Normal Cayley graphs, association schemes and spectra

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29th Nordic Congress of Mathematicians

July, 2023

Outline

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Spectra of graphs

For a graph X on n vertices, define its *adjacency matrix*, $A := A(X)$ by

$$A_{uv} = \begin{cases} 1 & \text{if } u \sim v, \\ 0 & \text{otherwise}, \end{cases}$$

where $u, v \in V(X)$.

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Definition

We define the *eigenvalues / eigenvectors of X* as the eigenvalues / eigenvectors of A.

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- Expander graphs
- Quantum walks

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Cayley graphs

Definition

Let G be a group and $\mathcal{C} \subseteq G \backslash \{e\}$ a subset with $\mathcal{C}^{-1} = \mathcal{C}$. The *Cayley graph*, $X := \mathrm{Cay}(G, \mathcal{C})$, has vertex set $V(X) := G$, and

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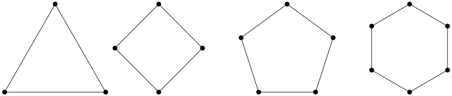
We say that $\mathrm{Cay}(G, \mathcal{C})$ is *normal* if $g^{-1}\mathcal{C}g = \mathcal{C}$ for all $g \in G$.

Examples

- Cycles



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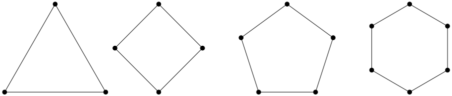

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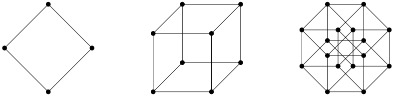

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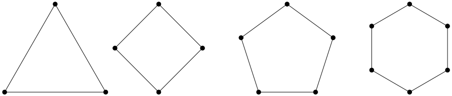

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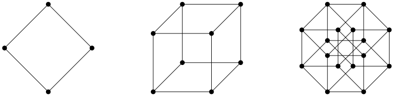

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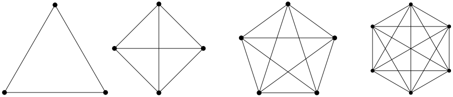
- K_n



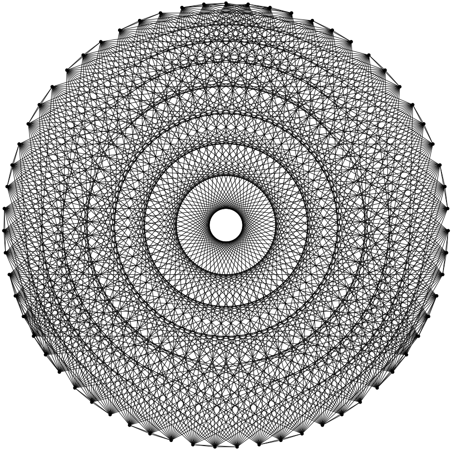
G *any group*, $\mathcal{C} = G \setminus e$

Theorem

Theorem (Árnadóttir & Godsil, 2023++)

If G is a group of odd order then any non-empty, normal Cayley graph for G with only integer eigenvalues has an odd eigenvalue.

Proof by example



Spectrum: $\{16^{(1)}, 13^{(2)}, 2^{(18)}, -1^{(36)}, -5^{(2)}, -8^{(4)}\}$.

Outline

Association schemes

Definition

An *association scheme* (with d classes) is a set of $n \times n$ matrices, $\mathcal{A} = \{A_0, \ldots, A_d\}$ with entries in $\{0, 1\}$ such that

- $A_0 = I$,
- $\sum_{r=0}^{d} A_r = J$,
- $A_r^T \in \mathcal{A}$ for all r,
- $A_r A_s = A_s A_r$ for all r, s, and
- $A_r A_s$ lies in the span of $\mathcal{A}$ for all r, s.

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- Any Hadamard idempotent can be viewed as the adjacency matrix of a (possibly directed) graph. These are the *graphs in the scheme*.

Example

Let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

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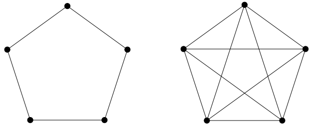

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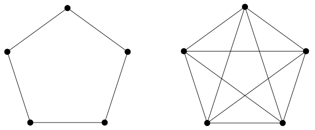

The only proper subscheme is $\mathcal{B} = \{I, A_1 + A_2\}$.

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There is another basis, $\mathcal{E} = \{E_0, \ldots, E_d\}$ of matrix idempotents satisfying

- $E_0 = \frac{1}{n}J$,
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The matrices $E_0, \ldots, E_d$ are the *minimal matrix idempotents* of $\mathbb{C}[\mathcal{A}]$.

Eigenvalues of a scheme

Since $\mathcal{E}$ is a basis of $\mathbb{C}[\mathcal{A}]$, there are scalars $p_r(s)$ such that

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We call the $p_r(s)$ the *eigenvalues of the scheme*, $\mathcal{A}$ and define the *matrix of eigenvalues* by $P = (p_r(s))_{s,r}$.

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Let v_r be the row sum of A_r and m_r be the rank of E_r.

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Fact 2

If $\mathcal{A}$ is an association scheme with matrix of eigenvalues $P_{\mathcal{A}}$ and $\mathcal{B}$ is a subscheme with matrix of eigenvalues $P_{\mathcal{B}}$, then $\det(P_{\mathcal{B}}) \mid \det(P_{\mathcal{A}})$.

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$$\begin{aligned}(A_r + A_s)E_j &= A_r E_j + A_s E_j \\ &= p_r(j)E_j + p_s(j)E_j \\ &= (p_r(j) + p_s(j))E_j\end{aligned}$$

so $p_r(j) + p_s(j)$ is an eigenvalue of $A_r + A_s$. □

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A *normal Cayley graph* of G is a graph in its conjugacy class scheme.

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This is the *integral conjugacy class scheme* of G.

The integral conjugacy class scheme

Theorem 1 (Bridges & Mena, 1981)

A normal Cayley graph of G is integral if and only if it lies in the integral conjugacy class scheme of G.

Integral normal Cayley graphs

Theorem 2 (Árnadóttir & Godsil, 2023++)

If G is a group of odd order then any non-empty, integral, normal Cayley graph for G has an odd eigenvalue.

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- We show that $\det(P_{\mathcal{A}}^* P_{\mathcal{A}})$ is an odd integer, using

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  and therefore $\det(P_{\mathcal{B}})$ is odd.
- The entries of $P_{\mathcal{B}} x$ cannot all be even since $P_{\mathcal{B}}$ is invertible mod 2.

□

Thank you