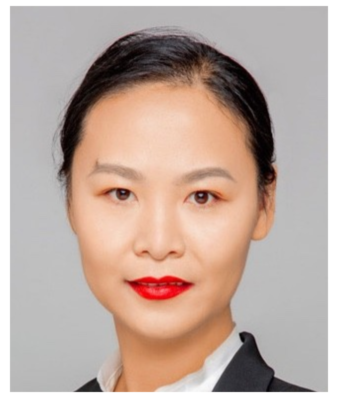
Inferring Functions with Uncertain Regularity

SIAM UQ23 - Trieste

B. M. Afkham, N. A. Riis, Y. Dong, P. C. Hansen (DTU), A. Carpio (UCM)

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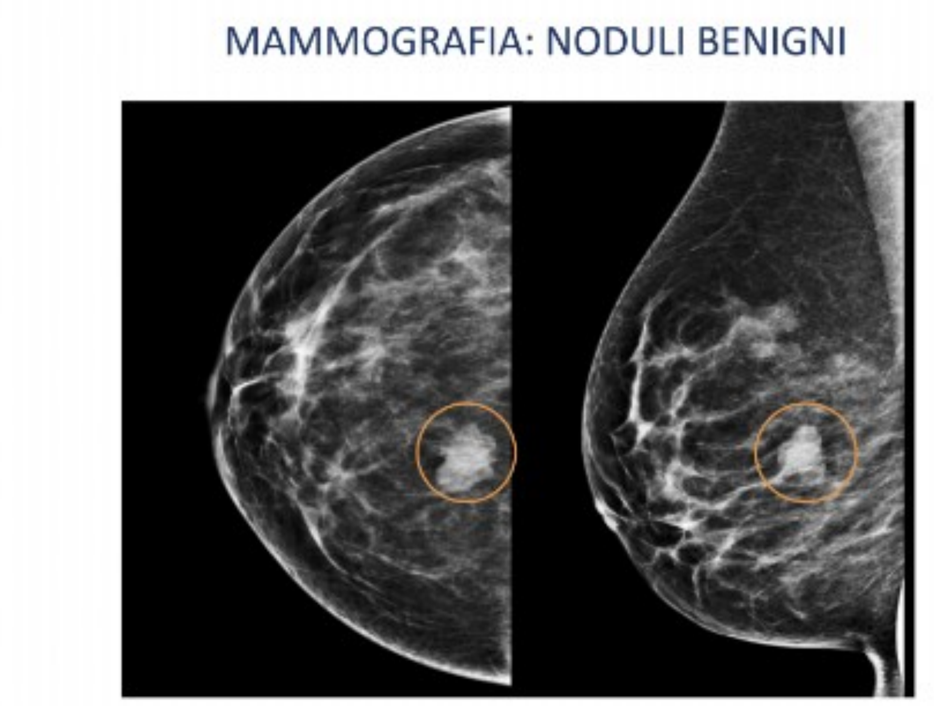
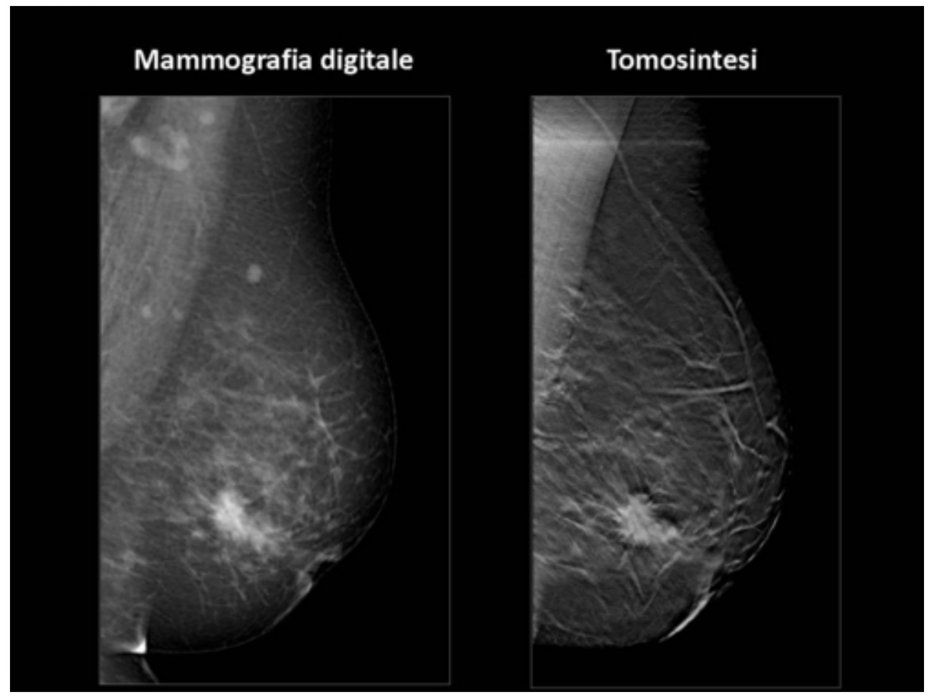
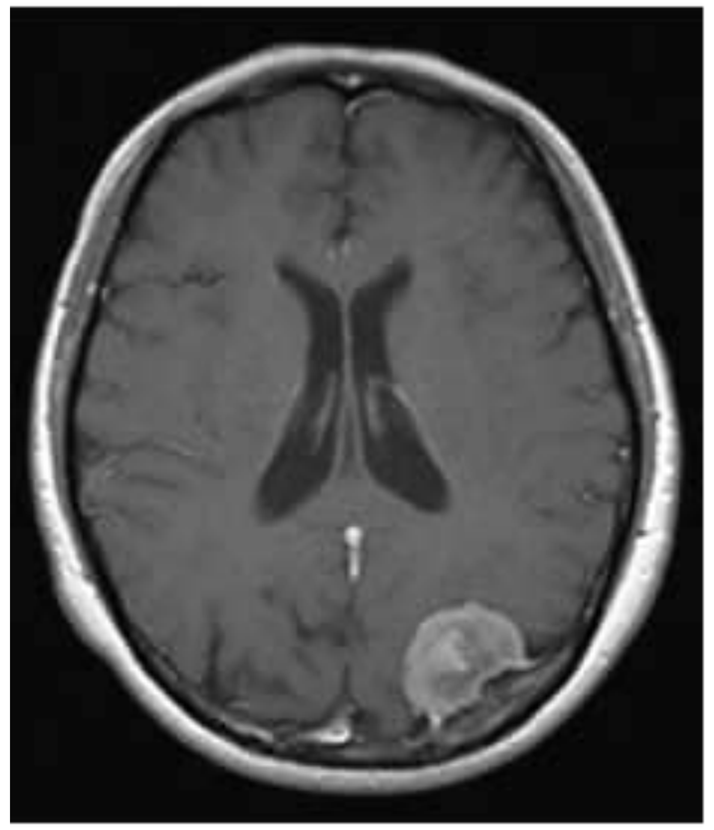
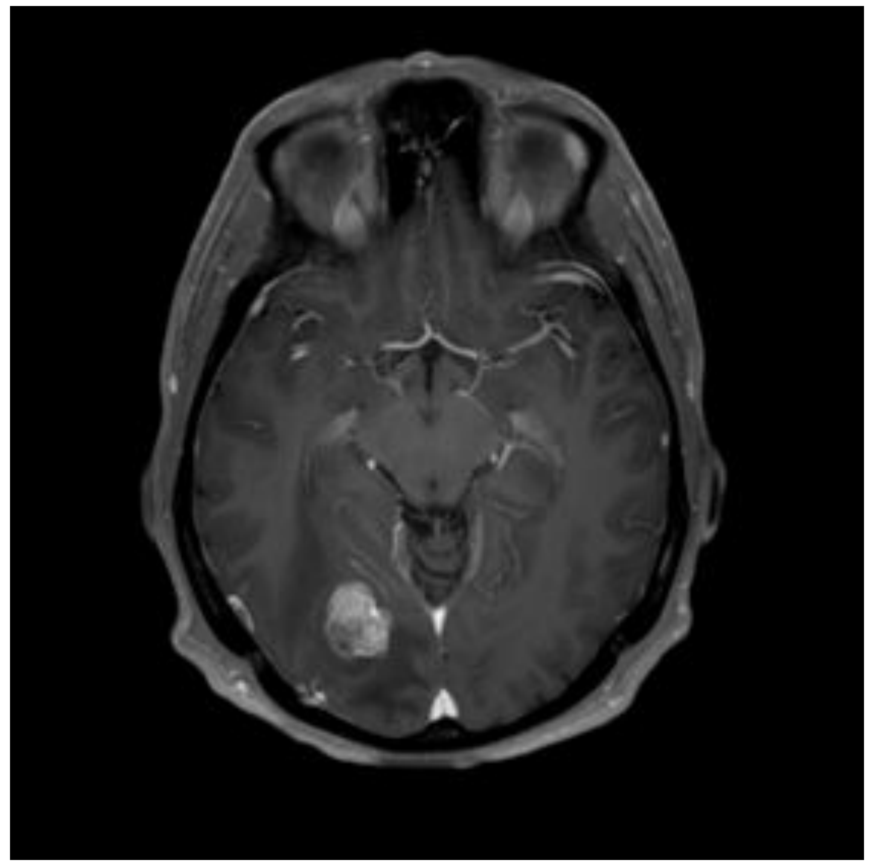
Importance of Regularity

