Cubic vertex-transitive graphs with infinite vertex stabilizers

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Number theory & Combinatorics seminar

This talk is based on joint work with Waltraud Lederle and Rögnvaldur Möller:
https://arxiv.org/abs/2101.04064.

Outline

Examples

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- The 3-regular tree.

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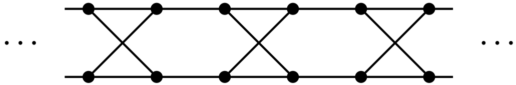
- This graph



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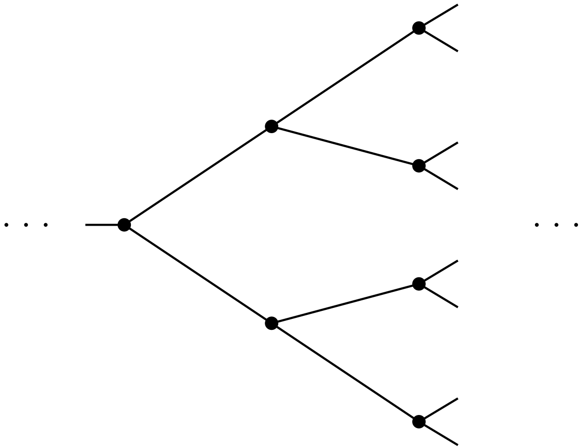

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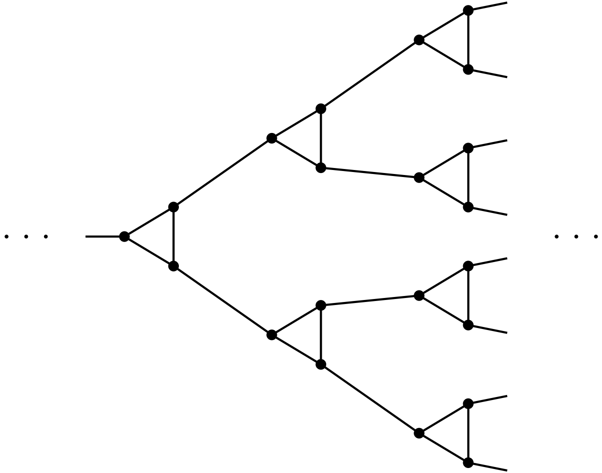

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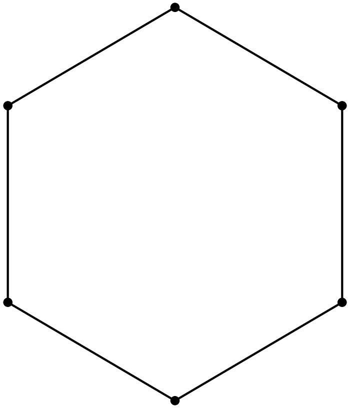

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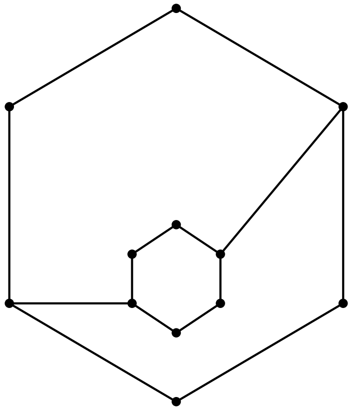

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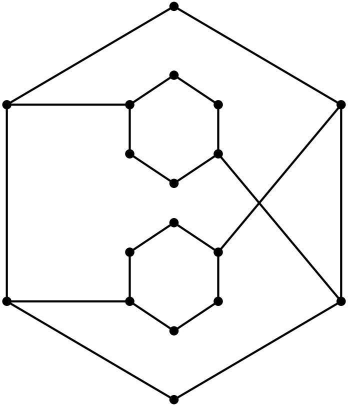

