Spectra of normal Cayley graphs

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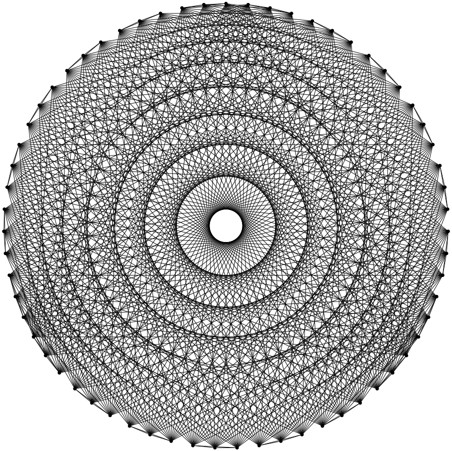
Abstract

Theorem (Árnadóttir & Godsil, 2023++)

If G is a group of odd order, then any non-empty, normal Cayley graph for G with integer eigenvalues has an odd eigenvalue.

Proof by example



Spectrum: $\{16^{(1)}, 13^{(2)}, 2^{(18)}, -1^{(36)}, -5^{(2)}, -8^{(4)}\}$.

Outline

Cayley graphs

Definition

Let G be a group and $\mathcal{C} \subseteq G \backslash \{e\}$ a subset with $\mathcal{C}^{-1} = \mathcal{C}$. The *Cayley graph*, $X := \mathrm{Cay}(G, \mathcal{C})$, has vertex set $V(X) := G$, and

$$g \sim h \quad \text{if} \quad hg^{-1} \in \mathcal{C}.$$

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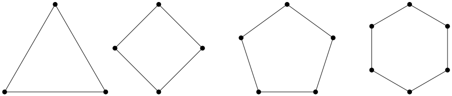
We say that $\mathrm{Cay}(G, \mathcal{C})$ is *normal* if $g^{-1}\mathcal{C}g = \mathcal{C}$ for all $g \in G$.

Examples

- Cycles



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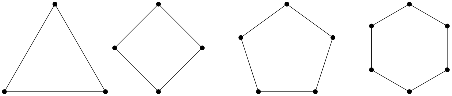

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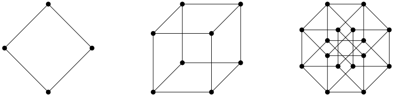

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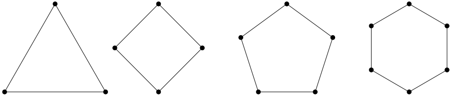

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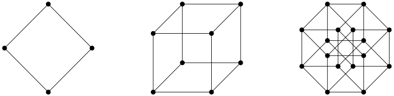

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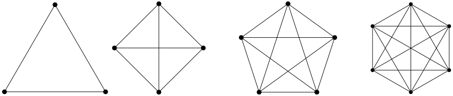

$$G \textit{ any group}, \mathcal{C} = G \setminus e$$

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Association schemes

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- Any Schur idempotent can be viewed as the adjacency matrix of a (possibly directed) graph. These are the *graphs in the scheme*.

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We call the $p_r(s)$ the *eigenvalues of the scheme*, $\mathcal{A}$ and define the *matrix of eigenvalues* by $P = (p_r(s))_{s,r}$.

Some basic properties of P

Recall that v_r is the row sum of A_r. We also define the *multiplicities*, $m_0, \ldots, m_d$ of the scheme by letting m_r be the rank of E_r.

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Fact 3

If X is a graph in a scheme with matrix of eigenvalues P, then there is a 01-vector x such that the eigenvalues of X are the entries of Px.

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The conjugacy class scheme

Let G be a group of order n with conjugacy classes $C_0, \ldots, C_d$ (where $C_0 = \{e\}$).

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The integral conjugacy class scheme

Theorem 1 (Bridges & Mena, 1981)

A normal Cayley graph of G is integral if and only if it lies in the integral conjugacy class scheme of G.

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Integral normal Cayley graphs

Theorem 2 (Árnadóttir & Godsil, 2023++)

If G is a group of odd order then any non-empty, integral, normal Cayley graph for G has an odd eigenvalue.

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Thank you