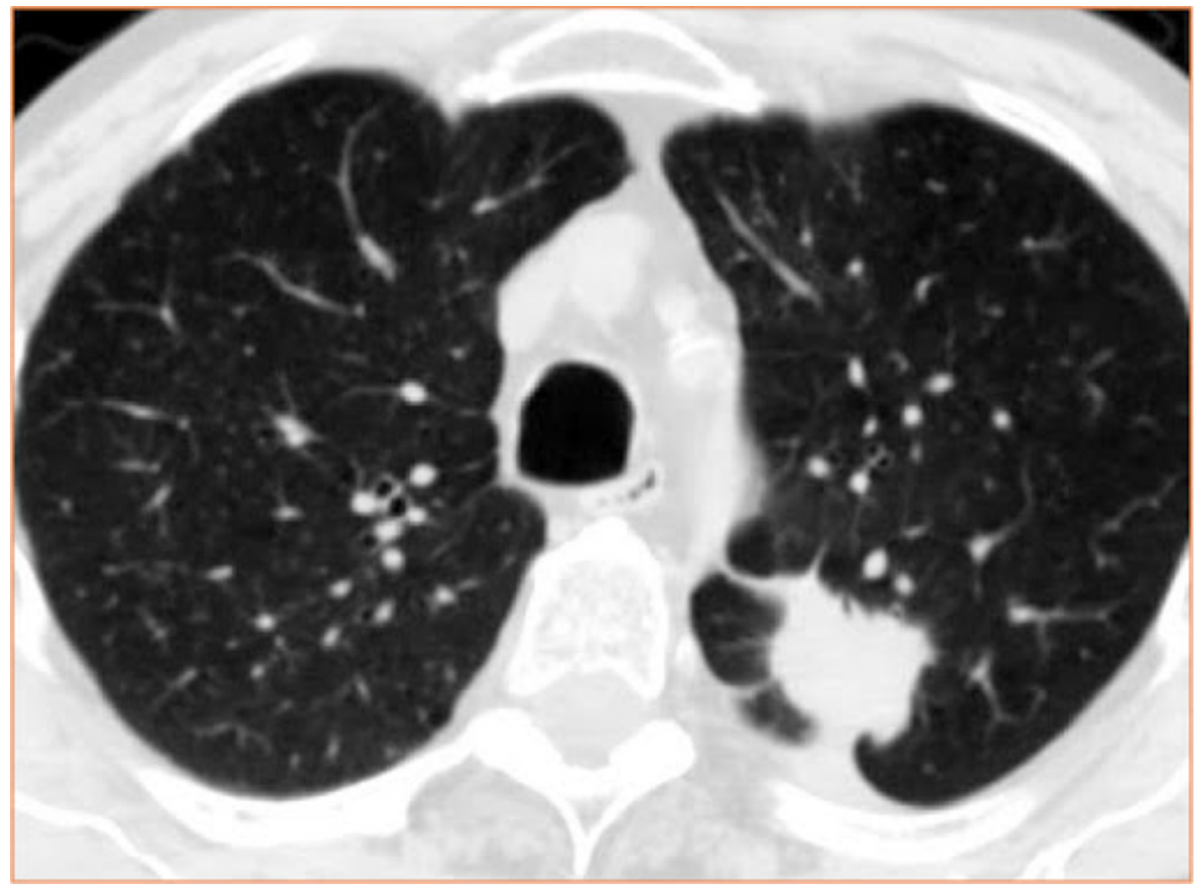
In X-ray CT









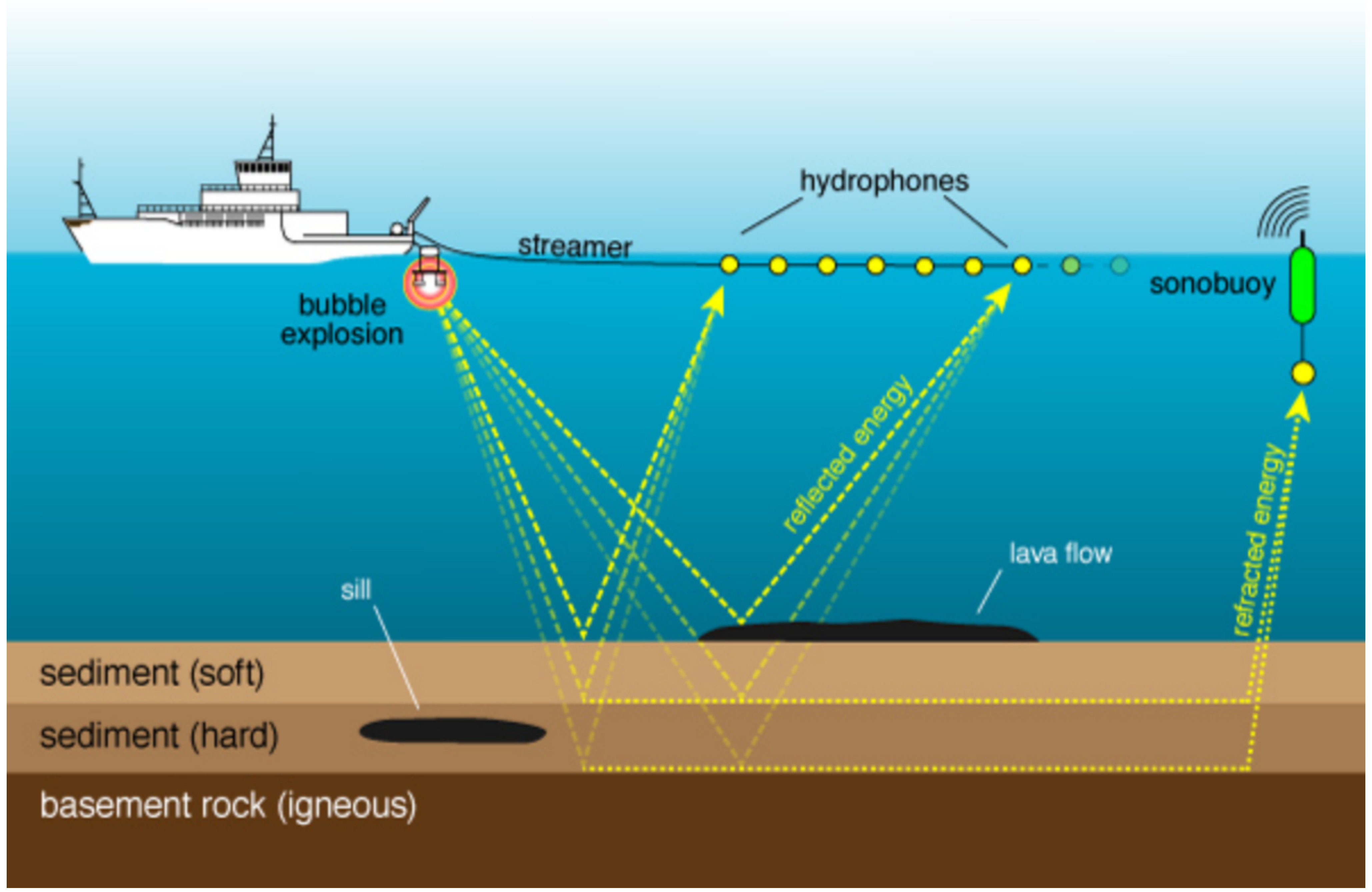


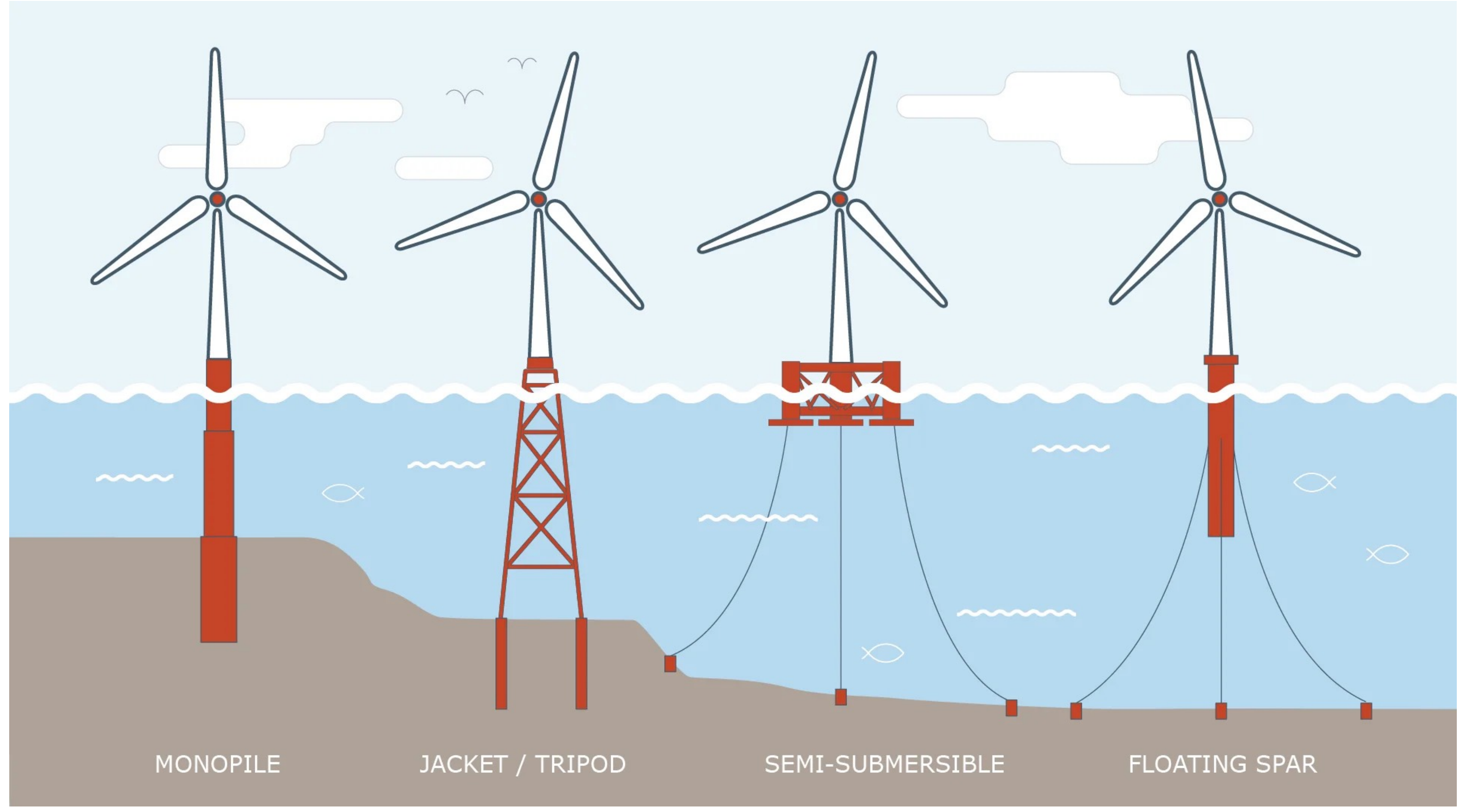
images: publicly available on Google
info: Dr. Riccardo R. Colciago

Importance of Regularity

In seabed tomography

Joint work with Ana Carpio (UCM)





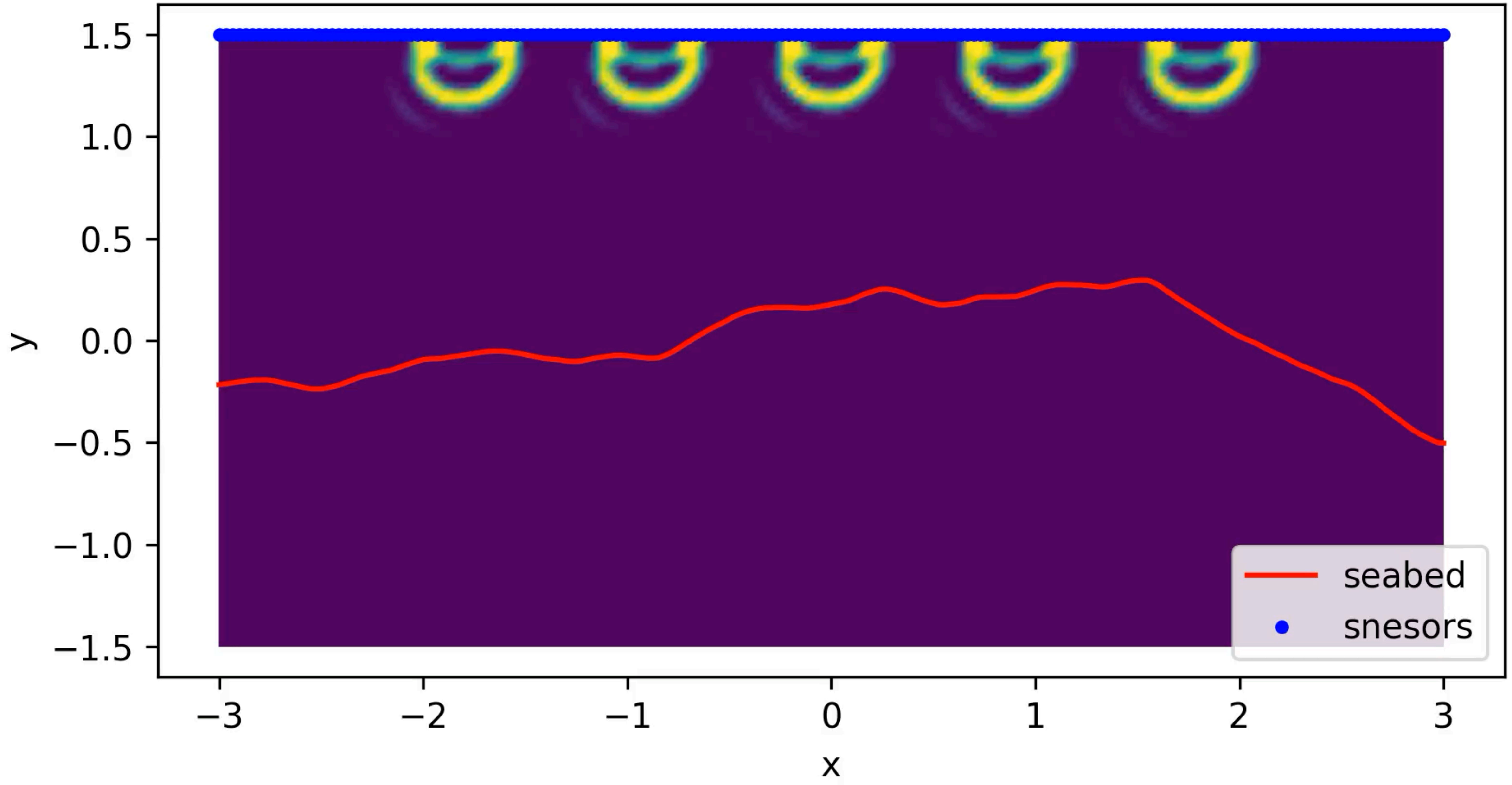
Source: cowi.com

Source: agcs.allianz.com

Importance of Regularity

In seabed tomography

Joint work with Ana Carpio (UCM)



