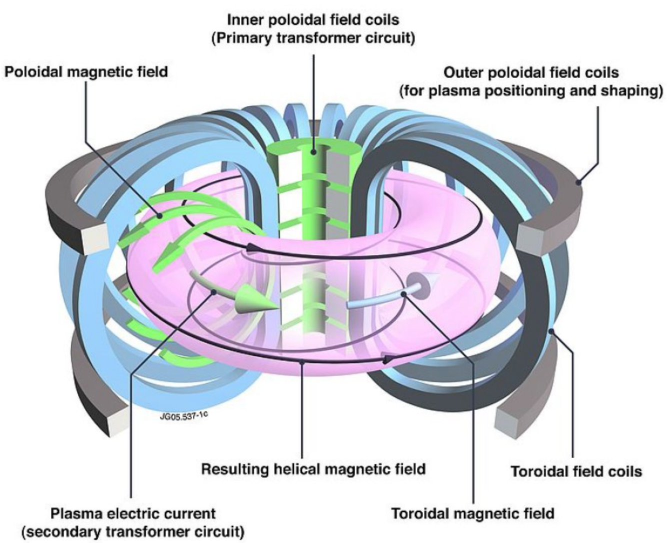
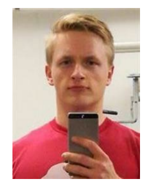
Tomography of the Fast-Ion Velocity Distribution in Tokamak Plasmas with a Physics-Based Prior

Per Christian Hansen
Technical University of Denmark



Joint work with

← Mirko Salewski
Professor, DTU Physics



← Bo Simmendefeldt
Postdoc, UC Irvine

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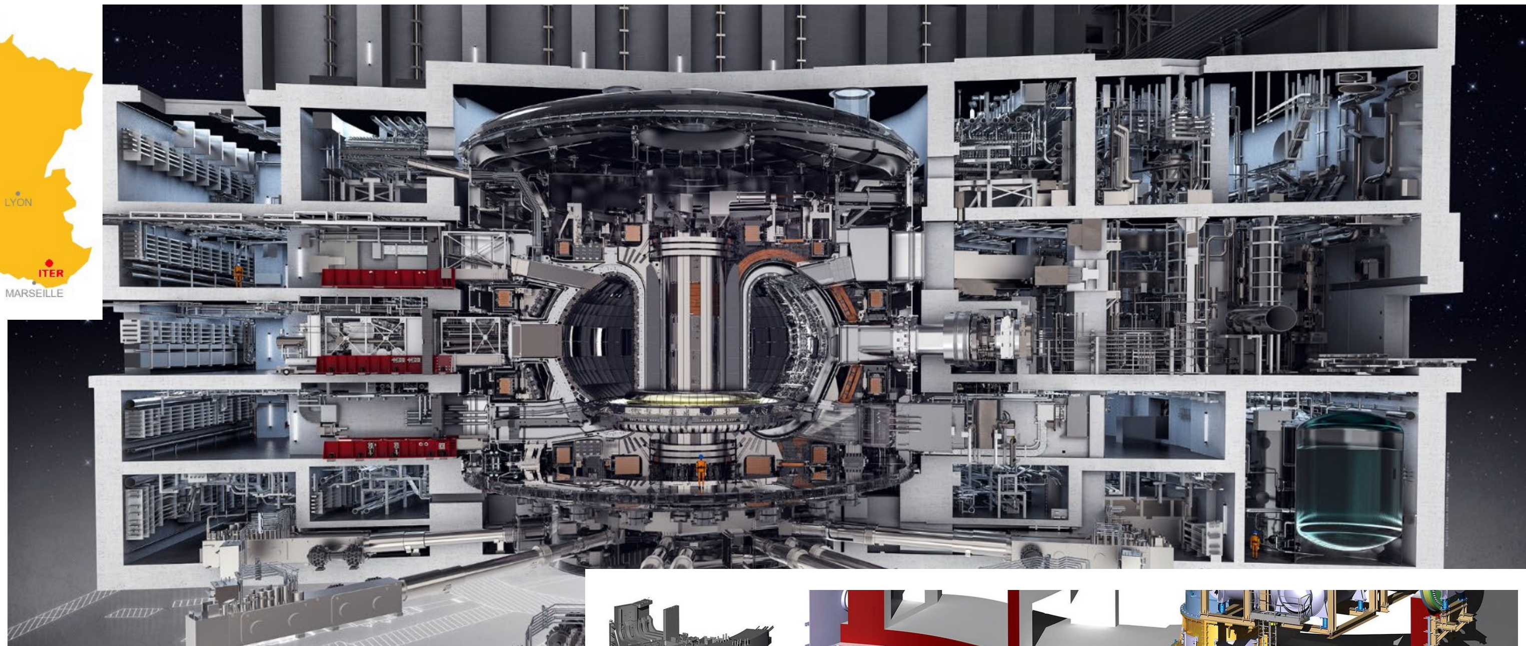
Please don't ask me about the physics!