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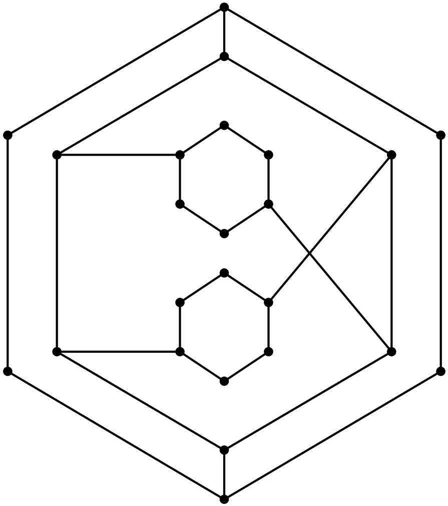

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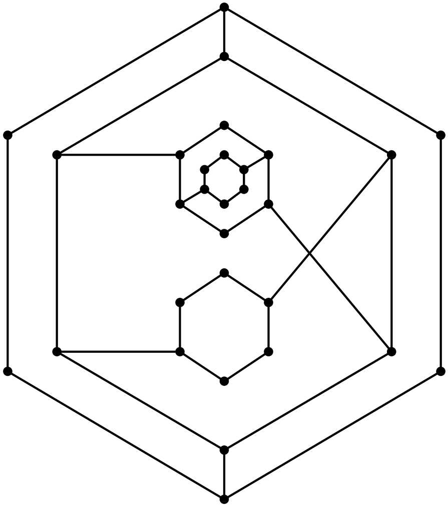

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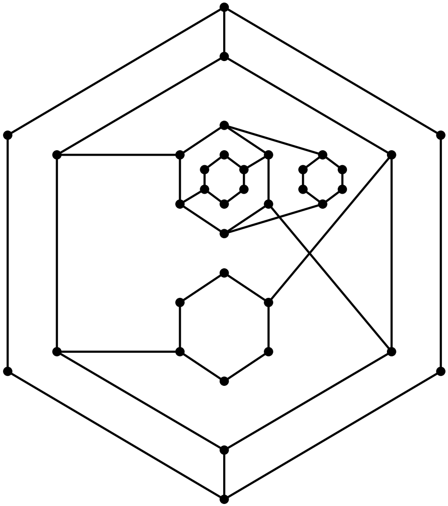

Outline

Graphs & digraphs

Definition

A *directed graph (digraph)* is an ordered pair of sets, $\Gamma = (V, A)$, where the elements of A, called *arcs*, are ordered pairs of distinct elements of V, called *vertices*. A *graph* is a digraph satisfying that if (α, β) is an arc, (β, α) is also an arc. In this case, we call the set $\{\alpha, \beta\}$ an *edge* and we think of the graph as a pair (V, E) where E is the set of edges. An *oriented graph* is a digraph satisfying that if (α, β) is an arc then (β, α) is not an arc.

Regular graphs

Definition

Let $\Gamma = (V, A)$ be a digraph. For a vertex, $\alpha \in V$, the sets of *in-neighbours* and *out-neighbours* are defined by

$$\mathrm{in}(\alpha) := \{\beta \in V : (\beta, \alpha) \in A\}, \quad \mathrm{out}(\alpha) := \{\beta \in V : (\alpha, \beta) \in A\}$$

respectively. Their cardinalities are called the *in-* and *out-degree* of α, respectively. If Γ is a graph, these sets are equal and we call it the set of *neighbours*, denoted $N(\alpha)$, and its cardinality the *degree* of α. If all vertices in a graph have the same degree, k, we call it *regular* (or k*-regular*), and refer to k as the degree of the graph. A 3-regular graph is also called *cubic*.

Actions

Definition

A digraph $\Gamma = (V, A)$ is said to be *vertex-transitive (respectively arc-transitive)* if its automorphism group, $\mathrm{Aut}(\Gamma)$, acts transitively on the set of vertices (respectively arcs). A graph is *edge-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on the set of edges.