Forward Model

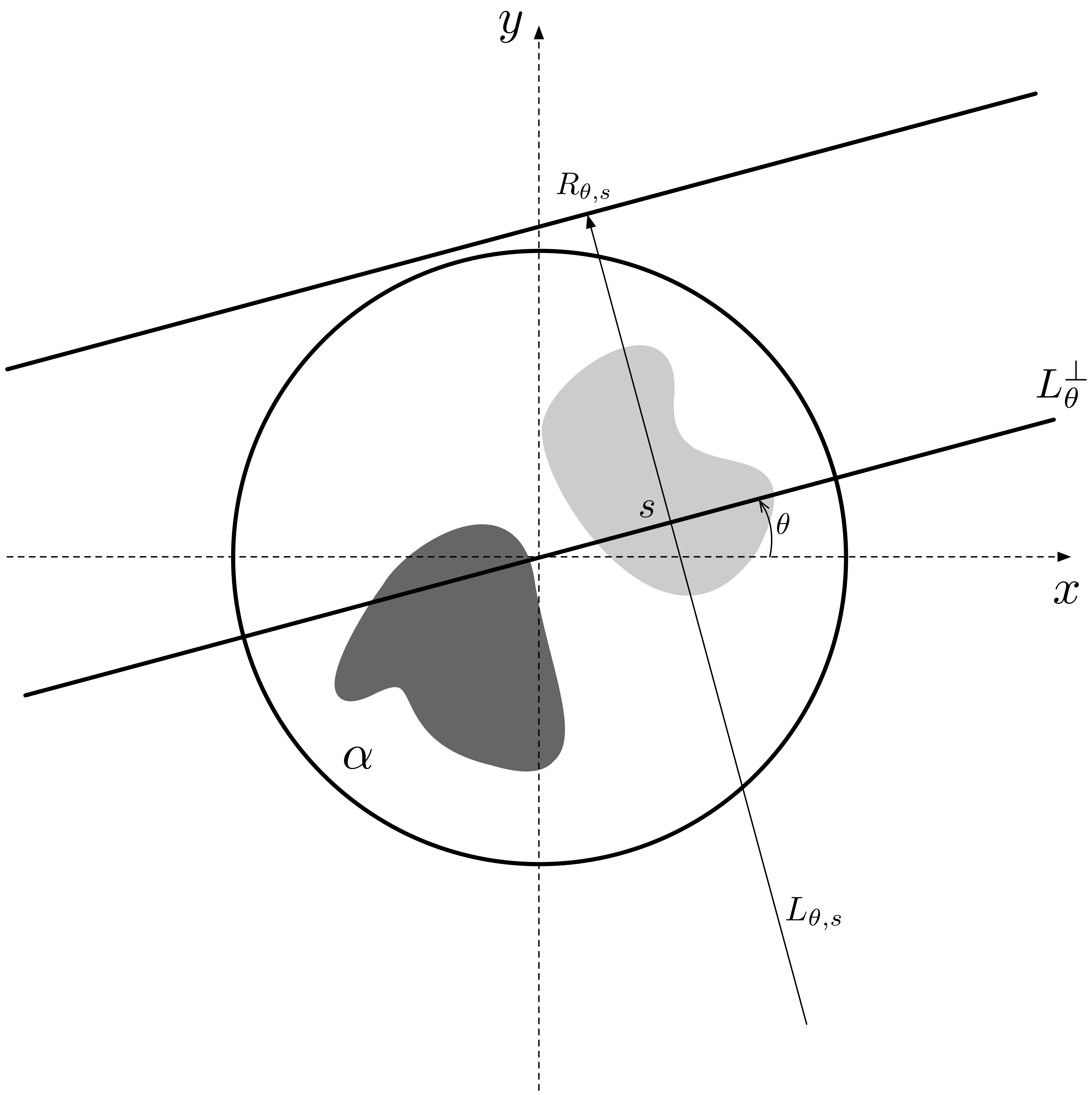
Radon transform

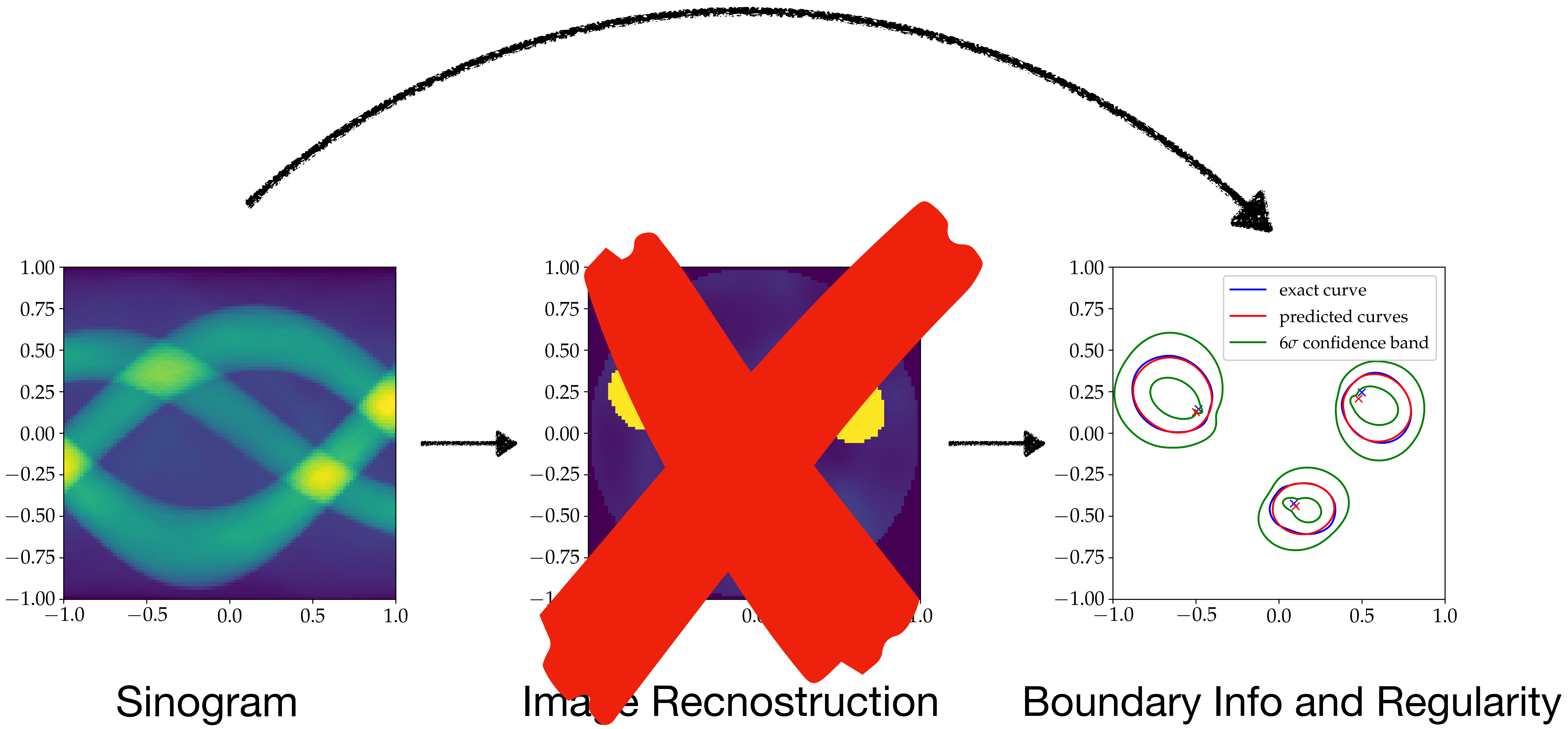
- Modelling X-ray interaction with line integrals:

$$R_{\theta,s}[\alpha] := \int_{L_{\theta,s}} \alpha(x,y)\ dl$$

- Collect in vector form:

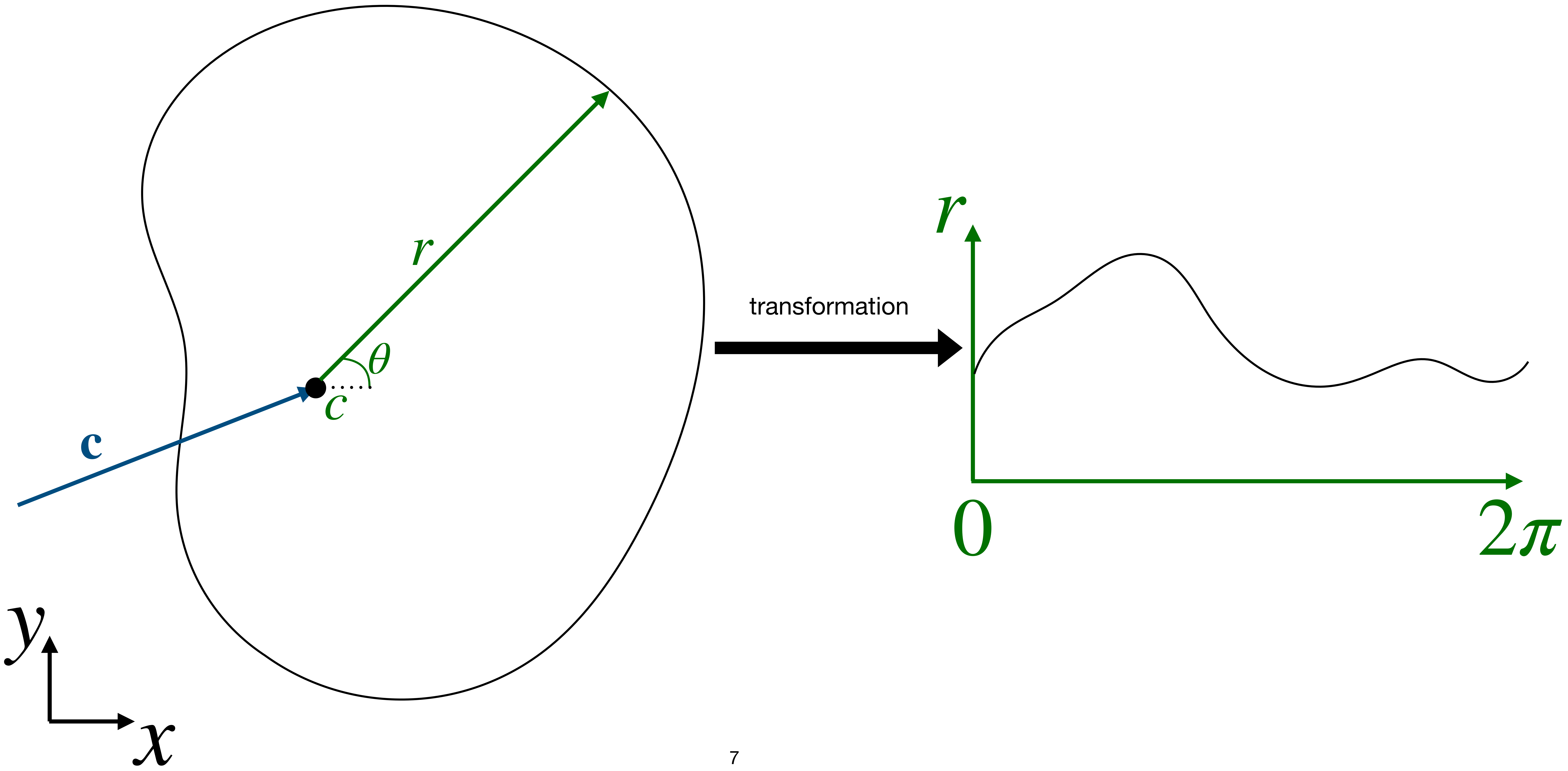
$$y(\alpha) = G(\alpha) + \varepsilon$$




exact curve
predicted curves
6σ confidence band
Sinogram
Ima e Recnostruction
Boundary Info and Regularity

