen boundary conditions, the solution exists, is unique, and admits the representation

$$u(x,\lambda) = \int_\Omega \Phi(x-y,\lambda)\,\mu(y)\,dy + \int_\Omega \Phi(x-y,\lambda)\,\sigma(y)\,dW_y,$$

where $\Phi(x-y,\lambda)$ is the Green's function (fundamental solution) of the operator. The first term is deterministic, while the second is a stochastic integral driven

4 Method and Reconstruction Formulas

For $x \in \Omega \subset \mathbb{R}^n$ and $j = 1 \ldots \infty$, $\tilde{s} = \max_k s_k$ consider

$$\Big(\lambda_j I + \sum_{k=1}^{N} c_k(-\Delta)^{s_k}\Big) u_j(x,\lambda_j) = \mu(x) + \sigma(x)W(x),$$

with our measurement when $x \in \partial\Omega$

$$(-\Delta)^\ell u_j(x,\lambda_j) = U_{j,\ell}(x,\lambda_j) \quad \text{for all } \ell = 0,\ldots,\tilde{s}-1$$

1. Multiply the main equation by ϕ_j and integrate over $\Omega \subset \mathbb{R}^n$ where $\{\phi_j\}_{j=1}^\infty$ are the eigenfunctions of $-\Delta$
2. Use

$$\langle(-\Delta)^{s_k}u,\phi_j\rangle_{L^2(\Omega)} = \sum_{d=1}^{s_k}\Big[\langle(-\Delta)^{s_k-d}u, \partial_\nu(-\Delta)^{d-1}\phi_j\rangle_{L^2(\partial\Omega)} - \langle(-\Delta)^{d-1}\phi_j, \partial_\nu(\,(-\Delta)^{s_k-d}u\rangle_{L^2(\partial\Omega)}\Big] + \langle u, (-\Delta)^{s_k}\phi_j\rangle_{L^2(\Omega)}$$

We get:

$$\langle f,\phi_j\rangle_{L^2(\Omega)} = \sum_{k=1}^{N} c_k \sum_{d=1}^{s_k} \kappa_j^{d-1}\langle U_{j,s_k-d}, \partial_\nu\phi_j\rangle_{L^2(\partial\Omega)}$$

Expression of $f(x)$

$$f(x) = \sum_{j=1}^{\infty}\Big[\sum_{k=1}^{N} c_k \sum_{d=1}^{s_k}\kappa_j^{d-1}\langle \underbrace{U_{j,s_k-d}}_{data}, \partial_\nu\phi_j\rangle_{L^2(\partial\Omega)}\Big]\phi_j(x)$$

Expression of $\mu(x)$

$$\mu(x) = \sum_{j=1}^{\infty}\Big[\sum_{k=1}^{N}\sum_{d=1}^{s_k} c_k\kappa_j^{d-1}\langle\mathbb{E}(U_{j,s_k-d}), \partial_\nu\phi_j\rangle_{L^2(\partial\Omega)}\Big]\phi_j(x)$$

Expression of $\sigma^2(x)$

$$\sigma^2(x) = \sum_{j=1}^{\infty}\Big[\sum_{k=1}^{N}\sum_{d=1}^{s_k} c_k^2(\kappa_j^{d-1})^2\,\mathrm{Var}\left(\left\langle U_{j,s_k-d},\partial_\nu\phi_j\right\rangle\right) + \sum_{(k,d)\neq(k',d')} c_k c_{k'}\kappa_j^{d-1}\kappa_j^{d'-1}\,\mathrm{Cov}\left(\left\langle U_{j,s_k-d},\partial_\nu\phi_j\right\rangle, \left\langle U_{j,s_{k'}-d'},\partial_\nu\phi_j\right\rangle\right)\Big]\phi_j^2(x)$$