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Definition

If $G \leq \mathrm{Aut}(\Gamma)$ and $\alpha \in V$, we denote by G_α, the set of elements in G that fix α and call it the *vertex-stabilizer* of α in G.

Actions

Definition

Let G be a group acting vertex-transitively on a graph Γ with degree k. The stabilizer in G of a vertex α acts on the set of neighbours of α like a subgroup of S_k, and we call this the *local action* of G on Γ.

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Let Γ be a connected, cubic, vertex transitive graph, let α be a vertex and $N(\alpha) = \{\beta, \gamma, \delta\}$ its neighbourhood.

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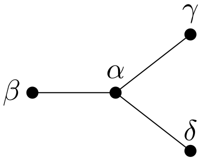

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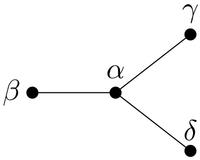

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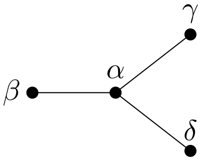

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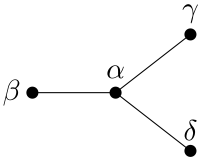
- Our options are $\{e\}, \mathbb{Z}_2, \mathbb{Z}_3, S_3$.



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- We are left with $\mathbb{Z}_2$ and S_3.

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In this case our graph is arc-transitive.

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In this case our graph is arc-transitive.

Theorem 1 (Tutte, 1959)

If Γ is a vertex- and arc-transitive cubic graph with infinite vertex-stabilizers, then Γ is the cubic tree.

The local action is $\mathbb{Z}_2$

Suppose that the local action is $\mathbb{Z}_2$.

In this case, the graph is not arc-transitive. Can it still be edge-transitive?

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Theorem 2 (Árnadóttir, Lederle, Möller)

Let Γ be a connected cubic graph and suppose $G \leq \mathrm{Aut}(\Gamma)$ acts vertex- and edge-transitively, but not arc-transitively on Γ such that the vertex-stabilizers of G are infinite. Then Γ is the tree.

Theorem 3 (Árnadóttir, Lederle, Möller)

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Proof.

G has exactly two orbits on the arcs. Define a digraph, Γ_+ with the same vertex set as Γ and one of the arc orbits as arcs. Note that Γ_+ is arc-transitive. We can choose the arc orbit so that every vertex has in-degree one and out-degree two.

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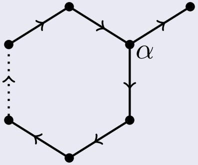

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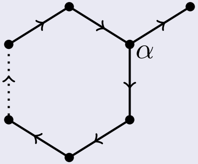

... impossible. □

Two edge-orbits

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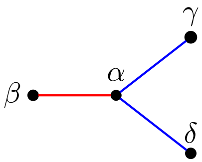

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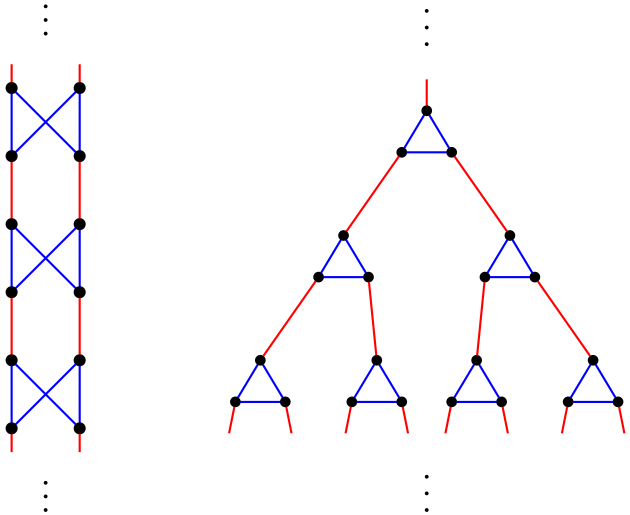