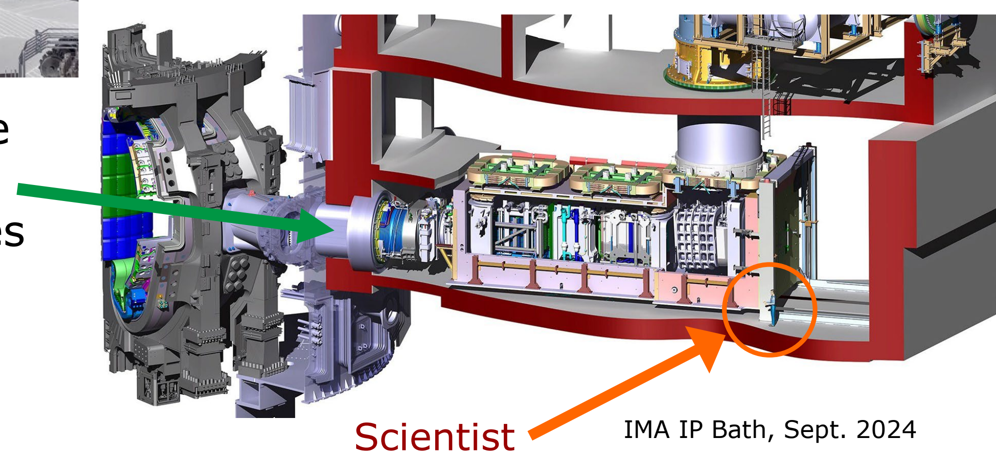
DTU Compute
Department of Applied Mathematics and Computer Science

The Tokamak Reactor

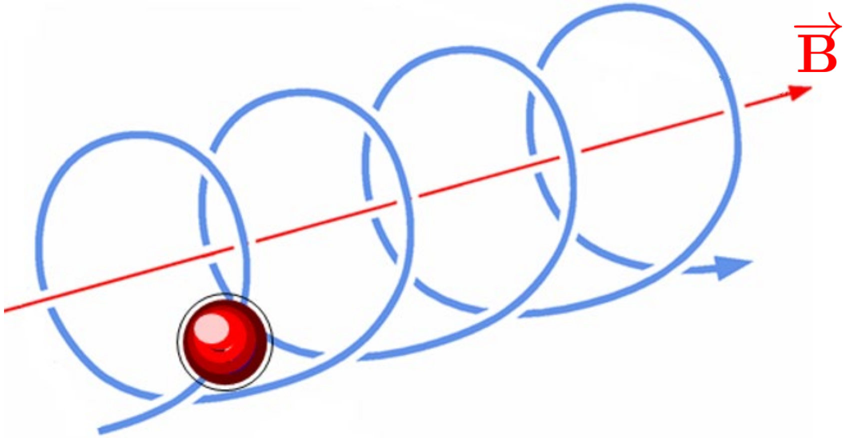
A **tokamak** uses a magnetic field to confine *plasma* in the shape of a torus.
Energy produced through *fusion* is absorbed as heat → *electricity* by turbines.

