Association schemes and spectra of normal Cayley graphs

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25th Conference of the International Linear Algebra Society

June, 2023

Outline

Cayley graphs

Definition

Let G be a group and $\mathcal{C} \subseteq G \backslash \{e\}$ a subset with $\mathcal{C}^{-1} = \mathcal{C}$. The *Cayley graph*, $X := \mathrm{Cay}(G, \mathcal{C})$, has vertex set $V(X) := G$, and

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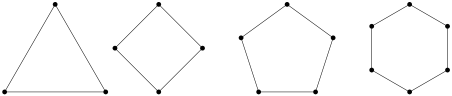
We say that $\mathrm{Cay}(G, \mathcal{C})$ is *normal* if $g^{-1}\mathcal{C}g = \mathcal{C}$ for all $g \in G$.

Examples

- Cycles



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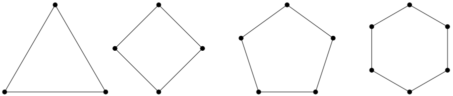

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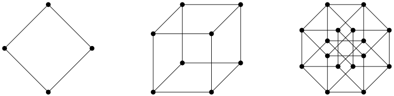

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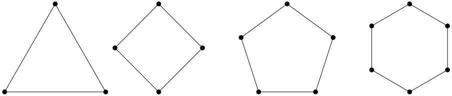

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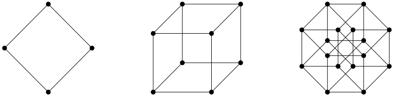

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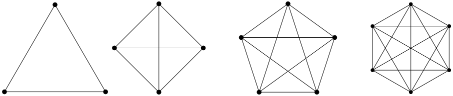

$$G \textit{ any group}, \mathcal{C} = G \setminus e$$

Outline

Association schemes

Definition

An *association scheme* (with d classes) is a set of $n \times n$ matrices, $\mathcal{A} = \{A_0, \ldots, A_d\}$ with entries in $\{0, 1\}$ such that

- $A_0 = I$,
- $\sum_{r=0}^{d} A_r = J$,
- $A_r^T \in \mathcal{A}$ for all r,
- $A_r A_s = A_s A_r$ for all r, s, and
- $A_r A_s$ lies in the span of $\mathcal{A}$ for all r, s.

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- Any Schur idempotent can be viewed as the adjacency matrix of a (possibly directed) graph. These are the *graphs in the scheme*.

Examples

Let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

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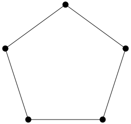

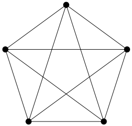

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Two bases

The association scheme, $\mathcal{A} = \{A_0, \ldots, A_d\}$ is a basis for $\mathbb{C}[\mathcal{A}]$.

There is another basis, $\mathcal{E} = \{E_0, \ldots, E_d\}$ of matrix idempotents satisfying

- $E_0 = \frac{1}{n} J$,
- $\sum_{r=0}^{d} E_r = I$,
- $E_r^T \in \mathcal{E}$ for all r,
- $E_r E_s = 0$ if $r \neq s$, and
- $E_r \circ E_s$ lies in the span of $\mathcal{A}$ for all r, s.

The matrices $E_0, \ldots, E_d$ are the *minimal matrix idempotents* of $\mathbb{C}[\mathcal{A}]$.

Eigenvalues of a scheme

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We call the $p_r(s)$ the *eigenvalues of the scheme*, $\mathcal{A}$ and define the *matrix of eigenvalues* by $P = (p_r(s))_{s,r}$.

Eigenvalues of graphs in a scheme

Observation

If X is a graph in a scheme with matrix of eigenvalues P, then there is a 01-vector x such that the eigenvalues of X are the entries of Px.

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Idea of proof. The adjacency matrix of X can be written $\sum_{r \in R} A_r$ where $R \subseteq \{1, \dots, d\}$. Note that

$$\begin{aligned}(A_r + A_s)E_j &= A_r E_j + A_s E_j \\ &= p_r(j)E_j + p_s(j)E_j \\ &= (p_r(j) + p_s(j))E_j\end{aligned}$$

so $p_r(j) + p_s(j)$ is an eigenvalue of $A_r + A_s$. □

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The conjugacy class scheme

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A normal Cayley graph of G is a graph in its conjugacy class scheme.

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Integral graphs

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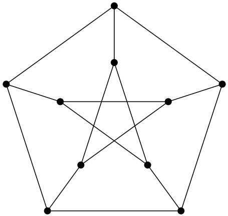
Example:



This graph

Spectrum: $\{3^{(1)}, 1^{(5)}, -2^{(4)}\}$.

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Theorem 1 (Bridges & Mena, 1981)

A normal Cayley graph of G is integral if and only if it lies in the integral conjugacy class scheme of G.

Outline

Integral normal Cayley graphs

Theorem 2 (Árnadóttir & Godsil, 2023++)

If G is a group of odd order then any non-empty, integral, normal Cayley graph for G has an odd eigenvalue.

Example

Let $G = H \times \mathbb{Z}_3$ where H is the unique non-abelian group of order 21. It has seven power-conjugacy classes, $C_0, C_1, \ldots, C_6$ of size $(1, 2, 6, 12, 14, 14, 14)$.

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Let $\mathcal{C} := C_1 \cup C_6$. Then $X := \mathrm{Cay}(G, \mathcal{C})$ is a connected, normal Cayley graph for a group of odd order and its spectrum is

$$\{16^{(1)},\ 13^{(2)},\ 2^{(18)},\ -1^{(36)},\ -5^{(2)},\ -8^{(4)}\}.$$

Thank you (and a picture of a graph)