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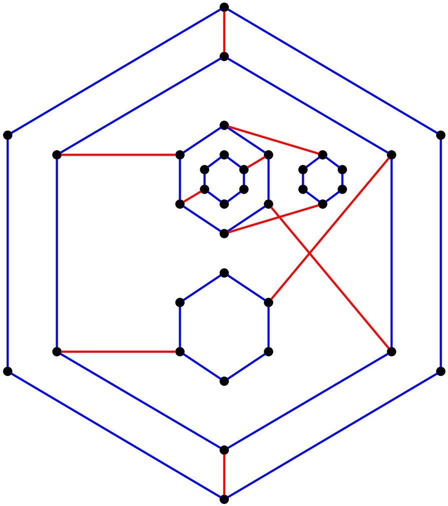

Can we characterize such graphs?

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No.

Outline

Rays and ends

Definition

A *ray* in a graph Γ is a subgraph with vertex set $\{\alpha_0, \alpha_1, \alpha_2, \dots\}$ (where the α_i are distinct), and edge set $\{\alpha_i\alpha_{i+1} : i = 0, 1, 2, \dots\}$.

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Definition

An *end* in a graph Γ is an equivalence class of rays, where two rays are equivalent if there is a third ray that intersects both in infinitely many vertices.

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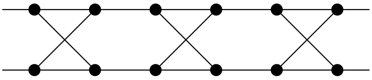

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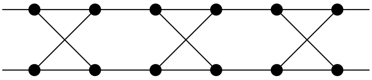

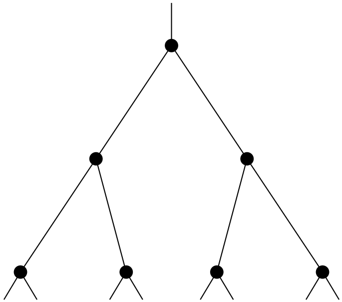

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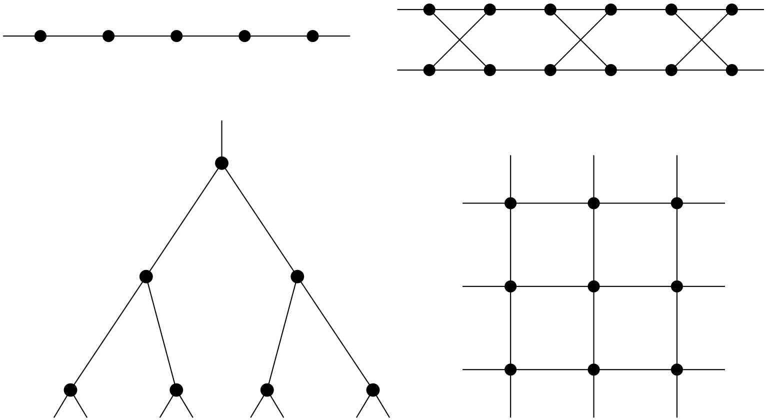


Number of ends

Theorem 4

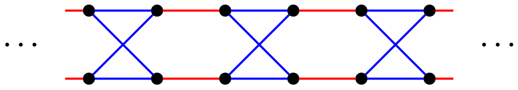
A vertex-transitive graph with finitely many ends has at most two ends.

We will now focus on graphs with two ends.

Outline

Digraphs with in-and out-degree two

We start with this graph again.



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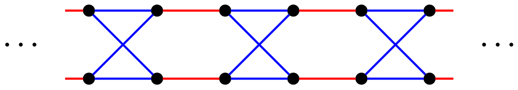

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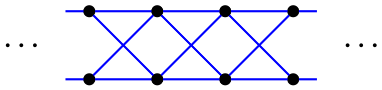

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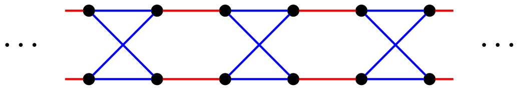

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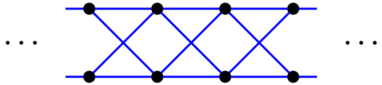
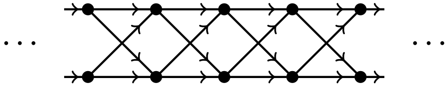
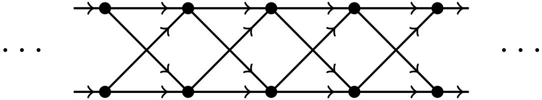


Then orient everything to the right



Δ_2

We call this digraph Δ_2. It is vertex-transitive with infinite vertex-stabilizers and in- and out-degree two. It also has two ends, and is *highly arc-transitive*.