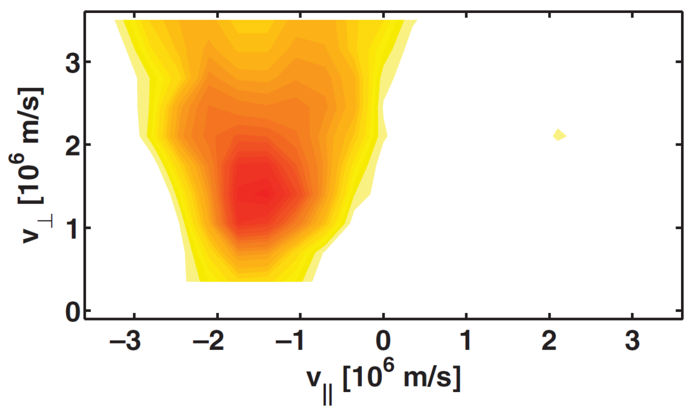


To reach fusion conditions the plasma is heated by *injecting* neutral particles (i.e., particles with no electric charge) into the plasma.



Distribution of Plasma Ion Velocities





Locally, the magnetic field is uniform and each *deuterium ion* ● in the plasma moves in a **helix** along the magnetic field **B**:

- $v_{||}$ denotes the velocity along the field.
- $v_\perp$ denotes the tangential speed (the radial speed is zero).

We want to know the probability distribution $f(v_{||}, v_\perp)$.

Towards the Inverse Problem

When fast plasma ions pass through the neutral beam, they undergo a reaction that creates a newly-born fast neutral and a photon.

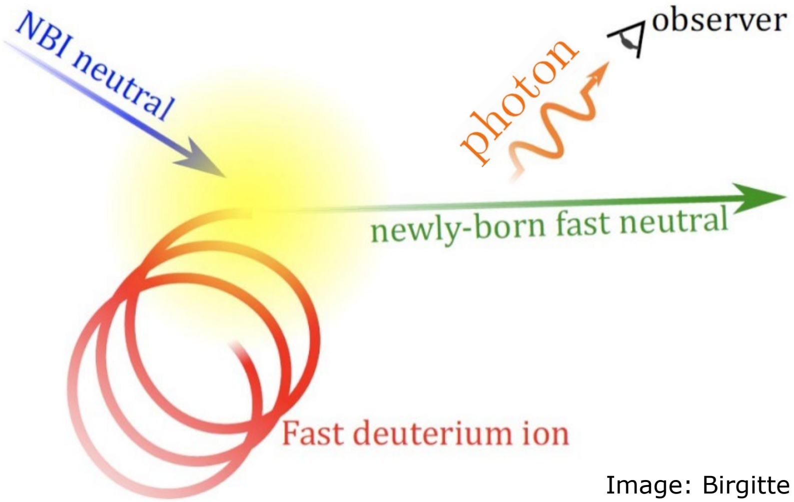


Image: Birgitte Madsen

The photons are *Doppler-shifted* depending on the velocity of the fast ion.

The inverse problem:
By measuring the Doppler-shift, we can infer about the *velocity of the fast ion*.

The Inverse Problem, Part I

On a spectral detector, we measure the *intensity per wavelength* $g(\lambda, \phi)$ of the photons, as a function of their wavelength λ.

This measurements also depend on the angle ϕ between the magnetic field $\mathbf{B}$ and the line-of-sight $\mathbf{u}$ to the photon detector.

The measurements are related to the velocity distribution $f(v_\|, v_\perp)$ via the integral

$$g(\lambda, \phi) = \int_0^\infty \int_{-\infty}^\infty k(\lambda, \phi; v_\|, v_\perp)\, f(v_\|, v_\perp)\, \mathrm{d}v_\| \, \mathrm{d}v_\perp \ ,$$

in which the kernel is given by

$$k(\lambda, \phi; v_\|, v_\perp) = R \cdot \pi(\lambda \,|\, \phi, v_\|, v_\perp)$$
$$\pi(\lambda \,|\, \phi, v_\|, v_\perp) = \text{probability density function for } \lambda$$

and R is a constant (for simplicity).