5 Stability Estimates

Assume

$$\sum_{j=1}^{\infty}[\mathbb{E}(U^{(1)}_{j,s_k-d} - U^{(2)}_{j,s_k-d})]^2 < S < \infty$$

and $r = 2s_k + \frac{n}{2} + \epsilon - 1$

$$\|\mu_1-\mu_2\|^2_{L^2(\Omega)} \leqslant \sum_{j=1}^{\infty}\sum_{k=1}^{N}\sum_{d=1}^{s_k}\tilde{C}\kappa_j^{-\frac{n}{2}-\epsilon}\cdot\|\mathbb{E}(U^{(1)}_{j,s_k-d}-U^{(2)}_{j,s_k-d})\|^2_{H^r(\partial\Omega)}$$

and for $\tilde{s} < \frac{n}{4}+1$ where $\tilde{s} = \max_k s_k$

$$\|\sigma_1^2(x) - \sigma_2^2(x)\|^2_{L^2(\Omega)} \leqslant D^2 < \infty$$

where $D = \varrho_2\cdot\zeta(\frac{4-4\tilde{s}}{n})$ and ϱ_2 is a constant and $\zeta(\alpha)$ is the Riemann zeta function.

6 Numerical Example (1D Case)

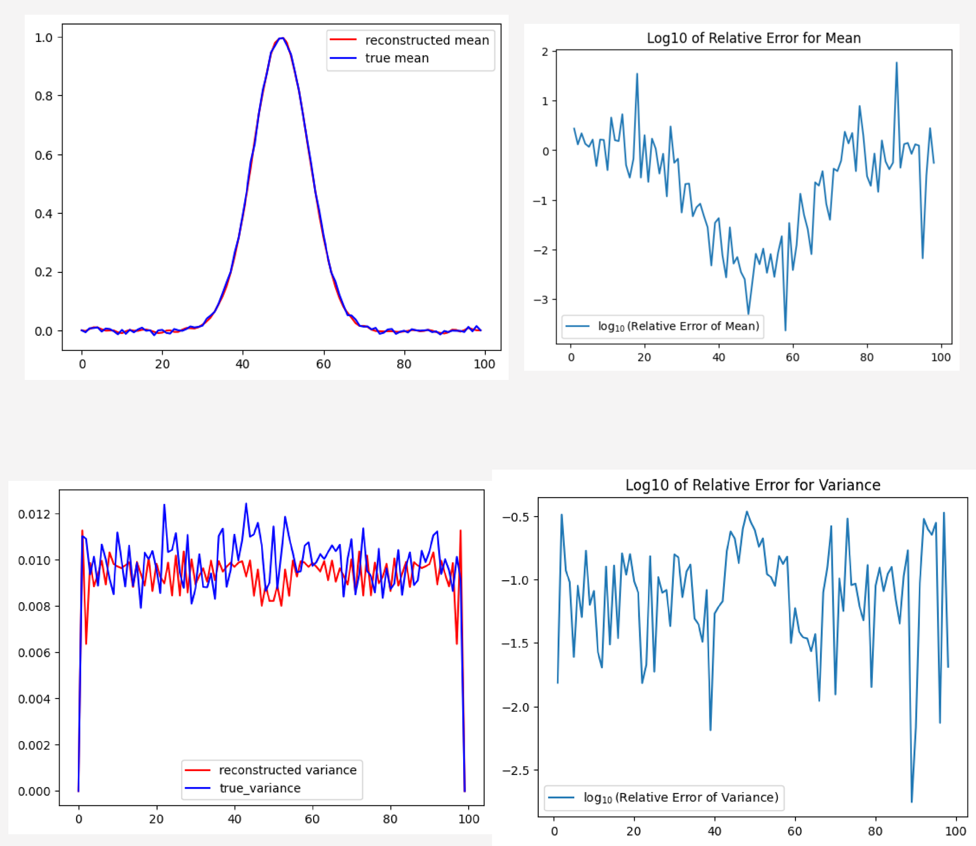
We solve a numerical example in one dimension where the equation takes the form:

$$(\lambda_j I - \Delta)u = f(x), \quad \text{in } (0,1) \subset \mathbb{R},$$

with eigenvalues $\lambda_j = -j^2\pi^2$ and the following boundary conditions:

$$\begin{cases} -u'(0) = i\sqrt{|\lambda|}\,u(0), \\ \;\;\,u'(1) = i\sqrt{|\lambda|}\,u(1). \end{cases}$$

We get



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