Prior Modelling

Star-shaped boundary



Notion of Regularity

Fractional differentiablity

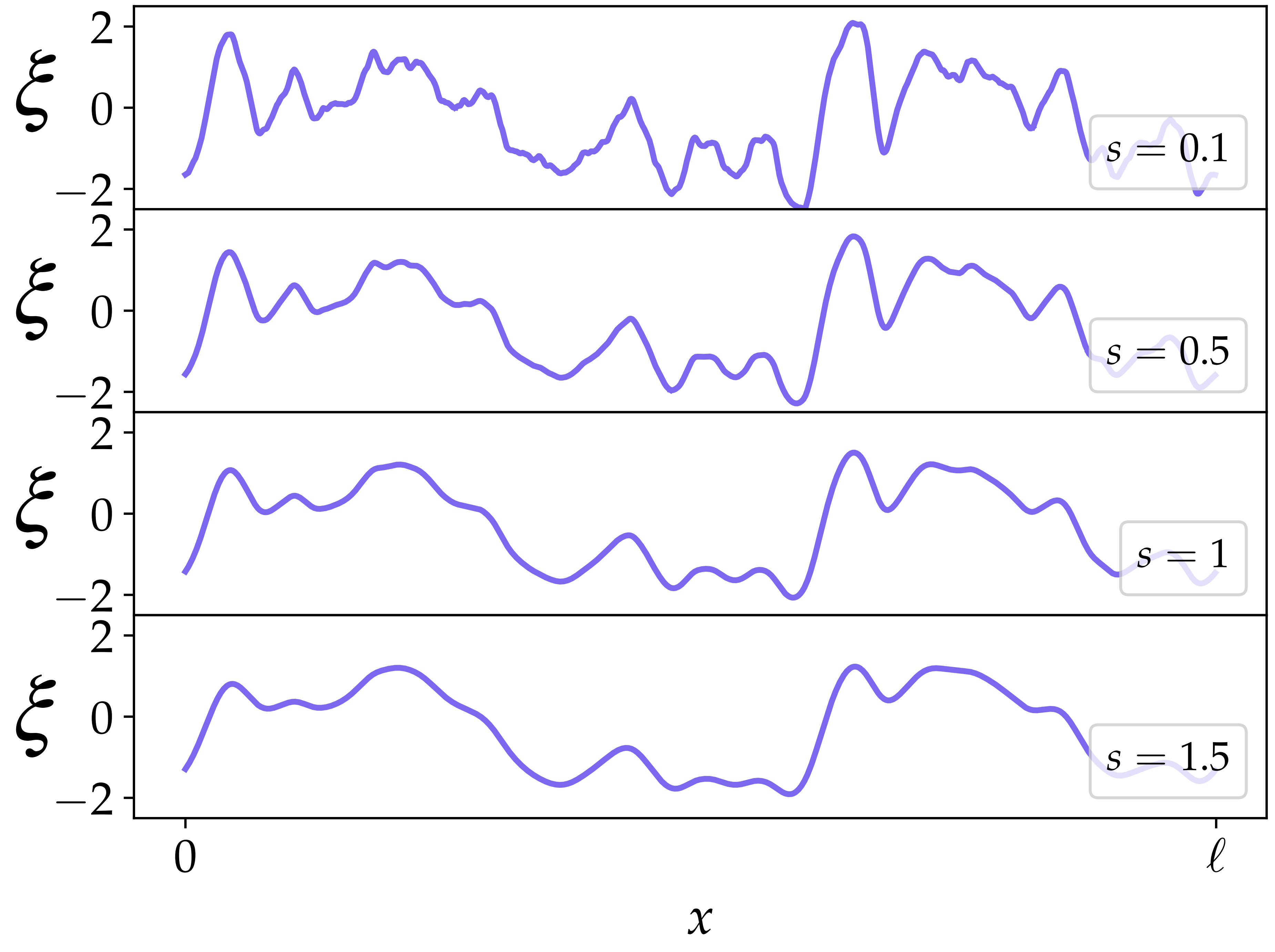
- Choose a basis (Fourier Basis) $\{\mathbf{e}_i\}$ and define

$$\xi = \sum_{i=1}^{\infty} x_i \mathbf{e}_i$$

- To make sense of the infinite series

$$\xi = \sum_{i=1}^{\infty} x_i \lambda_i^{s+\frac{1}{2}} \mathbf{e}_i, \qquad \lambda_i \asymp \frac{1}{i^2}$$

- $\xi \in H^{2t+1/2}$ for all $t < s$ [Dunlop et al. 2017]

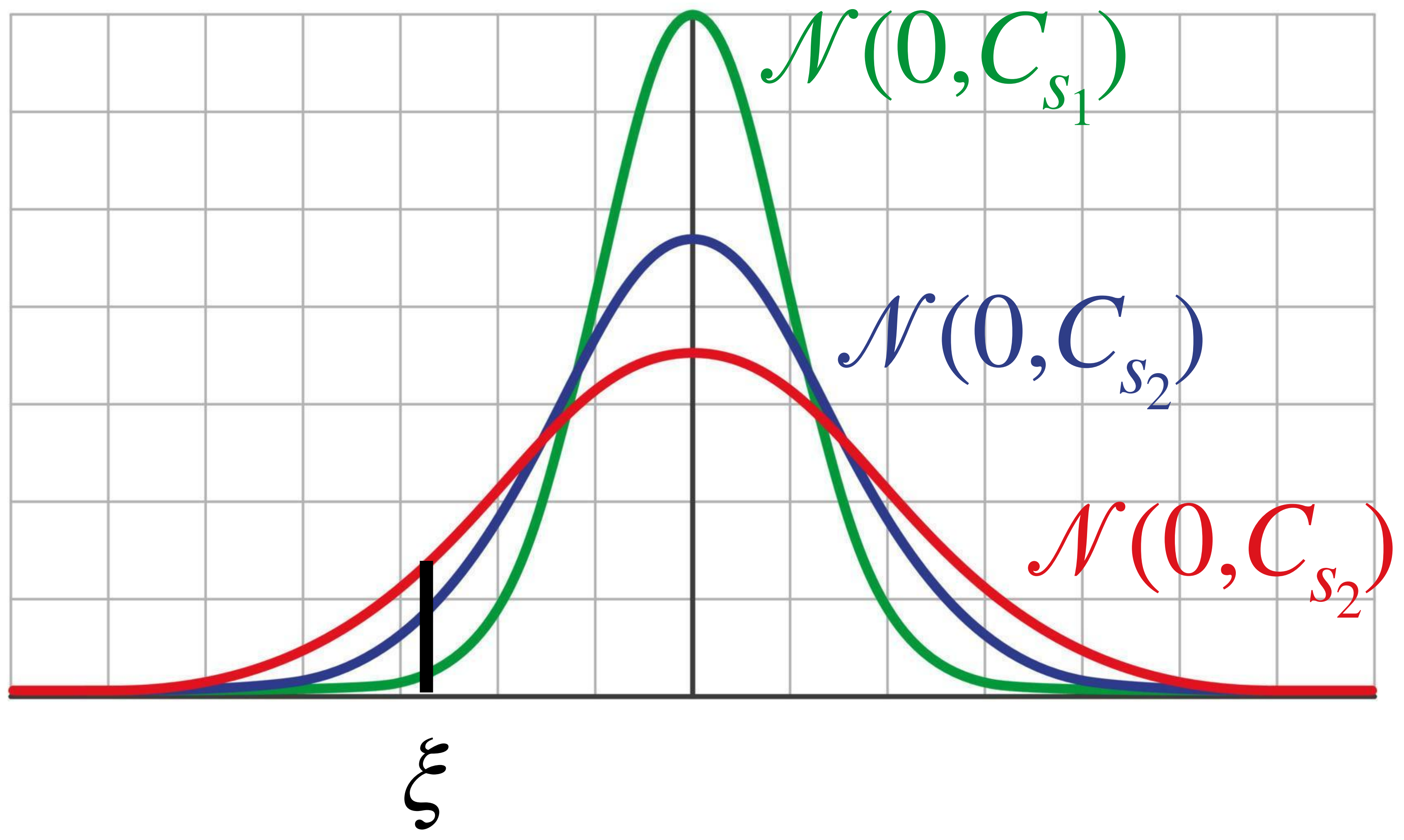


What can we say about the uncertainty in s?

Family of Gaussians

- We can construct a covariance as:

$$C_s = \sum_{i=1}^{\infty} \lambda_i^{s+1/2} \langle e_i, \cdot \rangle e_i$$



Goal Oriented Forward Map

boundary to image

- Recall:

$$y(\alpha) = G(\alpha) + \varepsilon$$

- We define the forward-map to be:

$$\mathscr{G} = G \circ F.$$

- New forward map:

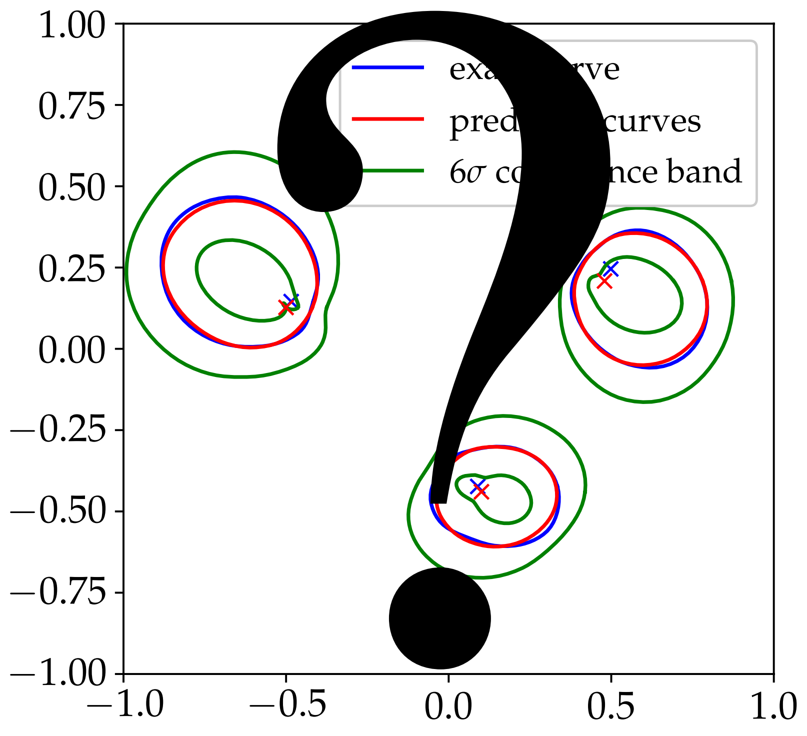
$$y = \mathscr{G}(\mathbf{x}, s) + \varepsilon$$