Arcs and s-arcs

Definition

Let $s \geq 0$ be an integer. An *s-arc* in a digraph Γ is an $(s+1)$-tuple, $(\alpha_0, \ldots, \alpha_s)$ of vertices of Γ such that (α_{i-1}, α_i) is an arc in Γ for $i = 1, \ldots, s$ and $\alpha_{i-1} \neq \alpha_{i+1}$.

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Definition

We say that a digraph Γ is *s-arc-transitive* if its automorphism group acts transitively on the set of s-arcs. If Γ is s-arc-transitive for all $s \geq 0$, we call it *highly arc-transitive*.

Arc-digraphs & s-arc-digraph

Definition

Let Γ be a digraph. We define its *arc-digraph*, $\mathrm{Arc}(\Gamma)$, as having the arcs of Γ as its vertices and $((\alpha, \beta), (\gamma, \delta))$ is an arc if and only if $\beta = \gamma$.

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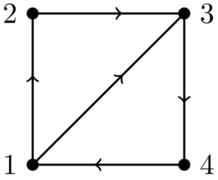

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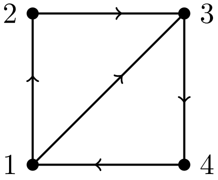

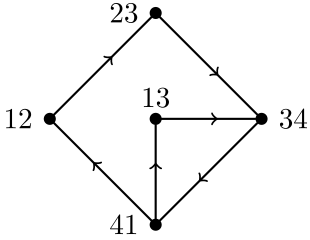

Arc-digraphs & s-arc-digraph

Definition

Let Γ be a digraph and s a non-negative integer. Define the *s-arc-digraph*, $\mathrm{Arc}^s(\Gamma)$, of Γ inductively:

- $\mathrm{Arc}^1(\Gamma) := \mathrm{Arc}(\Gamma)$, and
- $\mathrm{Arc}^s(\Gamma) := \mathrm{Arc}(\mathrm{Arc}^{s-1}(\Gamma))$.

The s-arc-digraphs of Δ_2

It is not too hard to see that the s-arc-digraphs of Δ_2 have in- and out-degree two, are highly arc-transitive and have two ends.

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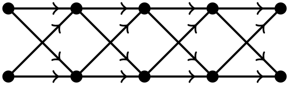

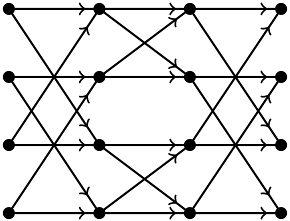

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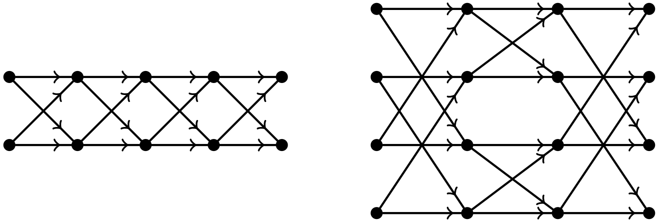

In fact, the converse holds.

Δ_2 and its s-arc-digraphs

Theorem 5 (Möller, Potočnik, Seifter, 2018)

Let Γ be a highly arc-transitive digraph with in- and out-degree two. If Γ has two ends, then it is isomorphic to $\mathrm{Arc}^s(\Delta_2)$ for some $s \geq 0$.

