Regularized Sampling of Fractional Gaussian Processes on R^d

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1 Introduction

In many applications of inverse problems, we face the problem of estimating continuous quantities from indirect measurements. In addition, we must quantify uncertainties in such an estimate. The infinite-dimensional Bayesian approach is a promising method for estimating continuous quantities in inverse problems.

- **The idea is to use pseudodifferential calculi and PDE theory to produce samplers in infinite dimensional function spaces with highly complex conditions built in.**
- **The pseudodifferential calculus is a method to model local irregularities and local inhomogeneity which would be challenging for current priors.**
- **This method helps reveal the singularities (e.g. discontinuities) in our estimate.**

2 Problem

Regularized sampling can be achieved, e.g., by solving pseudodifferential equations of the type

$$\left(k(x)I - \Delta\right)^{s(x)} u = \Psi \quad in\ R^n \qquad (1)$$

where:

- Ψ is a stochastic term, typically Gaussian white noise
- $k(x)$ and $s(x)$ carry information on the desired regularity of the sampled function u.

Goal: Pose the sampling problem in terms of a stochastic pseudodifferential equation,the numerical solution of each realization of the stochastic pseudodifferential equation amounts to a single sampling, where the solution function incorporates the desired properties.

The Sampling Method:
Construct an operator **Q** of the elliptic operator $P = \left(k(x)I - \Delta\right)^{s(x)}$ where

$$QP = I + K$$

where K is a smoothing operator and apply it to (1) to get

$$u = Q\Psi + Ku$$

with $Ku \in C^\infty$.

1. Find a symbol $p(x,\xi)$ of the pseudodifferential operator $\left(k(x)I - \Delta\right)^{s(x)}$
2. Construct the excision function, $\psi(\xi)$
3. Construct $q_0(x,\xi) = \frac{\psi(\xi)}{p(x,\xi)}$
4. Find $r_0(x,\xi) \sim \sum_{\alpha>0} D_\xi^\alpha q_0(x,\xi) D_x^\alpha p(x,\xi) - 1$
5. Calculate $s(x,\xi) \sim \sum_{j\in N} (\sigma(-r_0(x,D))^j)$
6. $q(x,\xi) \sim q_0(x,\xi) + \sum_{\alpha\in N_0^n} D_\xi^\alpha s(x,\xi) D_x^\alpha q_0(x,\xi)$

so we get:

$$Q\Psi(x) = (2\pi)^{-n} \int_{R^n} e^{ix\xi} q(x,\xi) \widehat{\Psi}(\xi) d\xi$$

.

The integral above can be evaluated numerically, yielding the sample $u(x)$.

$$u = Q\Psi + Ku$$

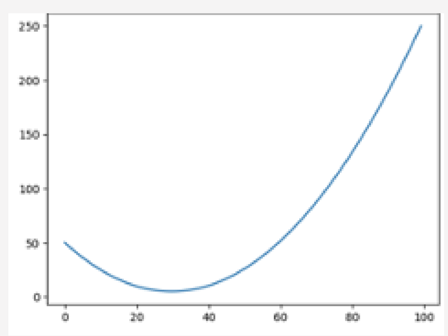
The term Ku does not carry singularities of u since $Ku \in C^\infty$.

Remarks:

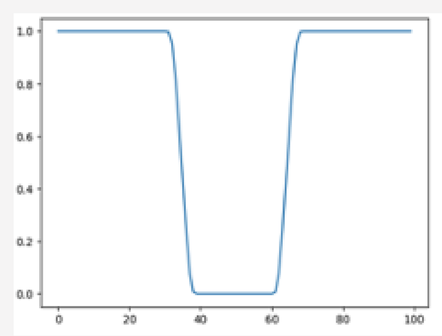
- The Whittle-Matern priors are a popular choice for such unknowns. However, incorporating spatial inhomogeneity, anisotropy, and local irregularities into such priors is challenging.
- This work has potential applications in many inverse problems with local inhomogeneity, e.g. in X-ray computed tomography, and fault detection in industrial applications.

3 Some of the Numerical Results

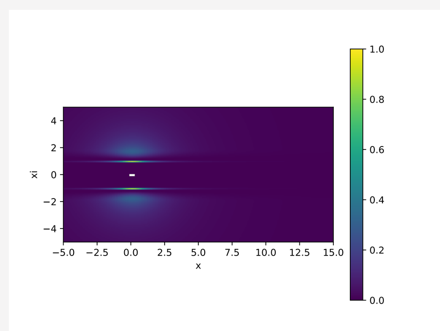
Particular Case: $s(x) = 1$, $k(x) = x^2$ **then:**



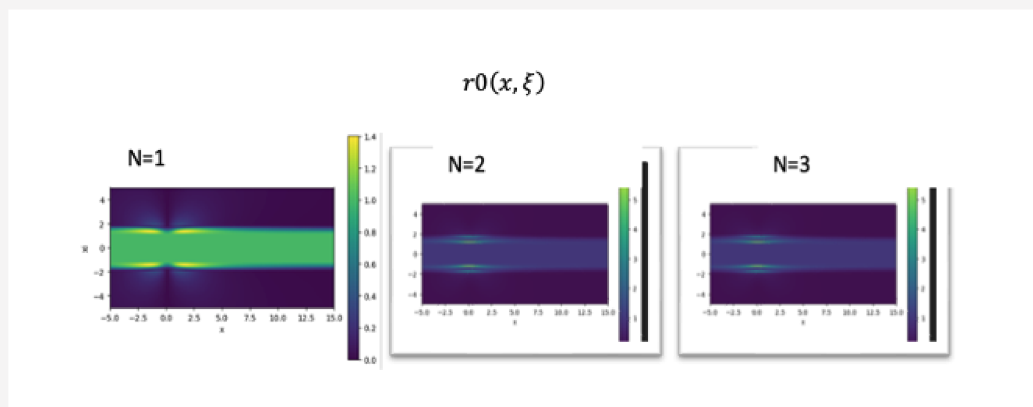
$p(x,\xi) = x^2 + \xi^2$



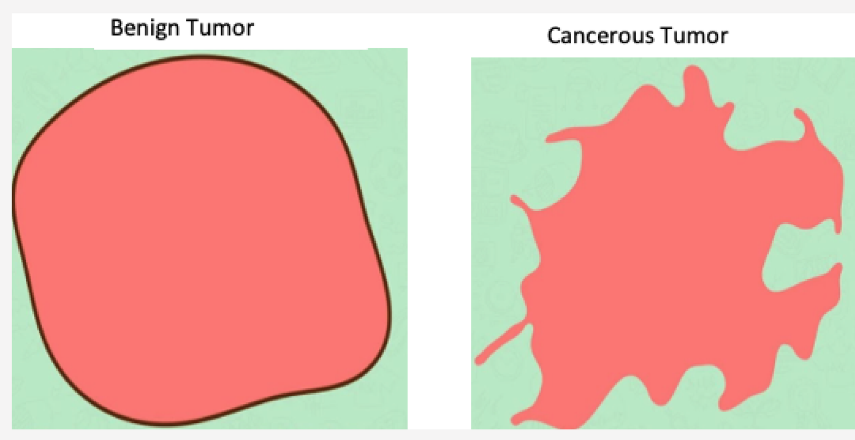
$\psi(\xi)$



$q_0(x,\xi) = \frac{\psi(\xi)}{p(x,\xi)}$



4 An Application



The boundary of the tumor is "u"

5 Conclusion

In conclusion, successfully solving the elliptic pseudodifferential operator outlined above facilitates the sampling process from function spaces.The sampler produces and construct functions from the correct space.