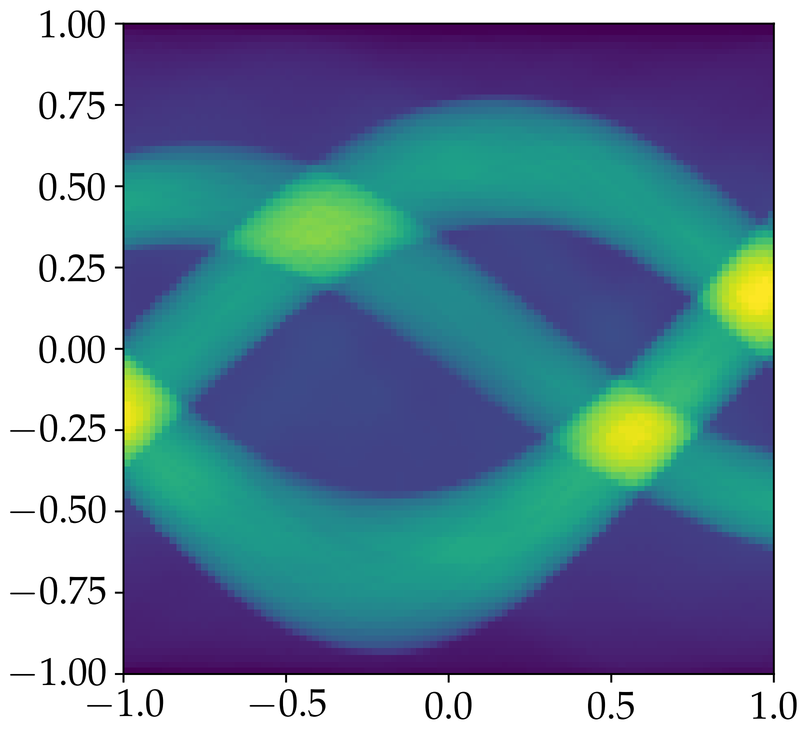
Bayesian Formulation

for the CT problem

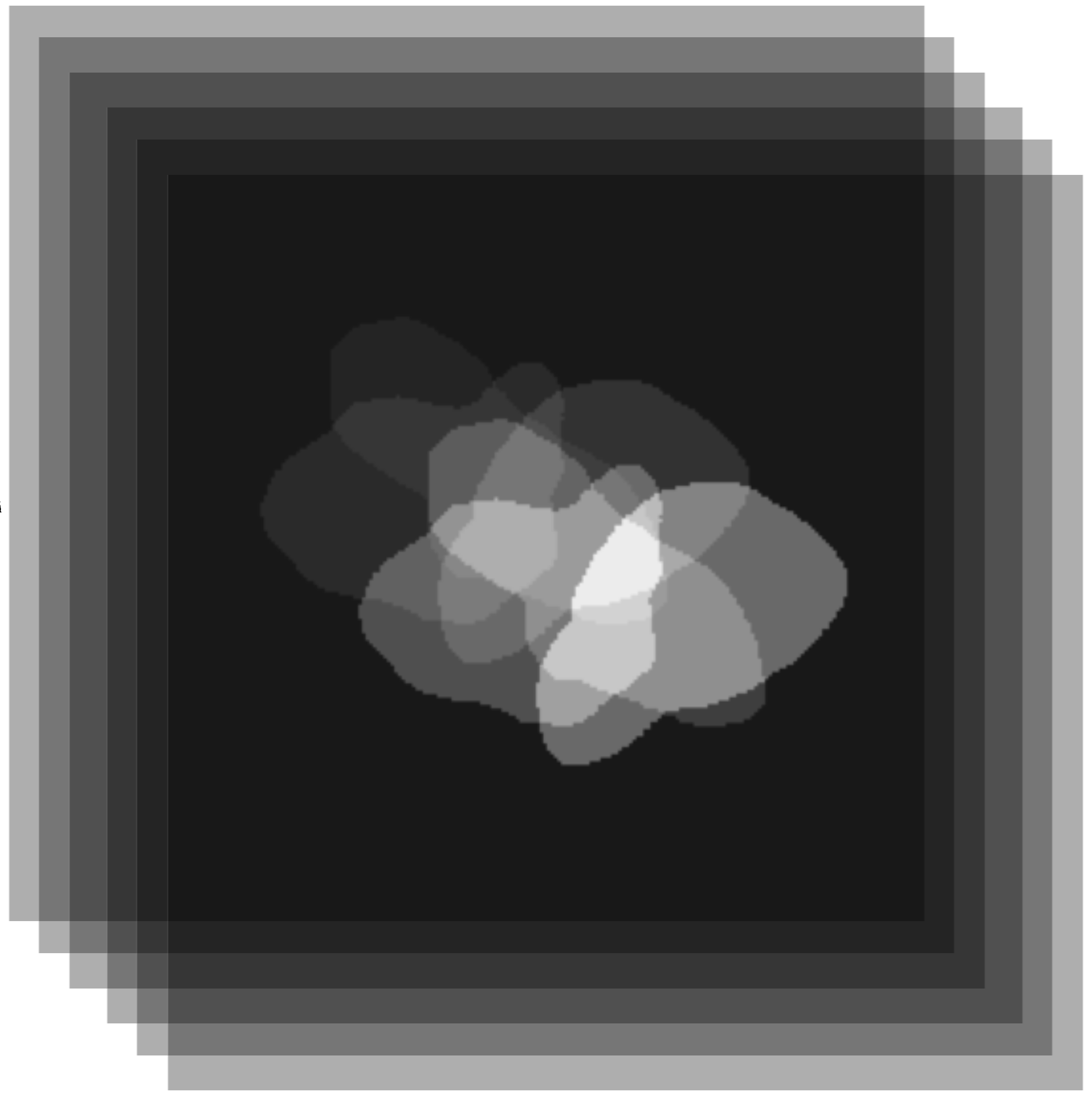
- Applying the Bayes' theorem:

$$\pi_{\text{post}}((\mathbf{x}, s) \mid y) \propto \pi_{\text{like}}(y \mid (\mathbf{x}, s))\pi_0(\xi \mid s)\pi_0(s)$$





$$y = \mathcal{G}(\mathbf{x}, s) + \varepsilon$$

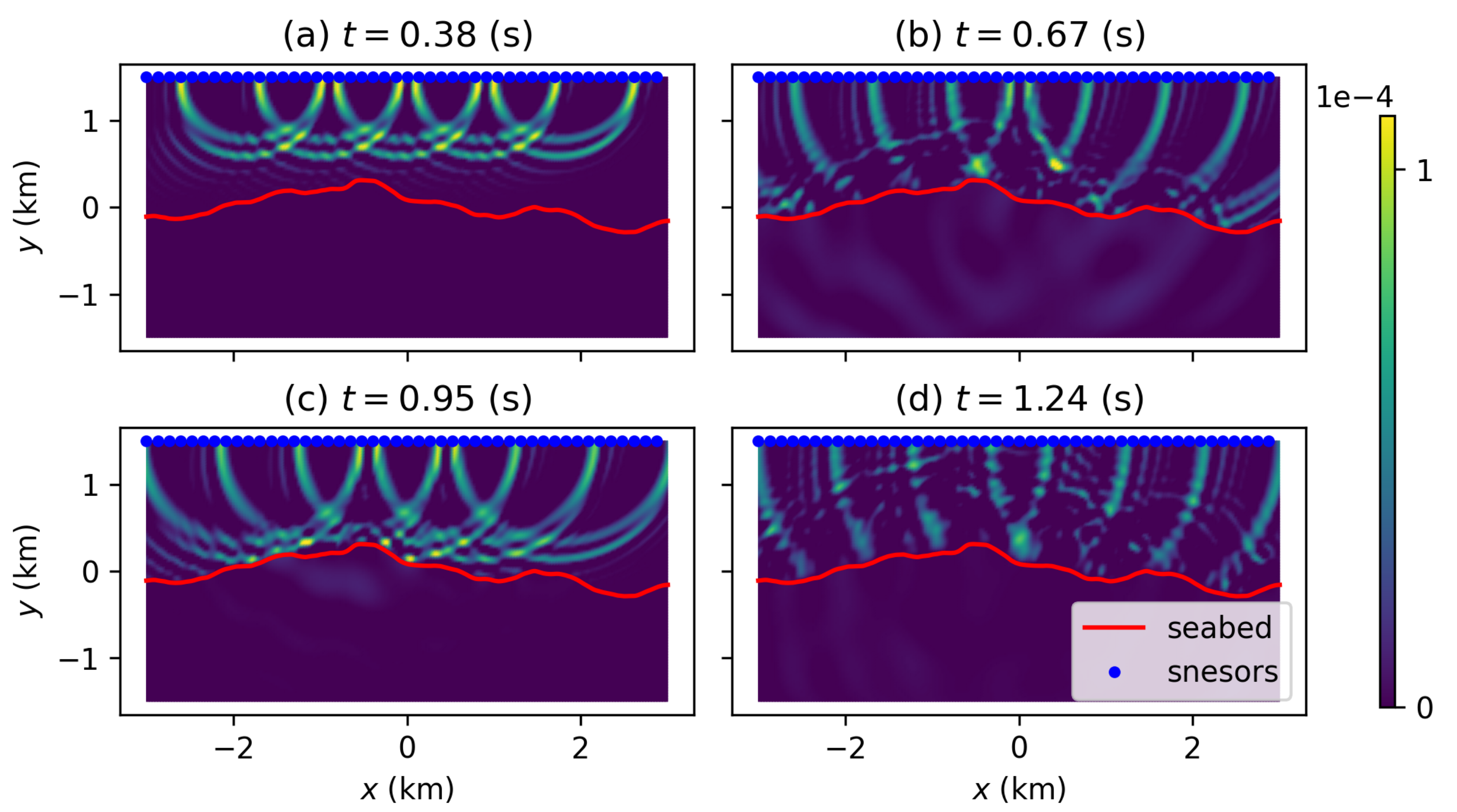


Posterior and The Gibbs Sampler

- We can form the posterior:

$$\pi_{\text{post}}((\mathbf{x}, s) \mid y) \propto \pi_{\text{like}}(y \mid (\mathbf{x}, s))\pi_0(\mathbf{x} \mid s)\pi_0(s)$$

- Gibbs sampler:
  - Sample from $\pi(\mathbf{x} \mid s, y)$
  - Maximum likelihood estimation for s



SIAM IS24

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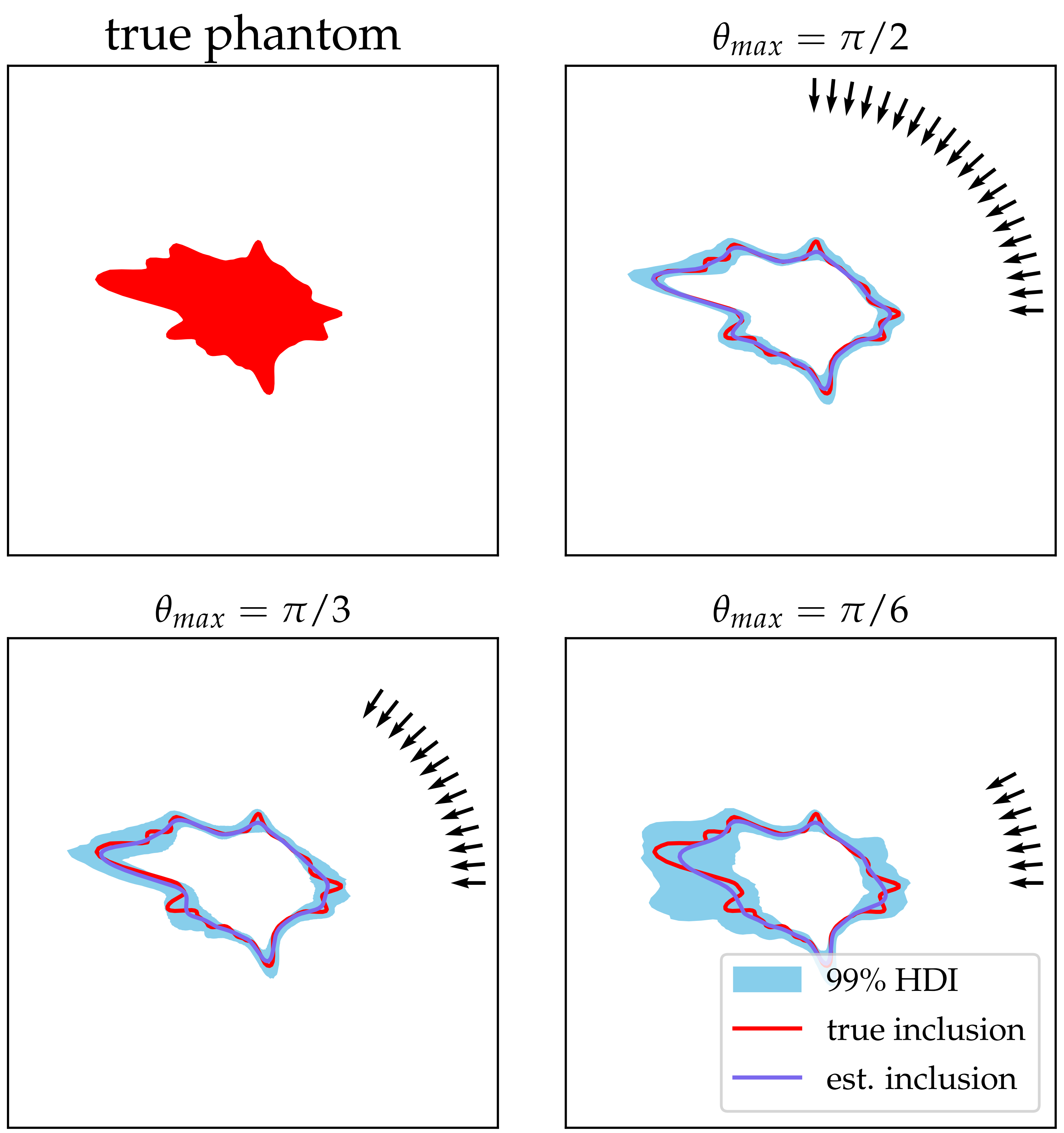
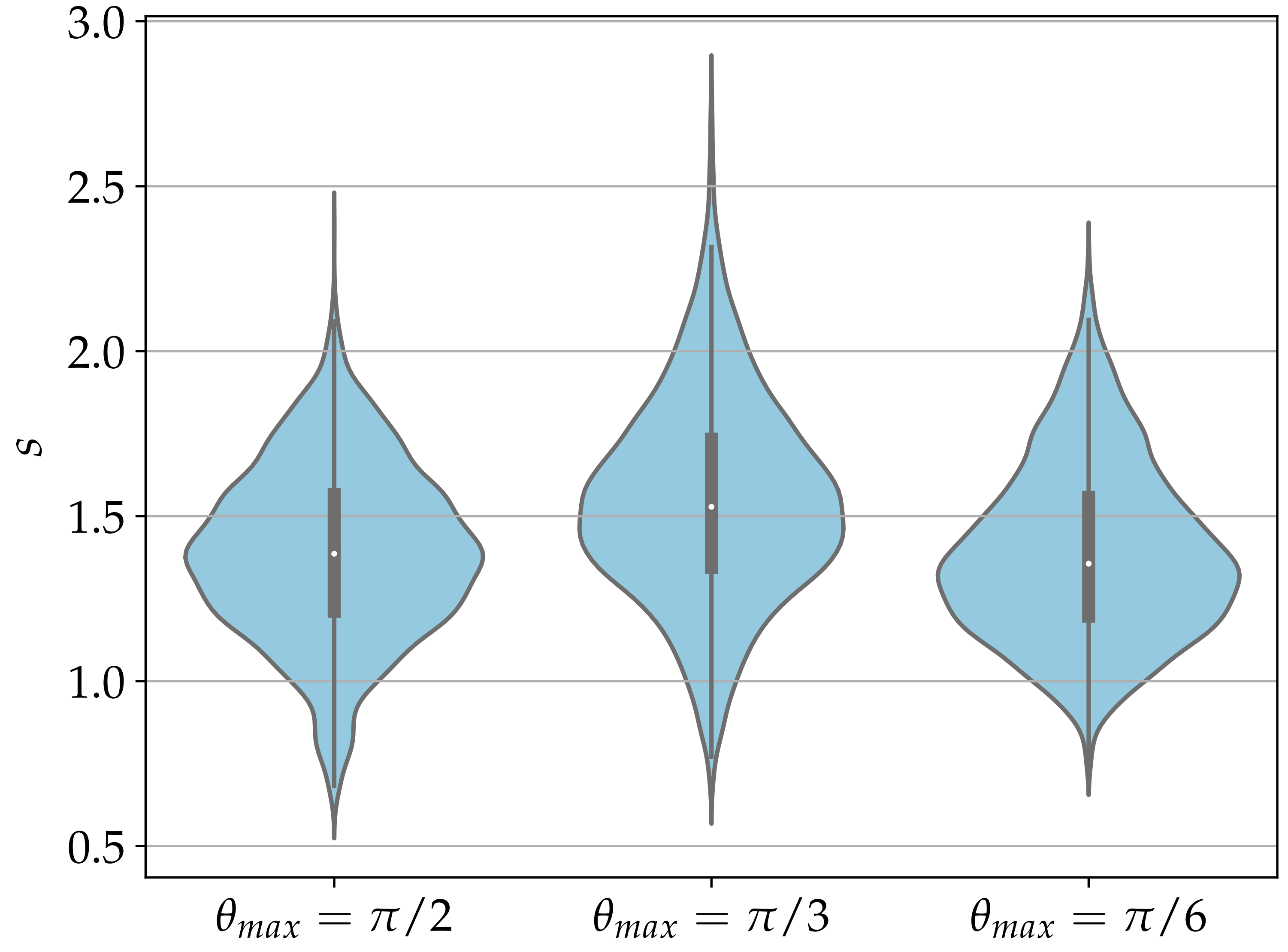
Performance