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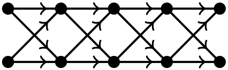
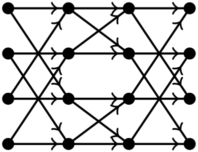
 ,  , . . .

From 4-regular to 3-regular

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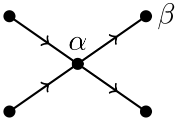

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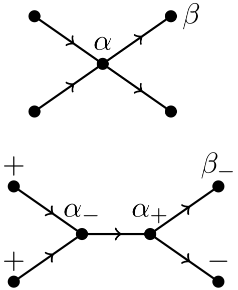

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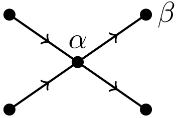

- Let the arcs of Δ_* be
  - (α_-, α_+)
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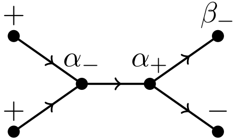
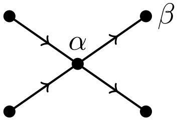


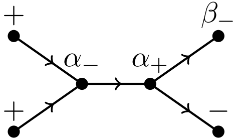
From 4-regular to 3-regular

Let Δ be a vertex-transitive digraph with in- and out degree two and infinite vertex stabilizers. We define the digraph Δ_* as follows.

- Replace each vertex, α in Δ by a pair of vertices, α_- and α_+.



- Let the arcs of Δ_* be
  - (α_-, α_+) for every vertex α of Δ, and
  - (α_+, β_-) for every arc (α, β) of Δ.



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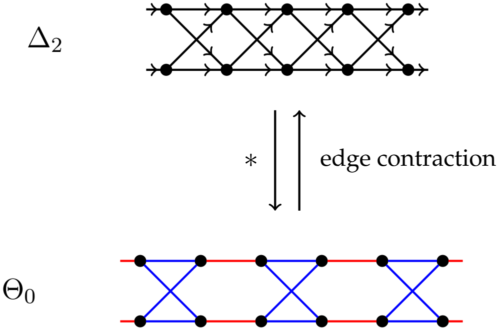
The graphs $\mathrm{Arc}^s(\Delta_2)$ satisfy these conditions.

Definition

For $s \geq 0$, denote by Θ_s the underlying undirected graph of $(\mathrm{Arc}^s(\Delta_2))_*$.

We can go in both directions



Outline

Main theorem

Theorem 6 (Árnadóttir, Lederle, Möller)

Let Γ be a connected, cubic, vertex-transitive graph with infinite vertex stabilizers. If Γ has two ends, then it is isomorphic to Θ_s for some $s \geq 0$.

Alternating s-arcs

Definition

Let Γ be a cubic, vertex-transitive graph with infinite vertex stabilizers and two orbits on its edges. Colour its edges as before. An s-arc $(\alpha_0, \ldots, \alpha_s)$ is called *alternating* if the edges $\{\alpha_{i-1}, \alpha_i\}$ and $\{\alpha_i, \alpha_{i+1}\}$ have different colours for all $i = 1, \ldots, s-1$.

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4. Construct a digraph, replacing the edges of Γ with the arcs of one of the blue arc-orbits and one of the red arc-orbits.
5. Contract the red arcs to get a digraph with in- and out-degree two and two ends.
6. By (2), this digraph is highly arc-transitive, and so it is isomorphic to $\mathrm{Arc}^s(\Delta_2)$ for some $s \geq 0$.

Conclusion

Therefore $\Gamma \simeq (\mathrm{Arc}^s(\Delta_2))_* = \Theta_s$ for some s.

Connections topological groups

We can use this to show the following.

Theorem 7 (Árnadóttir, Lederle, Möller)

Let G be a compactly generated, totally disconnected, locally compact group that acts transitively on the vertices of a cubic graph, such that the vertex stabilizers are compact open subgroups. If every $g \in G$ normalizes a compact open subgroup of G then G has a compact, open, normal subgroup.

Thank you