The Inverse Problem, Part II

Recall there is a Doppler shift, which we can write

$$\lambda - \lambda_{\mathrm{D}} = \frac{u\,\lambda_{\mathrm{D}}}{c} \qquad \Leftrightarrow \qquad u = c\,\frac{\lambda - \lambda_{\mathrm{D}}}{\lambda_{\mathrm{D}}}$$

where $\lambda_{\mathrm{D}} = 656.1\,\mathrm{nm}$, c = speed of light, and u = the ion's velocity component along the line-of-sight $\mathbf{u}$ to the detector. We now switch to the formulation

$$g(u,\phi) = \int_0^\infty \int_{-\infty}^\infty k(u,\phi;v_\parallel,v_\perp)\, f(v_\parallel,v_\perp)\,\mathrm{d}v_\parallel\,\mathrm{d}v_\perp \ ,$$

with

$$\boxed{k(u,\phi;v_\parallel,v_\perp) = \frac{R}{\pi\, v_\perp\ \sin\phi}\left(1-\left(\frac{u - v_\parallel\ \cos\phi}{v_\perp\ \sin\phi}\right)^2\right)^{-1/2} .}$$

When the argument to $(\ )^{-1/2}$ is negative, we set $k = 0$.

We use quadrature discretization to obtain a linear algebraic system $\boxed{A\,x = b}$.

Other Variables: *E* and *p*

Instead of working with the velocity distribution in the form $f(v_{\|}, v_{\perp})$, the physicists often prefer a *transformation of variables* to energy E and pitch p:

$$E = {}^1\!/_2\, m_{\mathrm{D}}\, v^2 \ , \qquad p = v_{\|}/v$$

$$v = \sqrt{v_{\|}^2 + v_{\perp}^2} \ , \qquad m_{\mathrm{D}} = \text{mass of deuterium ion}$$

and instead work with the distribution function $F(E, p)$:

$$I(u, \phi) = \int_{-1}^{1} \int_{0}^{\infty} K(u, \phi; E, p)\, F(E, p)\, \mathrm{d}E\, \mathrm{d}p \ .$$

From (E, p) to $(v_{\|}, v_{\perp})$:

$$v_{\perp} = p \sqrt{\frac{2E}{m_{\mathrm{D}}}} \ , \qquad v_{\perp} = \sqrt{\frac{2E}{m_{\mathrm{D}}}(1 - p^2)} \ .$$

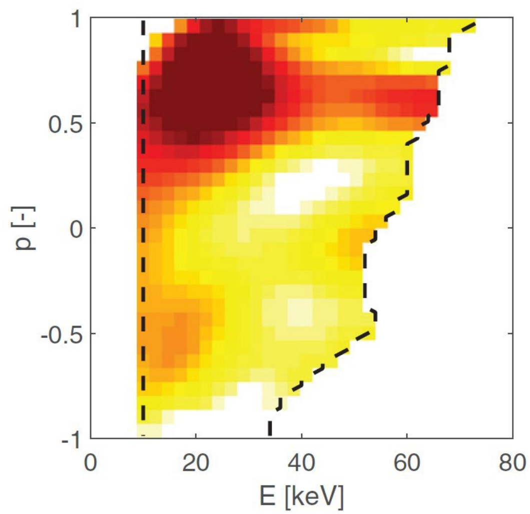
Utilizing Knowledge about the Physics

Tikhonov regularization in general form, with constraint:

$$\min_x \|A\,x - b\|_2 + \alpha^2 \|L\,x\|_2^2 \qquad \text{s.t.} \qquad x \geq 0\ ,$$

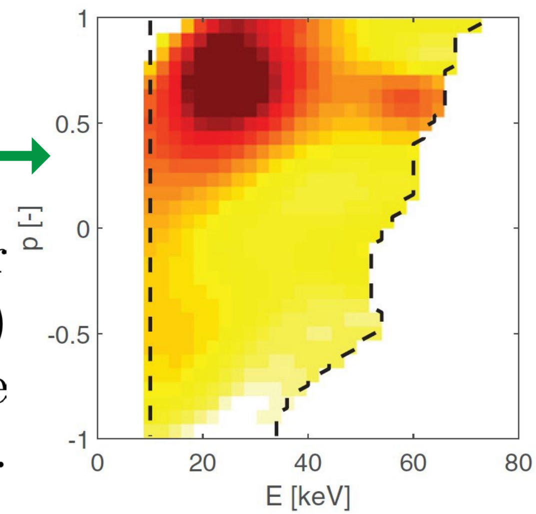
where $\alpha = \alpha(E, p)$ may depend on energy and pitch; Salewski et al. (2016).