Limited-angle X-ray CT (partial data)



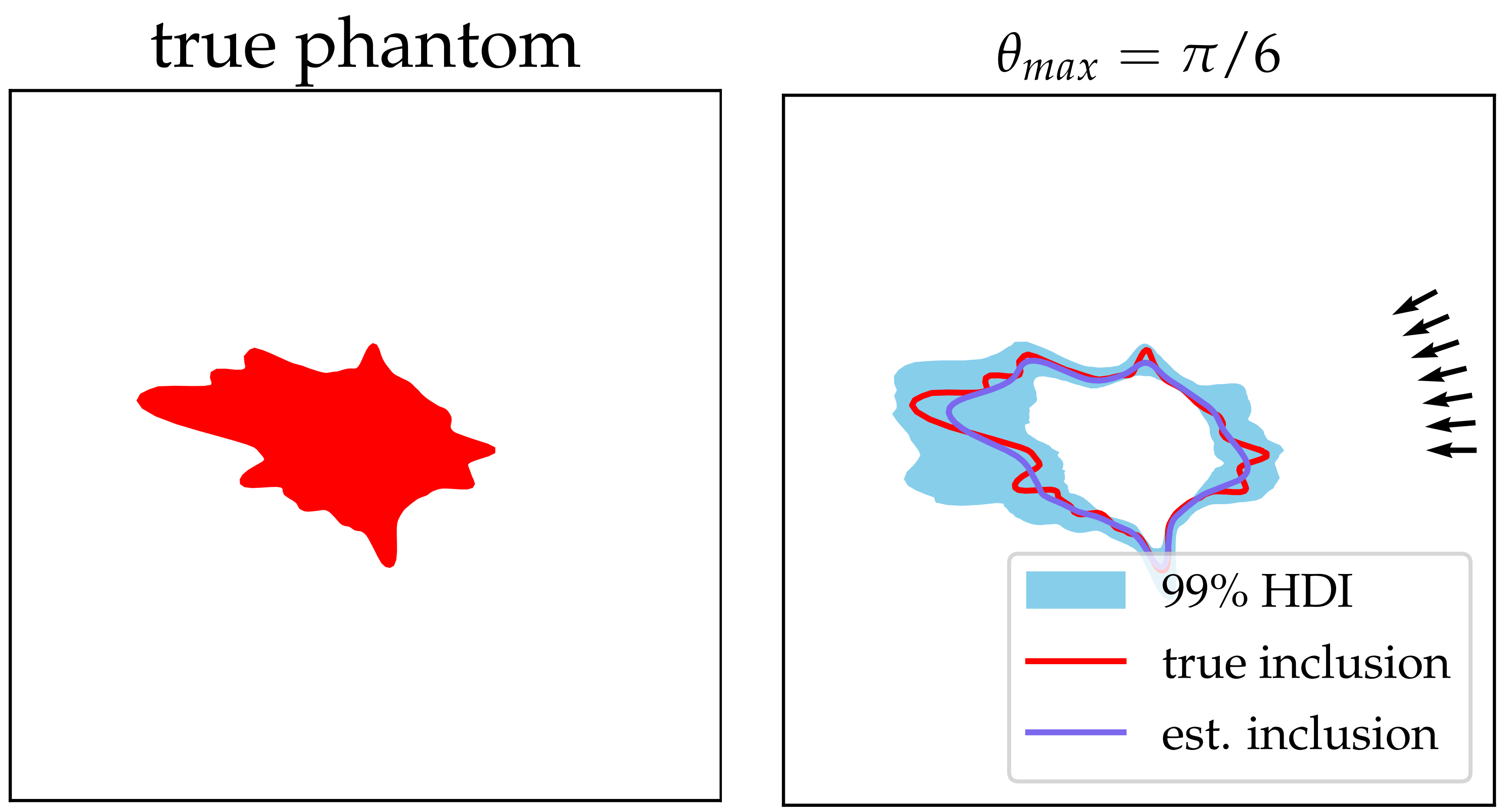
True s: 1.064



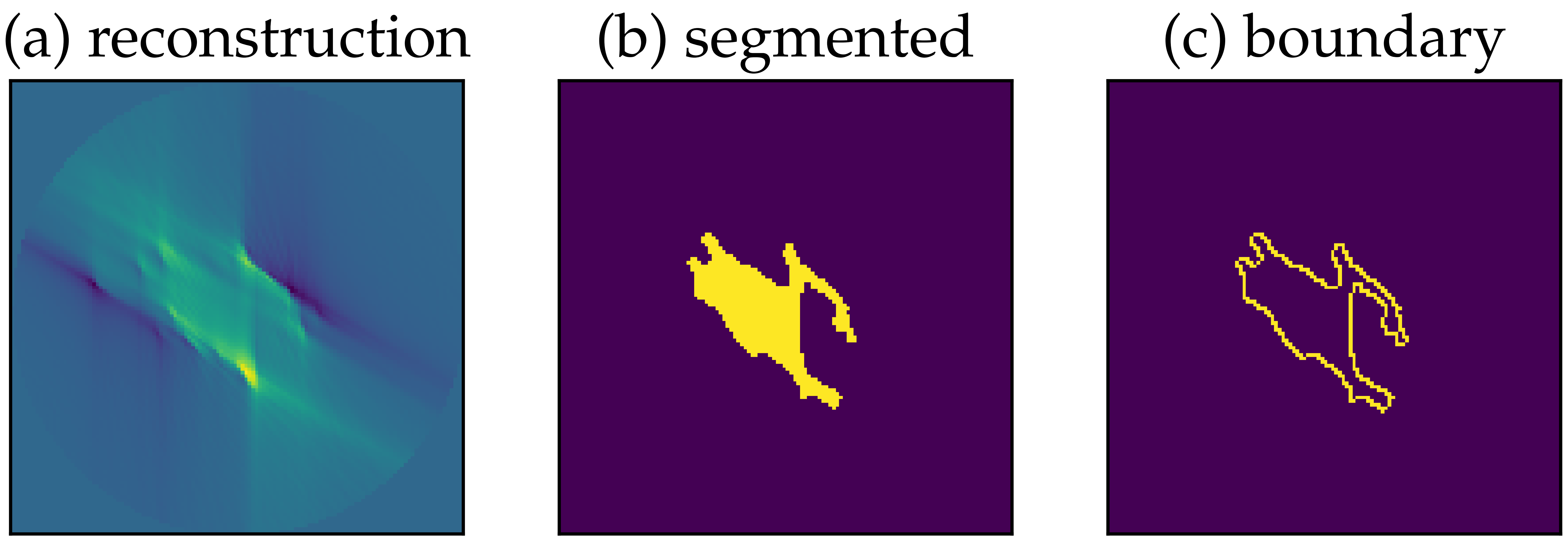
Performance

Comparison with the state-of-the-art

Hierarchical Bayesian



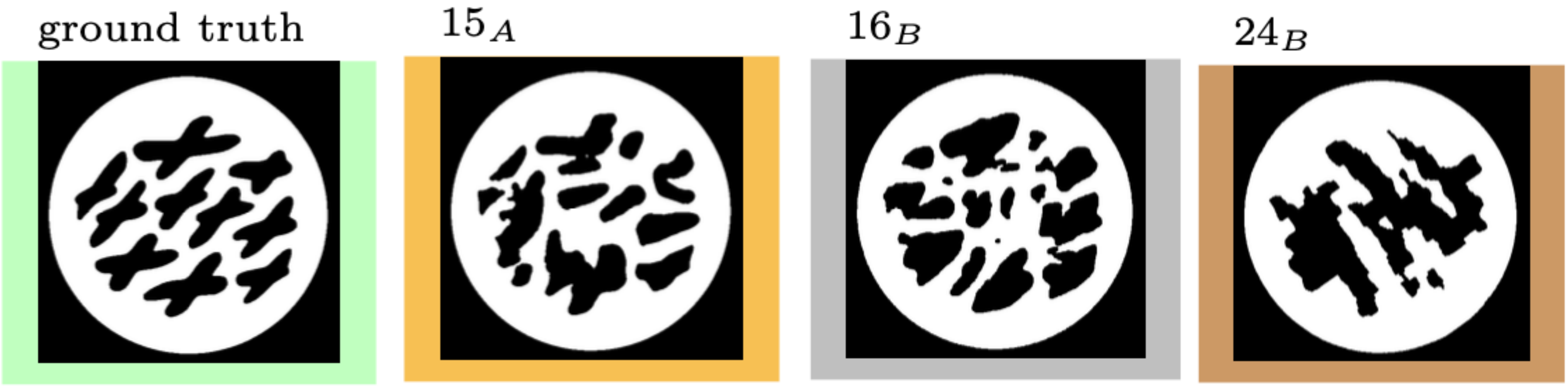
FBP



Performance

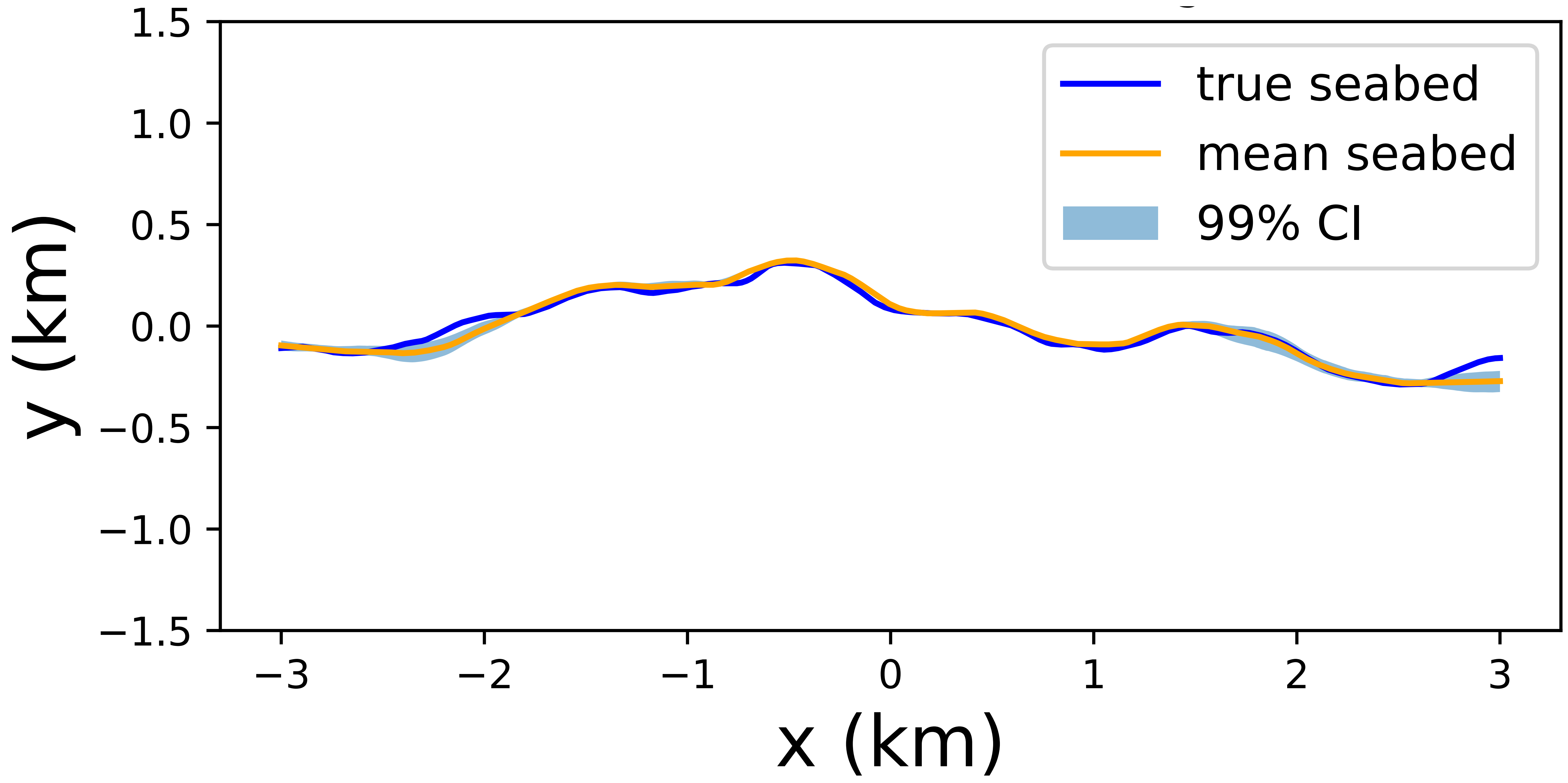
