Cayley graphs, association schemes & state transfer

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DTU

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Abstract

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Theorem

Characterization of perfect state transfer in 2-circulants.

Outline

Cayley graphs

Definition

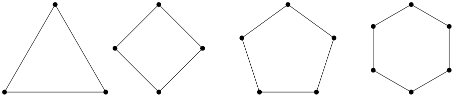
Let G be a group and $\mathcal{C} \subseteq G \backslash \{e\}$ a subset with $\mathcal{C}^{-1} = \mathcal{C}$. The *Cayley graph*, $X := \mathrm{Cay}(G, \mathcal{C})$, has vertex set $V(X) := G$, and

$$g \sim h \quad \text{if} \quad hg^{-1} \in \mathcal{C}.$$

The set $\mathcal{C}$ is called the *connection set* of the graph.

Examples

- Cycles



$$G = \mathbb{Z}_n,\ \mathcal{C} = \{\pm 1\}$$

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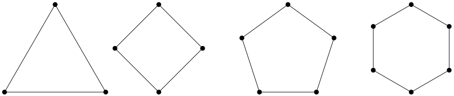

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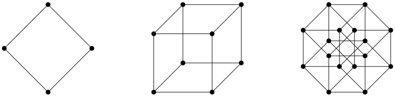

$$G = \mathbb{Z}_2^d,\ \mathcal{C} \text{ the standard basis}$$

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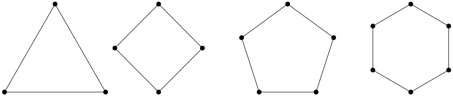

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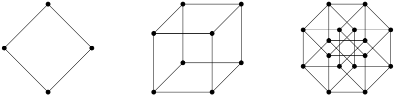

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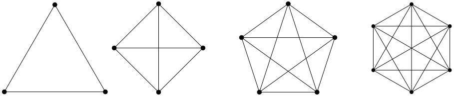
- K_n



$$G \text{ any group},\ \mathcal{C} = G \setminus e$$

Cayley graphs

Definition

We say that $\mathrm{Cay}(G, \mathcal{C})$ is *normal* if $g^{-1}\mathcal{C}g = \mathcal{C}$ for all $g \in G$. A Cayley graph for an abelian group is called a *translation graph*.

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Definition

A Cayley graph of $\mathbb{Z}_2^d$ is called a *cubelike graph* and a Cayley graph of a cyclic group is called a *circulant*.

Outline

Association schemes

Definition

An *association scheme* (with d classes) is a set of $n \times n$ matrices, $\mathcal{A} = \{A_0, \ldots, A_d\}$ with entries in $\{0, 1\}$ such that

- $A_0 = I$,
- $\sum_{r=0}^{d} A_r = J$,
- $A_r^T \in \mathcal{A}$ for all r,
- $A_r A_s = A_s A_r$ for all r, s, and
- $A_r A_s$ lies in the span of $\mathcal{A}$ for all r, s.

Association schemes

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- Any Schur idempotent can be viewed as the adjacency matrix of a (possibly directed) graph. These are the *graphs in the scheme*.

Outline

Two bases

The association scheme, $\mathcal{A} = \{A_0, \ldots, A_d\}$ is a basis for $\mathbb{C}[\mathcal{A}]$. There is another basis, $\mathcal{E} = \{E_0, \ldots, E_d\}$ of matrix idempotents satisfying

- $E_0 = \frac{1}{n}J$,
- $\sum_{r=0}^{d} E_r = I$,
- $E_r^T \in \mathcal{E}$ for all r,
- $E_r E_s = 0$ if $r \neq s$, and
- $E_r \circ E_s$ lies in the span of $\mathcal{A}$ for all r, s.

The matrices $E_0, \ldots, E_d$ are the *minimal matrix idempotents* of $\mathbb{C}[\mathcal{A}]$.

Eigenvalues of a scheme

Since $\mathcal{A}$ and $\mathcal{E}$ are a bases of $\mathbb{C}[\mathcal{A}]$, there are scalars $p_r(s)$ and $q_r(s)$ such that

$$A_r = \sum_{s=0}^{d} p_r(s) E_s \quad \text{and} \quad E_r = \frac{1}{n} \sum_{s=0}^{d} q_r(s) A_s.$$

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Since the E_r are pairwise orthogonal idempotents, this implies that

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We call the $p_r(s)$ the *eigenvalues of the scheme*, $\mathcal{A}$.

Matrix of eigenvalues

Define the matrices P and Q by

$$P_{sr} = p_r(s) \quad \text{and} \quad Q_{sr} = q_r(s).$$

We call P the *matrix of eigenvalues* of the scheme and Q the *matrix of dual eigenvalues*.

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If A is a Schur idempotent in $\mathbb{C}[\mathcal{A}]$, we can write

$$A = \sum_{r \in R} A_r$$

for some $R \subseteq \{0, \dots, d\}$. If x is the indicator vector for R, then the eigenvalues of A are the entries of Px.

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The conjugacy class scheme

Let G be a group of order n with conjugacy classes $C_0, \ldots, C_d$ (where $C_0 = \{e\}$).

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A normal Cayley graph of G is a graph in its conjugacy class scheme.

Translation schemes

If G is abelian, its conjugacy classes are singletons, so its conjugacy class scheme consists of permutation matrices.