Prior	Benefit	Risk
$x \geq 0$	Improves solution	$x < 0$ could diagnose data error
$L \sim 1.$ deriv	Gives smooth solutions	Misses spikes and ridges
$\alpha(E, p)$	Accounts for NBI peaks	Might introduce spurious peaks.



Constant α.

$\alpha = \alpha(E, p)$ with smaller values (less regularization) at known locations of the NBI peaks in (E, p)-space.



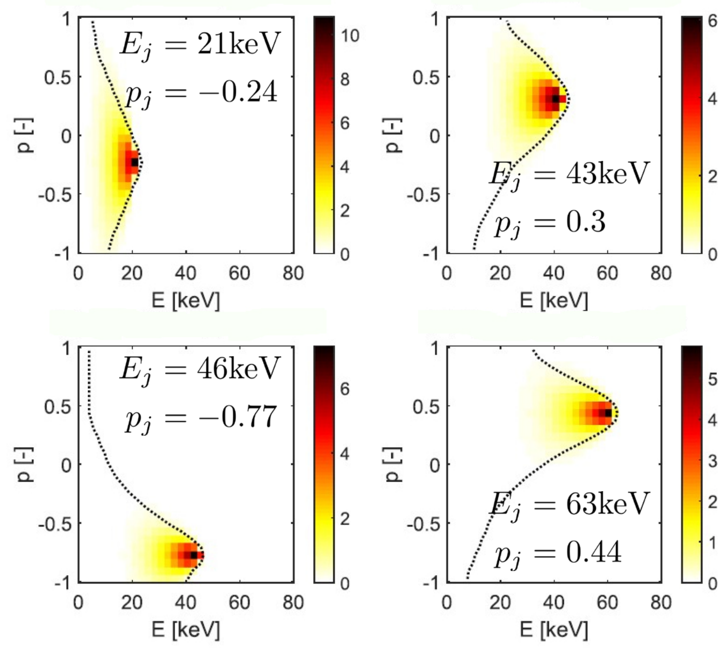
Physics Prior via Slowing-Down Functions

Instead of working with a standard "pixel basis" for $F(E,p)$, we can use suited expansion functions $\psi_1, \psi_2, \ldots, \psi_{n_{\rm sd}}$ and write the reconstruction

$$F(E,p) = \sum_{j=1}^{n_{\rm sd}} c_j \, \psi_j(E,p) \ .$$

The discretized and regularized problem for $\{c_j\}$ then takes the basic form

$$\min_c \|A\,\Psi\,c - b\|_2^2 + \alpha^2 \|c\|_2^2 \ , \qquad \Psi \in \mathbb{R}^{n \times n_{\rm sd}} \ , \qquad x = \Psi\,c \ ,$$



where the columns of Ψ are samples of the functions ψ_j = slowing down functions.

Each ψ_j is a distribution F excited by a δ-function $\delta(u_j, \phi_j)$ corresponding to (E_j, p_j), revealing how the plasma ions "slow down" due to collisions.

Hence, the basis functions ψ_j represent the physics of the plasma.

Interpretation of Basis with Physics Prior

The case $n_{\mathrm{sd}} = n \rightarrow$ as many slowing down functions as pixels (a basis)

Ψ is square and we assume it has full rank. Recall $x = \Psi\, c \;\Leftrightarrow\; c = \Psi^{-1} x$:

$$\min_c \|A\,\Psi\,c - b\|_2^2 + \alpha^2 \|c\|_2^2 \qquad \Leftrightarrow \qquad \min_x \|A\,x - b\|_2^2 + \alpha^2 \|\Psi^{-1} x\|_2^2 \ .$$

