Strongly cospectral vertices, Cayley graphs and other things

Arnbjörg Soffía Árnadóttir

Faculty of Mathematics
University of Waterloo

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Outline

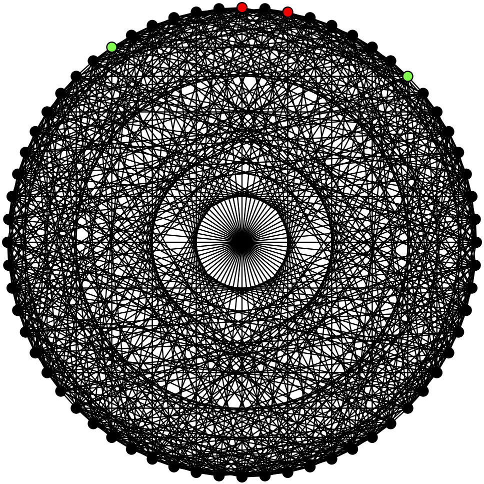

Figure: *This is a picture of a graph*

Strongly cospectral vertices

Definition

Let X be a graph with adjacency matrix A. We say that vertices $u, v \in V(X)$ are *strongly cospectral* if for each idempotent, E_r in the spectral decomposition,

$$A = \sum_r \theta_r E_r$$

we have $E_r e_u = \pm E_r e_v$.

Cayley graphs

Definition

Let G be a group and $\mathcal{C} \subseteq G \backslash e$ an inverse-closed subset. The *Cayley graph*, $X(G, \mathcal{C})$, of G with respect to $\mathcal{C}$ has vertex set G and

$$g \sim h \quad \text{if} \quad hg^{-1} \in \mathcal{C}.$$

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Definition

A Cayley graph of an abelian group is called a *translation graph*. Cayley graphs of cyclic groups and elementary abelian 2-groups are called *circulants* and *cubelike graphs*, respectively.

Characters

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The characters of an abelian group G, form a group under pointwise multiplication. This group is isomorphic to G.

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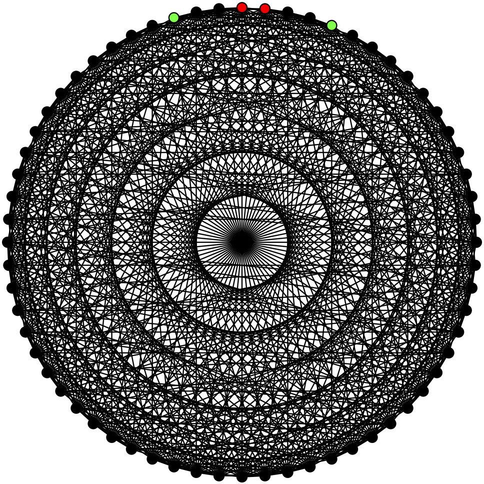

Figure: *Here is another graph*

Spectra of translation graphs

Let $X := X(G, \mathcal{C})$ be a translation graph with adjacency matrix A and let χ be a character of G. Then χ is an eigenvector of A and the corresponding eigenvalue is given by

$$\chi(\mathcal{C}) := \sum_{x \in \mathcal{C}} \chi(x).$$

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Strongly cospectral vertices in translation graphs

Lemma 1 (Coutinho & Godsil)

Let $X = X(G, \mathcal{C})$ be a translation graph. A vertex g is strongly cospectral to zero if and only if both of the following hold:

- *g has order at most two in G*
- *If ψ, φ are characters of G that have the same eigenvalue, then $\psi(g) = \varphi(g)$.*

Some immediate consequences

- A translation graph for a group of odd order has no strongly cospectral vertices.

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- A circulant has at most one vertex (other than zero) that is strongly cospectral to zero. So does any translation graph of a group with a cyclic Sylow-2-subgroup.

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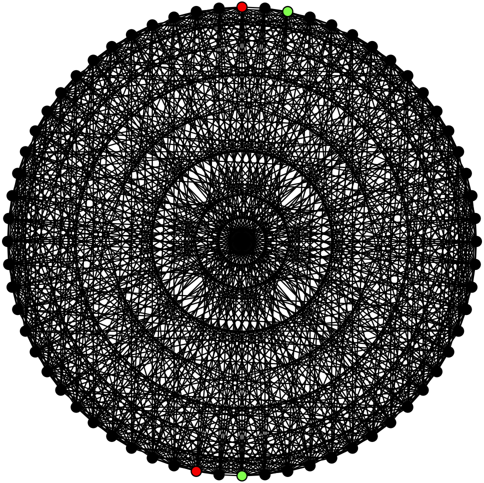
Some immediate consequences

- A translation graph for a group of odd order has no strongly cospectral vertices.
- A circulant has at most one vertex (other than zero) that is strongly cospectral to zero. So does any translation graph of a group with a cyclic Sylow-2-subgroup.
- In a translation graph, the set of vertices that are strongly cospectral to zero form a subgroup of G and it is an elementary abelian 2-group.

Questions?



Cubelike graphs

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Does this mean that all the vertices in a cubelike graph could be pairwise strongly cospectral?

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Does this mean that all the vertices in a cubelike graph could be pairwise strongly cospectral?