Comparison with the state-of-the-art

Helsinki tomography challenge 2022 (FIPS): 30° limited angle CT



Performance

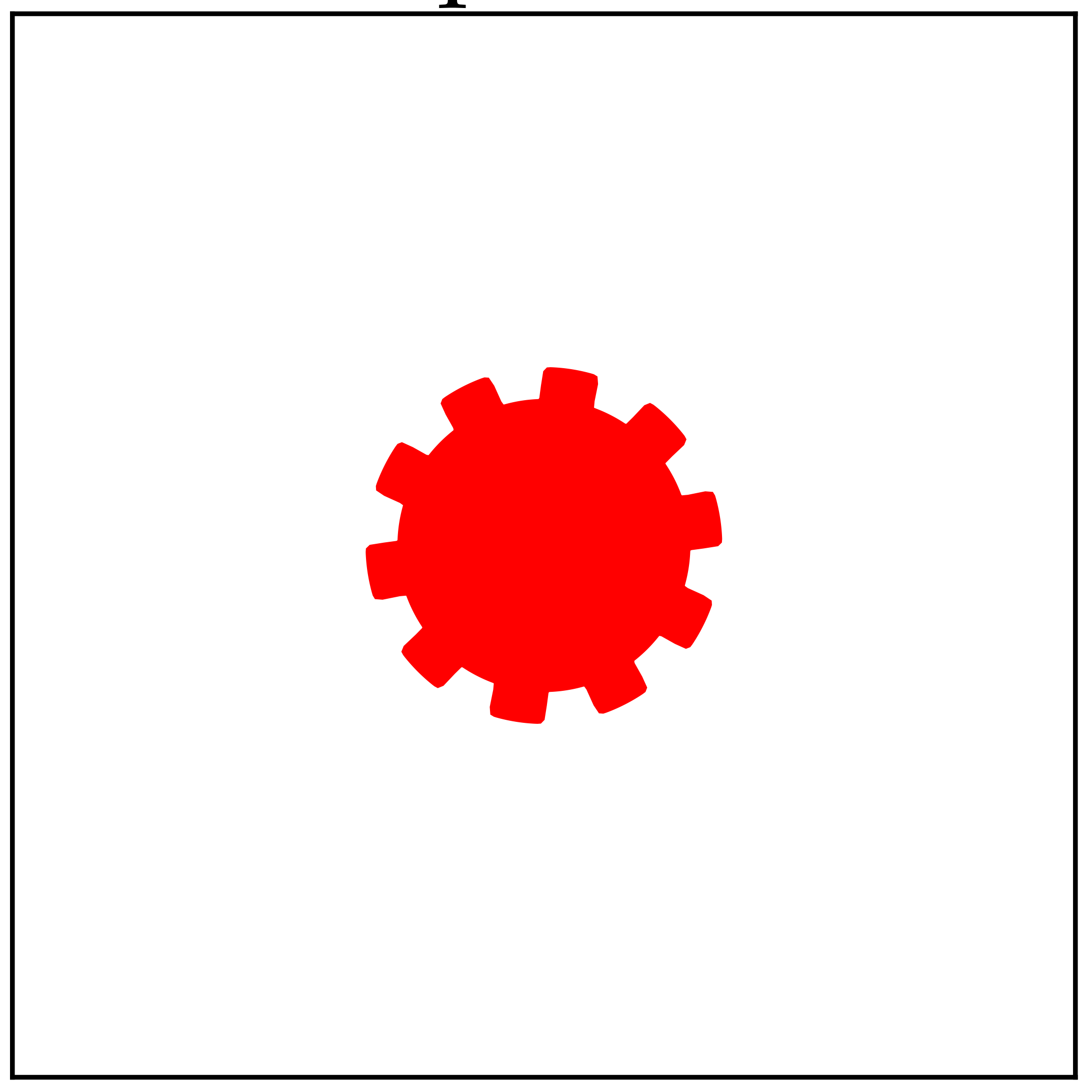
Seabed Tomography with Acoustic Waves



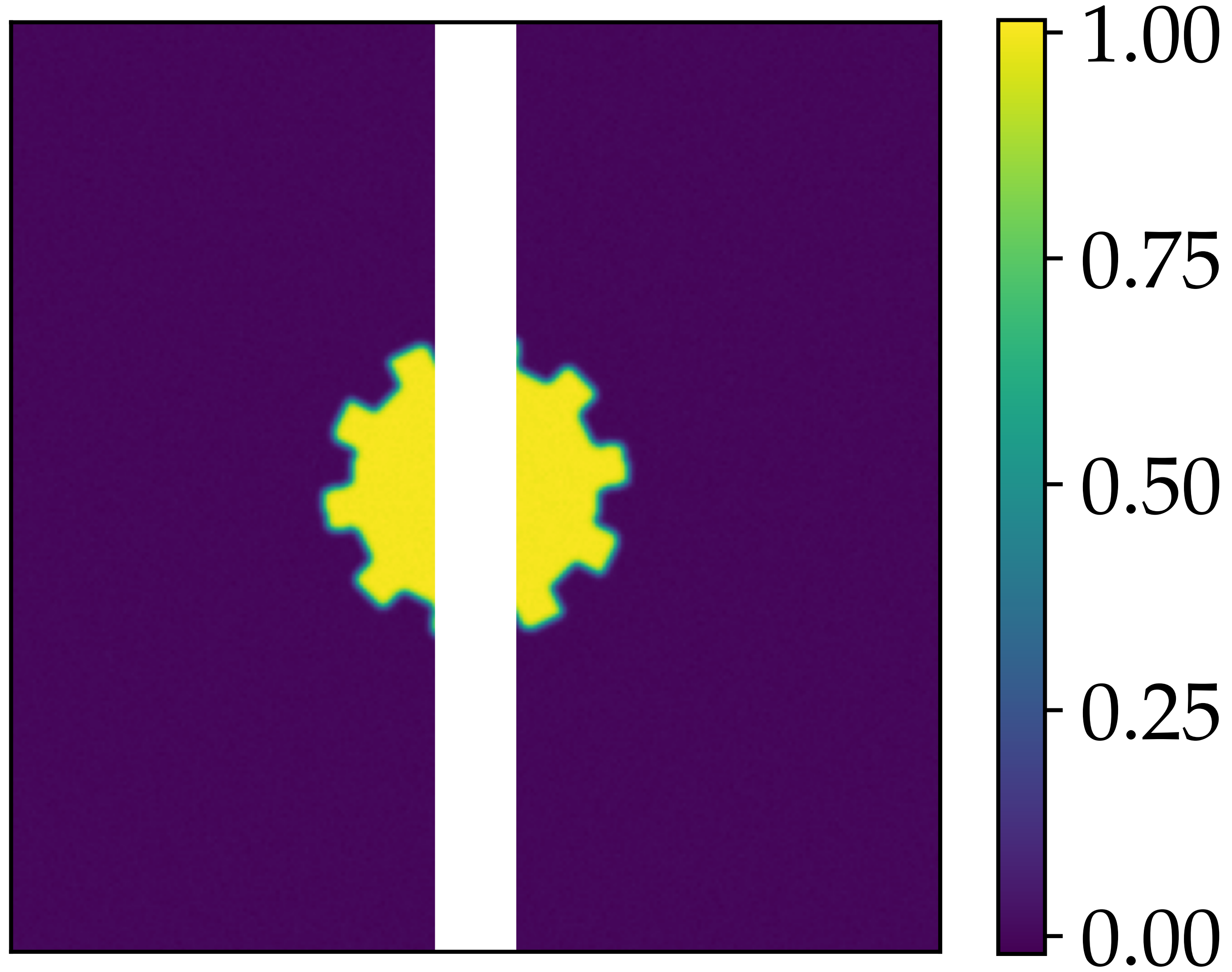
Performance

Image in-painting (Gaussian noise)