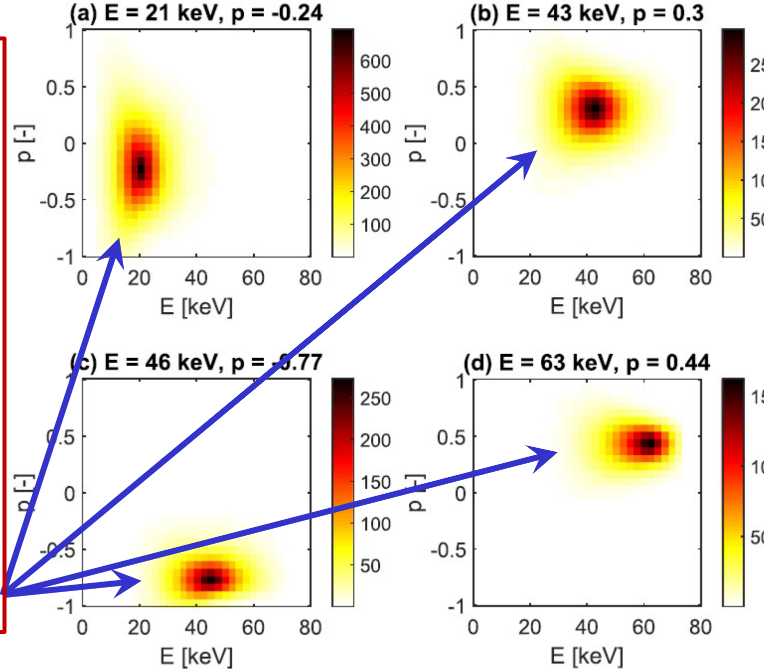
We can interpret the use of Ψ as a regularizer Ψ^{-1} for x in the reg. term.

In the Bayesian framework, the regularization term $\alpha^2 \|\Psi^{-1} x\|_2^2$ represents a Gaussian prior for x with covariance matrix

$$C_x = \left(\alpha^2 (\Psi^{-1})^T \Psi^{-1}\right)^{-1} = \alpha^{-2}\Psi\,\Psi^T \ .$$

This C_x ensures smoothness by correlating a given pixel to the pixels in its vicinity.

Rows of C_x reshaped to (E,p) domain



Interpretation, Part II

And now: the **standard-form transformation**

$$\min_x \|A\,x - b\|_2^2 + \alpha^2 \|L\,x\|_2^2 \qquad \Leftrightarrow \qquad \min_\xi \|(A\,L^{\#})\,\xi - b\|_2^2 + \alpha^2 \|\xi\|_2^2$$

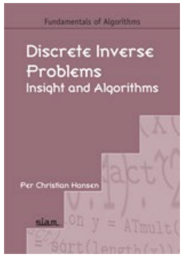
where $\xi = L\,x \;\Leftrightarrow\; x = L^{\#}\xi$, and $L^{\#}$ = some kind of inverse of L.

For the case on the previous slide ($n = n_{\mathrm{sd}}$) we immediately identify $L = \Psi^{-1}$.

The case $n_{\mathrm{sd}} > n$ → more exp. functions than pixels (overcomplete system)

The matrix Ψ = [box] is "obese" and we assume it has full rank. In this case

$$L^{\#} = L^{\dagger} = \text{pseudoinverse of } L, \qquad \text{cf. (H, 2010, §8.4 case 2).}$$



Hence we identify

$$L^{\dagger} = \Psi \qquad \text{and thus} \qquad L = (L^{\dagger})^{\dagger} = \Psi^{\dagger}$$

and

$$C_x = (\alpha^2 (\Psi^{\dagger})^T \Psi^{\dagger})^{-1} = \alpha^{-2} \Psi\,\Psi^T\,, \qquad \text{similar to before.}$$

Interpretation, Part III

The case $n_{\text{sd}} < n$ → fewer expansion functions than pixels

$\Psi = \Box$ is "tall & skinny" and we assume full row rank.

Write

$$\Psi = (\, Q \, , \, Q_0 \,) \begin{pmatrix} R \\ 0 \end{pmatrix} = Q \, R \, , \qquad \text{range}(Q_0) = \text{range}(\Psi)^{\perp} = \text{null}(\Psi^T) \ .$$

For a general x we must have:

$$x = \Psi \, c + Q_0 \, w \quad \Leftrightarrow \quad Q \, R \, c = x - Q_0 \, w \quad \Leftrightarrow \quad c = R^{-1} Q^T x = \Psi^{\dagger} x \ .$$

Let $P = Q_0 \, Q_0^T$ = orthogonal projector on $\text{null}(\Psi^T)$.