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A graph in a translation scheme of G is a translation graph of G, and any translation graph of G lies in a translation scheme of G.

Duality

Let $\mathcal{A}$ be an association scheme with matrix of eigenvalues and dual eigenvalues P and Q, respectively. We say that $\mathcal{A}$ is *formally self-dual* if $\overline{Q} = P$.

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Theorem 1

The abelian group scheme for an arbitrary abelian group is formally self-dual.

Idea of proof. The matrix of eigenvalues of the abelian group scheme is the character table of the group. Recall that we always have $PQ = nI$. □

Outline

The integral translation scheme

Define a relation on an abelian group G as follows. We say that $g, h \in G$ are *power-equivalent* if $\langle g \rangle = \langle h \rangle$. This is an equivalence relation; let $D_0, \dots, D_k$ be its equivalence classes.

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$\mathcal{A}$ is a scheme called the *integral translation scheme* of G.

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If G is an abelian group of odd order, n, then any non-empty, integral Cayley graph for G has an odd eigenvalue.

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I'm pretty sure this holds for normal Cayley graphs.

Outline

Quantum walks

For a graph X with adjacency matrix A, and $t \in \mathbb{R}$, define the unitary matrix,

$$U(t) := e^{itA} = \sum_{n \geq 0} \frac{(it)^n}{n!} A^n.$$

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A *quantum walk* on X is given by the collection

$$\{U(t) : t \in \mathbb{R}\},$$

and we call $U(t)$ the *transition matrix* of the walk at time t.

Perfect state transfer

Definition

For distinct vertices, x and y of X, we say that we have *perfect state transfer (PST)* from x to y at time t if

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Theorem 5

A regular graph that admits perfect state transfer is integral.

Examples of PST

- K_2

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- Some other Cayley graphs

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Circulants & 2-circulants

Recall that a *circulant* is a Cayley graph of the cyclic group $\mathbb{Z}_n$.

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Recall that a *circulant* is a Cayley graph of the cyclic group $\mathbb{Z}_n$.
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Definition

We call a *2-circulant* a Cayley graph of an abelian group that has a cyclic Sylow-2-subgroup.

2-circulants

Let G be an abelian group of order $2^d m$ where m is odd. Assume G has a cyclic Sylow-2-subgroup. Then

$$G \cong \mathbb{Z}_{2^d} \times H$$

where $|H| = m$.

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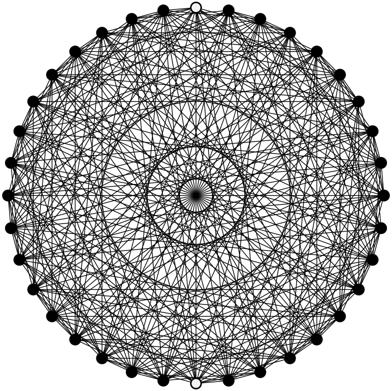
If $d \geq 1$, then G has a unique element, a of order two and if $d \geq 2$, it has a unique pair of inverse elements, $b, -b$ of order four.

Example

$G = \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3.$



Connection sets of 2-circulants

Let $\mathcal{C}$ be a power-closed subset of G. Define a partition, $\mathcal{C}_0, \ldots, \mathcal{C}_d$ of $\mathcal{C}$, by

$$\mathcal{C}_k := \{g \in \mathcal{C} : \operatorname{ord}(g) = 2^k m', m' \text{ odd}\}.$$

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Characterization

Theorem 6 (Árnadóttir & Godsil)

Let $X = \mathrm{Cay}(G, \mathcal{C})$ be a 2-circulant. Then X admits PST if and only if the following hold

1. *$\mathcal{C}$ is power-closed,*
2. *either a or b is in $\mathcal{C}$ but not both, and*
3. *$\mathcal{C}_0 = \mathcal{C}_1^* \setminus \{0\} = \mathcal{C}_2^* \setminus \{0\}$.* □

Proof idea

PST $\implies$ 3)

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- Define $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$ where

$$\mathcal{S}_1 := \mathcal{C}_0 \setminus (\mathcal{C}_1^* \setminus \{0\}) \text{ and } \mathcal{S}_2 := (\mathcal{C}_1^* \setminus \{0\}) \setminus \mathcal{C}_0.$$

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- $Y := \mathrm{Cay}(H, \mathcal{S})$ is an integral Cayley graph of odd order.
- We can show that Y has only even eigenvalues.

Proof idea

PST $\implies$ 3)

Idea of proof.

- Define $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$ where

$$\mathcal{S}_1 := \mathcal{C}_0 \setminus (\mathcal{C}_1^* \setminus \{0\}) \text{ and } \mathcal{S}_2 := (\mathcal{C}_1^* \setminus \{0\}) \setminus \mathcal{C}_0.$$

- $Y := \mathrm{Cay}(H, \mathcal{S})$ is an integral Cayley graph of odd order.
- We can show that Y has only even eigenvalues.
- Therefore $\mathcal{S}$ is empty, so $\mathcal{C}_0 = \mathcal{C}_1^* \setminus \{0\}$.

Future directions

How much of this can we generalize to normal Cayley graphs?

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How much of this can we generalize to normal Cayley graphs?

- At least some!

Future directions

How much of this can we generalize to normal Cayley graphs?

- At least some!
- Perhaps all?

Thank you