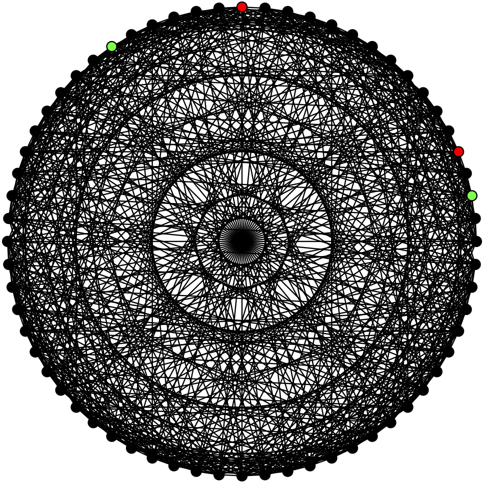
Spoiler: no.

Multiplicity bound

Lemma 2

Let $X = X(G, \mathcal{C})$ be a translation graph, let $H \leq G$ consist of the elements that are strongly cospectral to zero in X, and let ℓ be the index of H in G. If X has an eigenvalue with multiplicity m, then $m \leq \ell$.

Questions?



Spectrum of a cubelike graph

- Recall that the eigenvalues of a translation graph $X(G, \mathcal{C})$ are of the form

$$\sum_{x \in \mathcal{C}} \chi(x)$$

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- Since every non-zero element of $\mathbb{Z}_2^d$ has order two, its characters take only values ± 1.
- It follows that the eigenvalues of a cubelike graph of degree n are integers with the same parity as n.

Cubelike graphs

Claim

The spectrum of a cubelike graph sort of looks normally distributed.

Multiplicity bound

Lemma 3

Let $X = X(\mathbb{Z}_2^d, \mathcal{C})$ be a cubelike graph where $d \geq 3$ and let $k := d/2$. Then X has an eigenvalue with multiplicity larger than 2^k.

Cubelike graphs

Theorem 4

A cubelike graph on 2^d vertices, where $d \geq 3$, has at most

$$2^{\lceil d/2 \rceil - 1}$$

pairwise strongly cospectral vertices.

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Cubelike graphs

Theorem 4

A cubelike graph on 2^d vertices, where $d \geq 3$, has at most

$$2^{\lceil d/2 \rceil - 1}$$

pairwise strongly cospectral vertices.

Remark

The theorem does not hold for $d = 1, 2$: K_2 and C_4 both have a pair of strongly cospectral vertices.

Is the bound tight?

For $d = 3, 4$:

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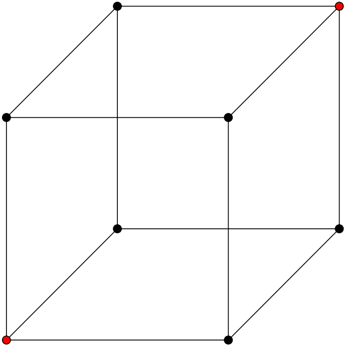
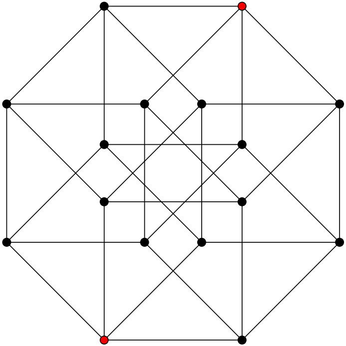
For $d = 3, 4$: yes.





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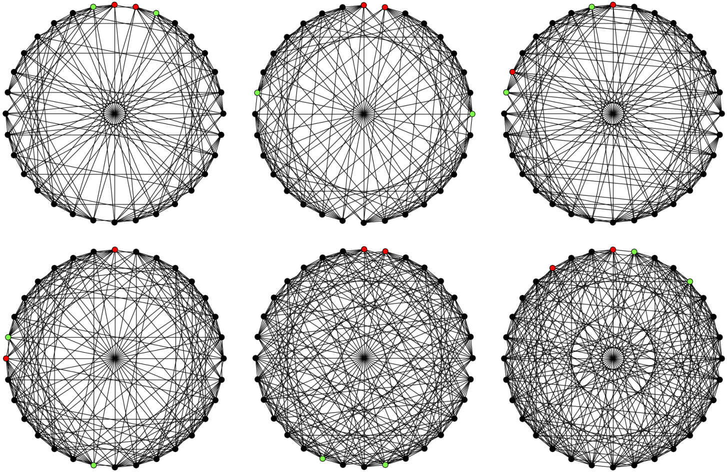

Is the bound tight?

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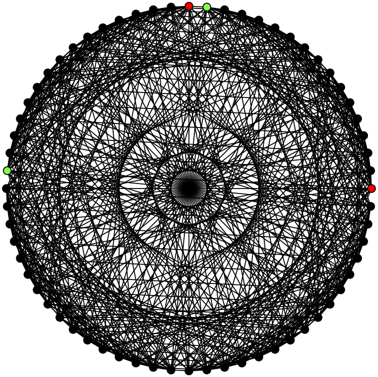

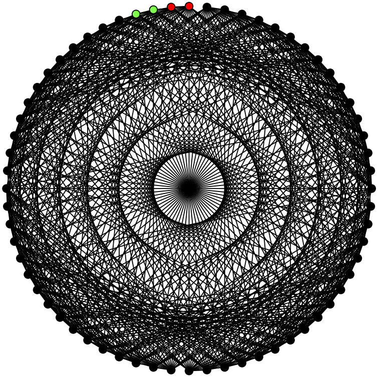

Is the bound tight?

For $d \geq 7$

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Thank you

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