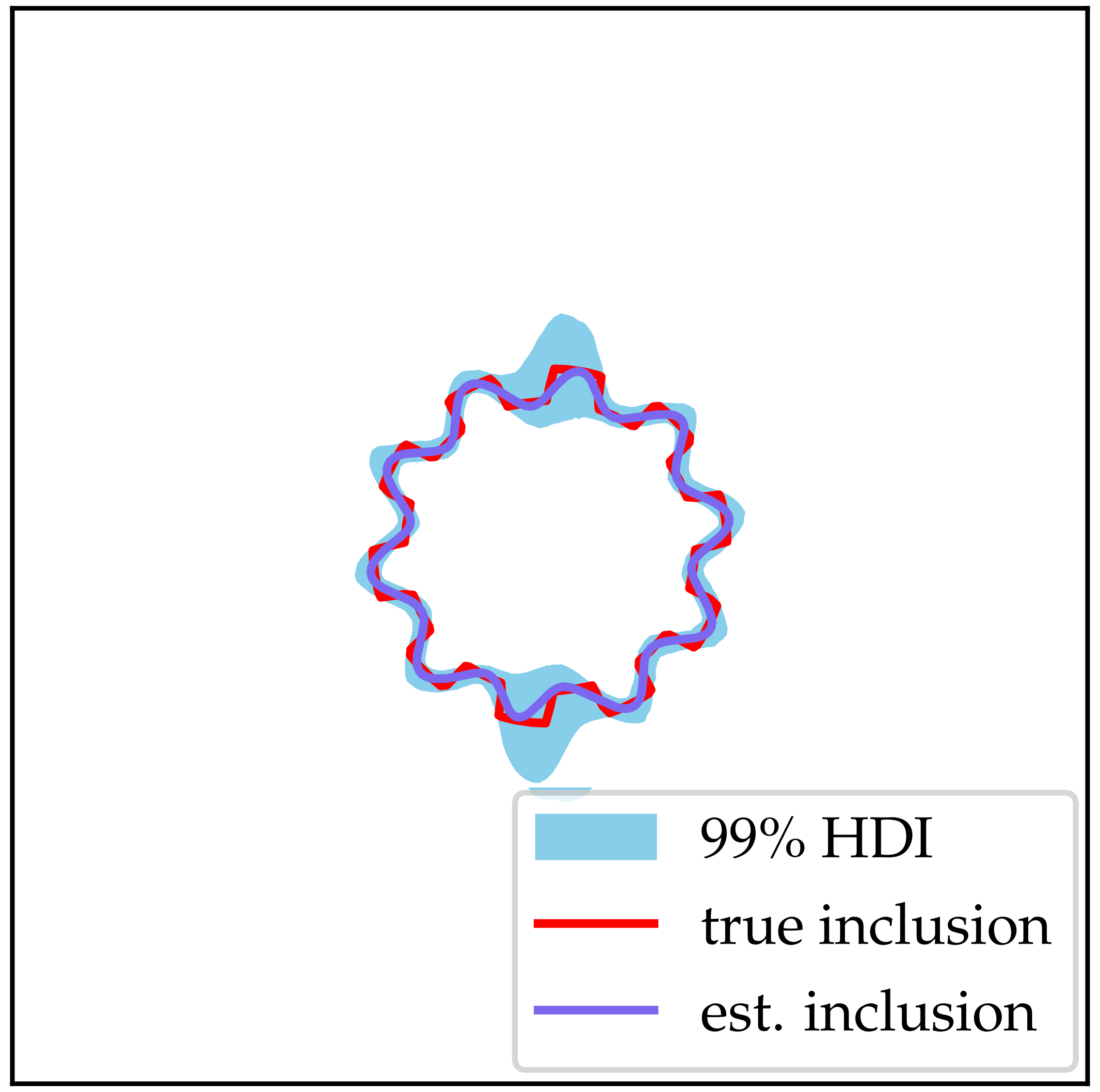
true phantom



Data with 2% noise



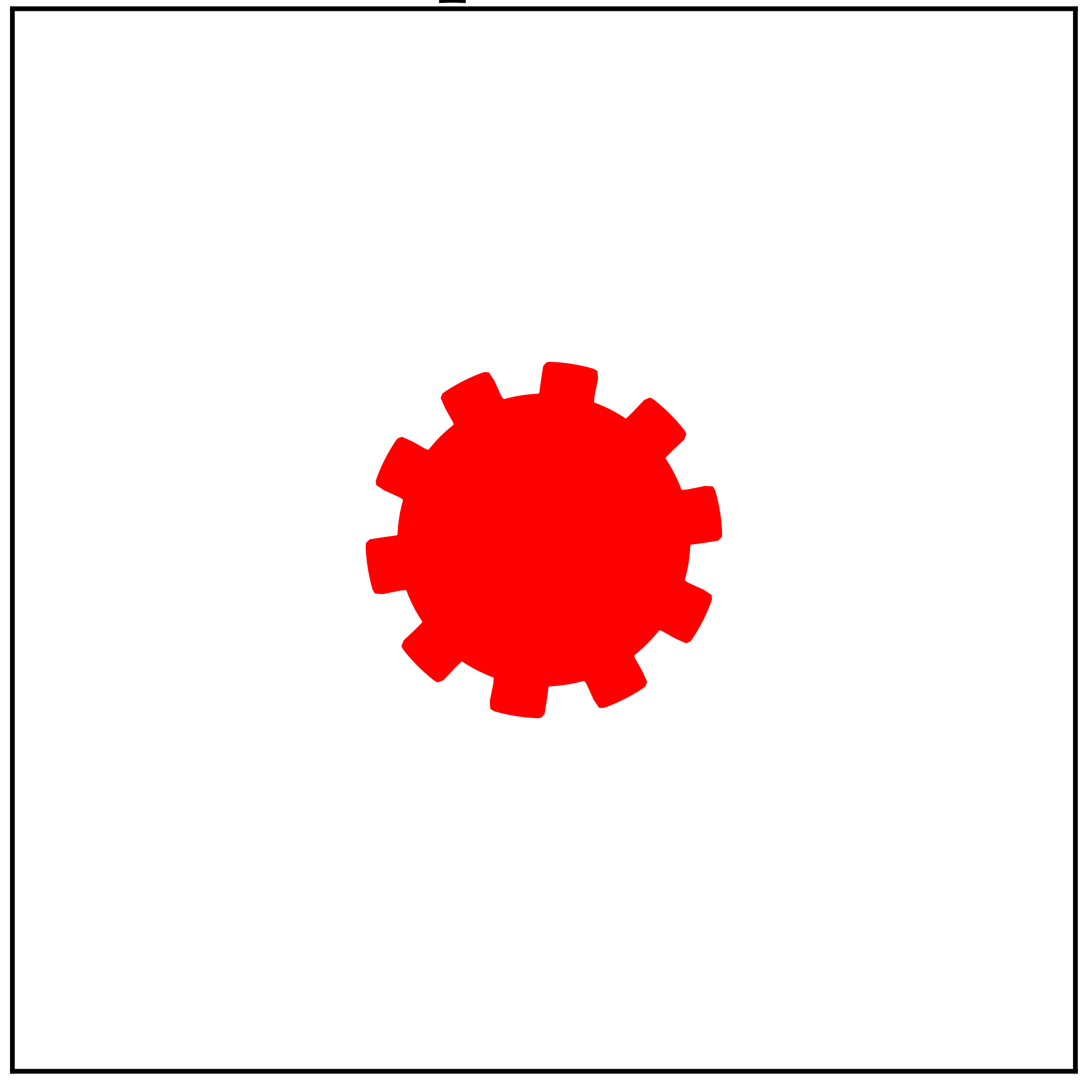
Estimate of the boundary



Performance

Image in-painting (Laplace noise)

true phantom



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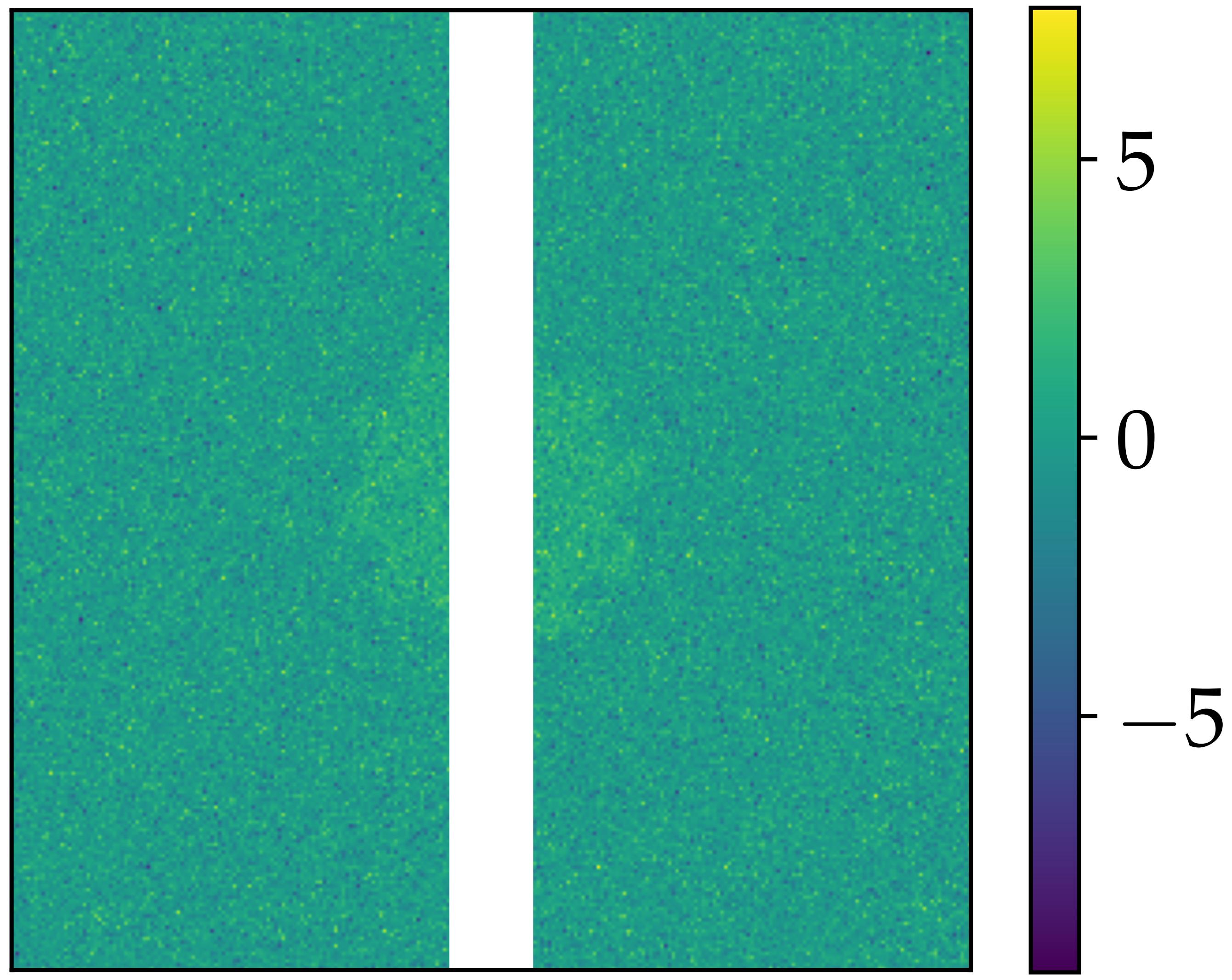
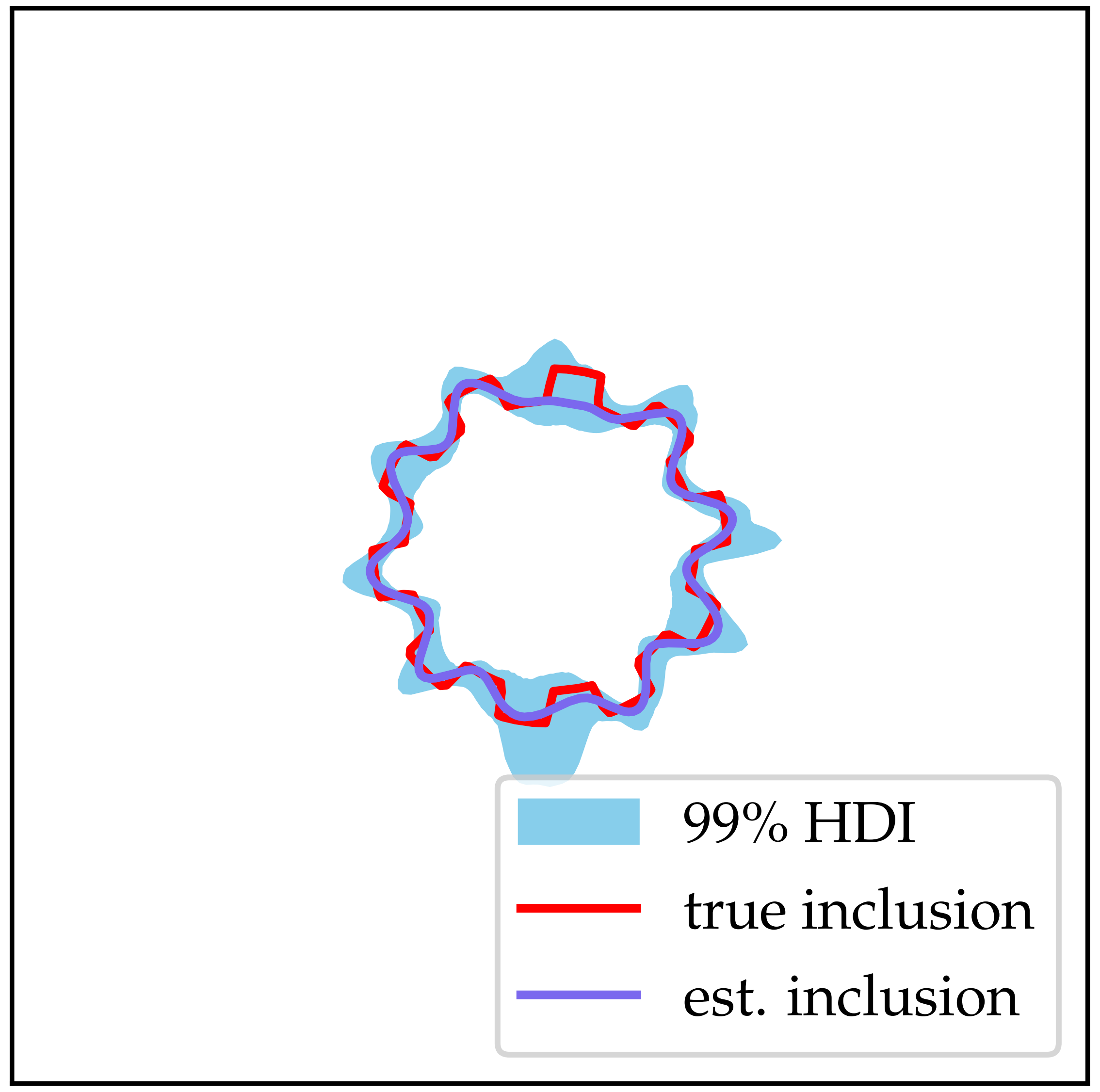


Estimate of the boundary



References

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