We want $x \in \text{range}(\Psi)$ and hence $Q_0 \, w = P \, x = 0$, and we arrive at

$$\min_x \| A \, x - b \|_2^2 + \alpha^2 \| \Psi^{\dagger} x \|_2^2 \qquad \text{s.t.} \qquad P \, x = 0 \ .$$

The constraint is always satisfied because $c = \Psi^{\dagger} x$. Thus, we get

$$C_x = (\alpha^2 (\Psi^{\dagger})^T \Psi^{\dagger})^{-1} = \alpha^{-2} \Psi \, \Psi^T \ , \quad \text{similar to before.}$$

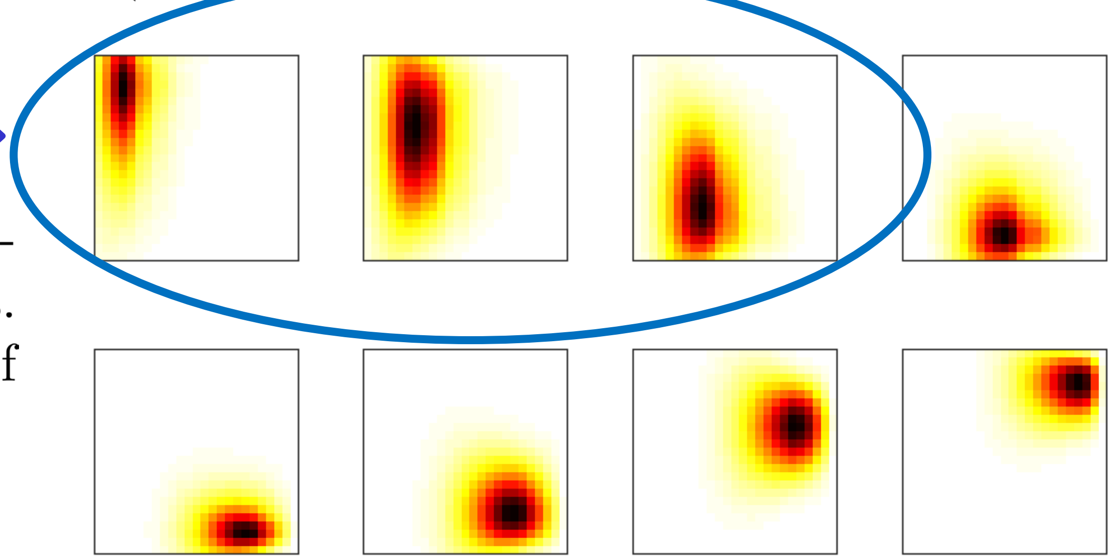
Ex: Covariance Matrix for a "Skinny" Basis

Recap: we want to *interpret the role of the slowing-down basis* Ψ via

- the regularization term $\alpha^2 \, \|\Psi^\dagger x\|_2^2$ ← Tikhonov approach
- the covariance matrix $C_x = \alpha^{-2} \Psi \, \Psi^T$ ← Bayesian approach

We continue with the case where $\Psi \in \mathbb{R}^{625 \times 170}$ is "tall & skinny," i.e., we have fewer basis functions ψ_j than the number of unknowns (pixels in the image).

We show **rows** of C_x reshaped to (E, p) domain; as before, they represent local averaging.



For lower energy E the correlation in pitch p increases. This confirms the intuition of the physicists.

Conclusion

- The slowing-down functions provide a set of basis vectors that represent the behavior of the ions in the plasma.
- We can interpret this as a regularization term in the Tikhonov formulation.
- Specifically, the use of these basis functions imposes local smoothing.
- The smoothing that we observe confirms the intuition of the physicists.
- Next step: deal with linearly dependent expansion functions.

References

- B. Madsen, J. Huang, M. Salewski, H. Järleblad, P. C. Hansen + 19, *Fast-ion velocity-space tomography using slowing-down regularization in EAST plasmas with co- and counter-current neutral beam injection*, Plasma Physics and Controlled Fusion, 62 (2020), 115019, doi 10.1088/1361-6587/abb79b.
- B. S. Schmidt, M. Salewski, D. Moseev, M. Baquero-Ruiz, P. C. Hansen + 16, *4D and 5D phase-space tomography using slowing-down physics regularization*, Nuclear Fusion, 63 (2023), 076016, doi 10.1088/1741-4326/acd6a6.